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TOTALLY REAL SURFACES IN THE COMPLEX 2-SPACE

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Introduction

Let M be an immersed oriented surface in the complex 2-space $\mathbb{C}^2 = (\mathbb{R}^4, \langle\ ,\ \rangle, J)$, where $\mathbb{C}^2$ is identified with the real 4-space $\mathbb{R}^4$, and $\langle\ ,\ \rangle$ denotes the standard inner product and J the standard almost complex structure on $\mathbb{R}^4$. A point p in M is called a complex point if the tangent space T_pM is J-invariant. If there is no complex point on M, the surface M is said to be *totally real*, and we obtain that $T_pM \oplus JT_pM = \mathbb{C}^2$ at each point $p \in M$. Especially, if $T_pM \perp JT_pM$ at each point $p \in M$, the surface M is said to be *Lagrangian*.

In this article, we prove that any totally real conformal immersion from M into $\mathbb{C}^2$ can be given merely by an algebraic combination of the components of a solution of a linear system of first order differential equations, which system is a specific Dirac-type equation on M. This equation and the combination are given by means of the Kähler angle function $\alpha : M \to (0, \pi)$ and the Lagrangian angle function $\beta : M \to \mathbb{R}/2\pi\mathbb{Z}$ for the constructed totally real immersed surface M in $\mathbb{C}^2$. Moreover, the pair of α and β describes the self-dual part of the generalized Gauss map of the immersed surface M in the Euclidean 4-space $(\mathbb{R}^4, \langle\ ,\ \rangle)$.

This representation formula for the totally real surfaces in $\mathbb{C}^2$ gives a new method of constructing surfaces in $\mathbb{R}^4$. The particular known methods are the Weierstrass-Kenmotsu formulas for surfaces with prescribed mean curvature in $\mathbb{R}^3$ and $\mathbb{R}^4$ ([Ke1, Ke2]) and their spin versions ([Ko, KL]) (cf. [AA]). The spin versions of Weierstrass-Kenmotsu formulas represent conformal immersions of surfaces by *integrating* a combination of the components of solutions of a similar Dirac-type equation to ours. In [HR], Hélein and Romon have given such a Weierstrass type representation formula for Lagrangian surfaces in $\mathbb{C}^2$. We note that their method does not directly imply the following known result in [CM1]: *Minimal Lagrangian orientable surfaces in $\mathbb{C}^2$ can be represented as holomorphic curves by exchanging the orthogonal complex structure on* $\mathbb{R}^4$, however ours implies this fact as a simple corollary.

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1. Angle functions on a surface in $\mathbb{C}^2$ and the generalized Gauss map

We consider $\mathbb{C}^2$ as the Euclidean 4-space $(\mathbb{R}^4, \langle\ ,\ \rangle)$ with the orthonormal complex structure $J(x_1, x_2, x_3, x_4) = (-x_3, -x_4, x_1, x_2)$, that is, a complex vector $\mathbf{x} = (x_1 + \mathbf{i}x_3, x_2 + \mathbf{i}x_4) \in \mathbb{C}^2$ is identified with the real vector $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. Let $f : M \to \mathbb{C}^2$ be a conformal immersion from a Riemann surface M into $\mathbb{C}^2$. For a given oriented orthonormal basis $\{\mathsf{e}_1, \mathsf{e}_2\}$ of the tangent space f_*T_pM, we put

$$\alpha(T_pM) = \cos^{-1}\langle J\mathsf{e}_1, \mathsf{e}_2\rangle.$$

Then $\alpha(T_pM) \in [0, \pi]$ is independent of the choice of the oriented orthonormal basis of f_*T_pM. $\alpha(p) = \alpha(T_pM)$ is called the *Kähler angle* at $p \in M$. f is totally real if and only if $0 < \alpha < \pi$ at all point of M, and in this case $\alpha : M \to (0, \pi)$ is a smooth function. f is Lagrangian if and only if $\alpha \equiv \pi/2$. Regarding e_1 and e_2 as the complex column vectors in $\mathbb{C}^2$, we can obtain that $|\det(\mathsf{e}_1, \mathsf{e}_2)| = |\sin\alpha|$. Then, if f is totally real, we can define a function $\beta : M \to \mathbb{R}/2\pi\mathbb{Z}$ by, at $p \in M$,

$$\mathsf{e}_1 \wedge \mathsf{e}_2 = e^{\mathbf{i}\beta(p)} \sin\alpha(p)\, \mathbf{e}_1^C \wedge \mathbf{e}_2^C,$$

where $\mathbf{e}_1^C = (1, 0), \mathbf{e}_2^C = (0, 1) \in \mathbb{C}^2$. We call β the *Lagrangian angle* function for f.

Regarding e_1 and e_2 as the real vectors in $\mathbb{R}^4$, we can define the normalization of the real wedge product $\mathsf{e}_1 \wedge \mathsf{e}_2$ and identify it with the real 2-subspace $\mathcal{G}(p)$ parallel to the tangent plane f_*T_pM in $\mathbb{R}^4$. So we obtain the *generalized Gauss map* $\mathcal{G} : M \to G_{2,2}$ of the immersed surface in $\mathbb{R}^4$, where $G_{2,2}$ stands for the Grassmann manifold of oriented 2-planes in $\mathbb{R}^4$. According to the direct sum decomposition of the real wedge product space $\bigwedge^2(\mathbb{R}^4)$ between the self-dual subspace $\bigwedge^2_+$ and the anti-self-dual subspace $\bigwedge^2_-$, $\mathcal{G}$ can be decomposed into the self-dual part $\mathcal{G}_+$ and the anti-self dual part $\mathcal{G}_-$. We consider each of the real 3-spaces $\bigwedge^2_\pm$ as the Euclidean 3-subspace $\mathbb{R}^3$ in $\mathbb{C}^2 \cong \mathbb{R}^4$ defined by $x_1 = 0$, identifying the basis $\{E_1^\pm, E_2^\pm, E_3^\pm\}$ of $\bigwedge^2_\pm$ with the standard basis $\{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ of $\mathbb{R}^3$, where

$$\begin{aligned}
&E_1^\pm = \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_2 \mp \mathbf{e}_3 \wedge \mathbf{e}_4), \quad E_2^\pm = \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_3 \pm \mathbf{e}_2 \wedge \mathbf{e}_4),\\
&E_3^\pm = \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_4 \pm \mathbf{e}_3 \wedge \mathbf{e}_2),\\
&\mathbf{e}_1 = (1, 0, 0, 0), \cdots, \mathbf{e}_4 = (0, 0, 0, 1) \in \mathbb{R}^4.
\end{aligned}$$

Then $\mathcal{G}_+$ and $\mathcal{G}_-$ are maps from M to the unit 2-sphere S^2 in the real 3-space $\mathbb{R}^3$.

Proposition 1. *For a totally real immersed oriented surface M in $\mathbb{C}^2$, the self-dual part $\mathcal{G}_+$ of the generalized Gauss map can be represented in terms of the Kähler angle function α and the Lagrangian angle function β as follows:*

$$\mathcal{G}_+ = (\mathbf{i}\cos\alpha, e^{\mathbf{i}\beta}\sin\alpha) : M \to S^2 \subset \mathbb{C}^2.$$

This proposition follows from the framing method below.

Assume $f : M \to \mathbb{C}^2$ is totally real conformal immersion with the Kähler angle $\alpha : M \to (0, \pi)$ and the Lagrangian angle $\beta : M \to \mathbb{R}/2\pi\mathbb{Z}$. Let $\{e_1, e_2\}$ be an oriented orthonormal tangent frame defined on a neighborhood U in the immersed surface M in $\mathbb{R}^4$. Then we can choose a local orthonormal normal frame $\{e_3, e_4\}$ on U such that

$$e_3 = \frac{1}{\sin\alpha}\big(Je_1 - (\cos\alpha)e_2\big), \quad e_4 = \frac{1}{\sin\alpha}\big(Je_2 + (\cos\alpha)e_1\big). \tag{1.1}$$

Since the identity component $\mathrm{Isom}_0(\mathbb{R}^4)$ of the isometry group of $\mathbb{R}^4$ acts transitively also on the oriented orthonormal frame bundle on $\mathbb{R}^4$, we can take a smooth map $\mathcal{E} : U \to \mathrm{Isom}_0(\mathbb{R}^4)$ such that $f = \mathcal{E}\cdot\mathbf{0}$, $e_a = \mathcal{E}\cdot\mathbf{e}_a - f$ $(a = 1, 2, 3, 4)$. We call this map $\mathcal{E}$ the *adapted framing* of f. Making the most of the complex structure on $\mathbb{C}^2$ and M, we will use the Lie group $G = \mathbb{R}^4 \rtimes (SU(2)\times SU(2))$ instead of $\mathrm{Isom}_0(\mathbb{R}^4) = G/\mathbb{Z}_2$. Identify $\mathbb{C}^2$ with the linear hull $\mathbb{R}{\cdot}SU(2)$ of the special unitary group $SU(2)$ by the map

$$\mathbf{x} = (x_1^C, x_2^C) = (x_1 + \mathbf{i}x_3, x_2 + \mathbf{i}x_4) \ \mapsto\ \underline{\mathbf{x}} = \begin{pmatrix} x_1^C & -\overline{x_2^C} \\ x_2^C & \overline{x_1^C} \end{pmatrix}.$$

So the standard vectors $\mathbf{e}_a$ $(a = 1, 2, 3, 4)$ in $\mathbb{R}^4$ corresponds the following matrices:

$$\underline{\mathbf{e}_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =: \mathbf{I}, \quad \underline{\mathbf{e}_2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \underline{\mathbf{e}_3} = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix} =: \mathbf{J}, \quad \underline{\mathbf{e}_4} = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}.$$

G acts isometrically and transitively on $\mathbb{C}^2$ by

$$\underline{\mathrm{g}\cdot\mathbf{x}} = \mathrm{g}_1\underline{\mathbf{x}}\mathrm{g}_2^* + \underline{\mathbf{v}} \qquad (\mathrm{g} = (\mathbf{v}, (\mathrm{g}_1, \mathrm{g}_2)) \in G = \mathbb{R}^4 \rtimes (SU(2)\times SU(2))\ ).$$

Now we can take the adapted framing $\mathcal{E} : U \to G = \mathbb{R}^4 \rtimes (SU(2)\times SU(2))$ of f as follows:

$$\mathcal{E} = \big(f, (\mathcal{E}_-, \mathcal{E}_+)\big) \quad \text{such that} \quad \underline{e_a} = \mathcal{E}_-\underline{\mathbf{e}_a}\mathcal{E}_+^* \ (a = 1, 2, 3, 4). \tag{1.2}$$

The complex structure $J(\in \mathrm{Isom}_0(\mathbb{R}^4))$ corresponds the action of $(\mathbf{0}, \pm(\mathbf{I}, \mathbf{J})) \in G$. We remark that $\underline{e_{i+2}} = J\cdot(\mathcal{E}\cdot\underline{\mathbf{e}_i}) = \mathcal{E}\cdot(J\cdot\underline{\mathbf{e}_i})$ $(i = 1, 2)$. This fact implies that $\mathcal{E}_+$ can be written as follows and hence it is defined globally:

$$\mathcal{E}_+ = \begin{pmatrix} e^{-\mathbf{i}\beta/2}\cos(\alpha/2) & -\mathbf{i}e^{-\mathbf{i}\beta/2}\sin(\alpha/2) \\ -\mathbf{i}e^{\mathbf{i}\beta/2}\sin(\alpha/2) & e^{\mathbf{i}\beta/2}\cos(\alpha/2) \end{pmatrix} T, \tag{1.3}$$

where

$$T := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathbf{i} \\ \mathbf{i} & 1 \end{pmatrix} \in SU(2),$$

and moreover $(\mathbf{0}, (T, T)) \in G$ acts on $\mathbb{C}^2$ as

$$T\underline{\mathbf{e}_1}T^* = \underline{\mathbf{e}_1},\ T\underline{\mathbf{e}_2}T^* = \underline{\mathbf{e}_3},\ T\underline{\mathbf{e}_3}T^* = -\underline{\mathbf{e}_2},\ T\underline{\mathbf{e}_4}T^* = \underline{\mathbf{e}_4}.$$

Now, we can show that the generalized Gauss map $\mathcal{G} = (\mathcal{G}_+, \mathcal{G}_-) : M \to S^2 \times S^2$ of f is represented as

$$\mathcal{G}_\pm = [(e_1 \wedge e_2)^\pm] = (\mathcal{E}_\pm T^*)\underline{\mathbf{e}_3}(\mathcal{E}_\pm T^*)^* : M \to S^2 \subset \mathbb{R}^3 \cong \mathfrak{su}(2).$$

Moreover, regarding S^2 as the extended complex plane $\hat{\mathbb{C}}$ by the stereographic projection from the north pole $\mathbf{e}_3 \in S^2$, we represent them as

$$\mathcal{G}_\pm = \frac{P_\pm}{Q_\pm} : M \to \hat{\mathbb{C}}, \quad \text{where} \quad \mathcal{E}_\pm T^* = \begin{pmatrix} P_\pm & -\overline{Q_\pm} \\ Q_\pm & \overline{P_\pm} \end{pmatrix} \quad (|P_\pm|^2 + |Q_\pm|^2 = 1).$$

Then we can obtain $\mathcal{G}_+ = \mathbf{i}e^{\mathbf{i}\beta}\cot(\alpha/2)$.

We also represent $\mathcal{G}_\pm$ by means of the complex projective line $\mathbb{C}P^1 \cong S^2$ as

$$\mathcal{G}_\pm = [P_\pm; Q_\pm] : M \to \mathbb{C}P^1.$$

Remark. For a totally real immersed surface M in $\mathbb{C}^2$, we can define a map $\mathcal{G}_0 : M \to S^1$ by $\mathcal{G}_0 = e^{\mathbf{i}\beta}$, where β is the Lagrangian angle. Let Ω be the volume form of S^1. The *Maslov form* defined in [CM2] and [B] coincides with the 1-form $\Phi = (\mathcal{G}_0)^*\Omega = (1/2\pi)d\beta$. The *Maslov class* is the first cohomology class defined by $[\Phi] \in H^1(M;\mathbb{Z})$.

2. Representation formula for totally real surfaces in $\mathbb{C}^2$

Now we give the representation formula of the totally real surfaces in $\mathbb{C}^2$.

Theorem. *Let M be a Riemann surface with an isothermal coordinate $z = x + \mathbf{i}y$. Given two smooth functions $\alpha : M \to (0,\pi)$ and $\beta : M \to \mathbb{R}/2\pi\mathbb{Z}$, put*

$$U_\pm = \frac{1}{2}(\mathbf{i}\alpha_z \pm \beta_z \sin\alpha), \qquad V = \frac{1}{2}\mathbf{i}\beta_z\cos\alpha.$$

Let $F = (F_1, F_2) : M \to \mathbb{C}^2$ be a solution of the Dirac-type equation

$$\begin{pmatrix} 0 & \partial_z \\ -\partial_{\overline{z}} & 0 \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} U_+ & V \\ -\overline{V} & \overline{U_+} \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \tag{2.1}$$

and define a smooth map $S = (S_1, S_2) : M \to \mathbb{C}^2$ as follows:

$$\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \left[\begin{pmatrix} 0 & \partial_z \\ -\partial_{\overline{z}} & 0 \end{pmatrix} + \begin{pmatrix} U_- & V \\ -\overline{V} & \overline{U_-} \end{pmatrix}\right]\begin{pmatrix} \overline{F_1} \\ F_2 \end{pmatrix}.$$

If S does not vanish on M, the following functions

$$f_1 + \mathbf{i}f_3 = \exp(\mathbf{i}\beta/2)\big[\cos(\alpha/2)F_1 - \mathbf{i}\sin(\alpha/2)\overline{F_2}\big],$$
$$f_2 + \mathbf{i}f_4 = \exp(\mathbf{i}\beta/2)\big[\cos(\alpha/2)F_2 + \mathbf{i}\sin(\alpha/2)\overline{F_1}\big]$$

define a conformal immersion $f = (f_1 + \mathbf{i}f_3, f_2 + \mathbf{i}f_4) : M \to \mathbb{C}^2$ with the Kähler angle α and the Lagrangian angle β. The induced metric on M by f takes the form

$$f^*ds^2 = e^{2\lambda}|dz|^2, \quad e^{2\lambda} = |S_1|^2 + |S_2|^2,$$

and the anti-self-dual part $\mathcal{G}_-$ of the generalized Gauss map is given by

$$\mathcal{G}_- = [-S_2; S_1]\big(= -S_2/S_1\big) : M \to S^2 \cong \mathbb{C}P^1(\cong \hat{\mathbb{C}}).$$

Conversely, every totally real conformal immersion $f: M \to \mathbb{C}^2$ with the Kähler angle α and the Lagrangian angle β is congruent with the one constructed as above.

Proof. For a totally real conformal immersion $f: M \to \mathbb{C}^2$ with the Kähler angle α and the Lagrangian angle β, we can define the smooth map $F = (F_1, F_2): M \to \mathbb{C}^2$ by

$$\underline{F} = \begin{pmatrix} F_1 & -\overline{F_2} \\ F_2 & \overline{F_1} \end{pmatrix} := \underline{f}(\mathcal{E}_+ T^*), \tag{2.2}$$

where $\mathcal{E}_+$ is given by (1.3).

Let $\{\omega^1, \omega^2\}$ be the dual coframe for $\{e_1, e_2\}$, and put $\phi = \omega^1 + \mathbf{i}\omega^2$. Locally we choose the isothermal coordinate $z = x + \mathbf{i}y$ on M such that $f_x = e^\lambda e_1$ and $f_y = e^\lambda e_2$. Hence $\phi = e^\lambda dz$ and the induced metric on M by f is given by $f^* ds^2 = \phi \cdot \overline{\phi} = e^{2\lambda}|dz|^2$. We compute that

$$\begin{aligned}
d\underline{f} &= \underline{e_1}\omega^1 + \underline{e_2}\omega^2 = \mathcal{E}_- \underline{\mathbf{e}_1} \mathcal{E}_+^* \omega^1 + \mathcal{E}_- \underline{\mathbf{e}_2} \mathcal{E}_+^* \omega^2 = \mathcal{E}_- T^* (\underline{\mathbf{e}_1}\omega^1 + \underline{\mathbf{e}_3}\omega^2) T \mathcal{E}_+^* \\
&= \mathcal{E}_- T^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T \mathcal{E}_+^* \phi + \mathcal{E}_- T^* \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} T \mathcal{E}_+^* \overline{\phi}, \\
(d\underline{f})\mathcal{E}_+ T^* &= \begin{pmatrix} P_- & 0 \\ Q_- & 0 \end{pmatrix} \phi + \begin{pmatrix} 0 & -\overline{Q_-} \\ 0 & \overline{P_-} \end{pmatrix} \overline{\phi}, \\
d\underline{F} &= \begin{pmatrix} (F_1)_z & * \\ (F_2)_z & * \end{pmatrix} dz + \begin{pmatrix} (F_1)_{\overline{z}} & * \\ (F_2)_{\overline{z}} & * \end{pmatrix} d\overline{z} \\
&= d\underline{f}(\mathcal{E}_+ T^*) + \underline{f} d(\mathcal{E}_+ T^*) = d\underline{f}(\mathcal{E}_+ T^*) + \underline{F}(\mathcal{E}_+ T^*)^{-1} d(\mathcal{E}_+ T^*),
\end{aligned}$$

$$(\mathcal{E}_+ T^*)^{-1} d(\mathcal{E}_+ T^*) = -\begin{pmatrix} V & U_+ \\ U_- & -V \end{pmatrix} dz + \begin{pmatrix} \overline{V} & \overline{U_-} \\ \overline{U_+} & -\overline{V} \end{pmatrix} d\overline{z}, \qquad \text{and} \tag{2.3}$$

$$d\underline{F} = \begin{pmatrix} P_- e^\lambda - VF_1 + U_-\overline{F_2} & * \\ Q_- e^\lambda - VF_2 - U_-\overline{F_1} & * \end{pmatrix} dz + \begin{pmatrix} \overline{V}F_1 - \overline{U_+ F_2} & * \\ \overline{V}F_2 + \overline{U_+ F_1} & * \end{pmatrix} d\overline{z}.$$

Then we obtain that

$$\begin{aligned}
&(F_1)_{\overline{z}} = \overline{V}F_1 - \overline{U_+ F_2}, \quad (F_2)_{\overline{z}} = \overline{U_+ F_1} + \overline{V}F_2, \\
&(F_1)_z = P_- e^\lambda - VF_1 + U_-\overline{F_2}, \quad (F_2)_z = Q_- e^\lambda - U_-\overline{F_1} - VF_2.
\end{aligned}$$

Put $S_1 = Q_- e^\lambda$ and $S_2 = -P_- e^\lambda$, then we have

$$\begin{aligned}
S_1 &= (F_2)_z + U_-\overline{F_1} + VF_2 \big(= e^{-\mathbf{i}\beta}(f_2 + \mathbf{i}f_4)_z / \cos(\alpha/2)\big), \\
S_2 &= -(F_1)_z - VF_1 + U_-\overline{F_2} \big(= -e^{-\mathbf{i}\beta}(f_1 + \mathbf{i}f_3)_z / \cos(\alpha/2)\big).
\end{aligned} \tag{2.4}$$

This completes the proof of Theorem. □

Moreover, we obtain the following

Proposition 2. *The spinor representation $S = (S_1, S_2) : M \to \mathbb{C}^2$ of $\mathcal{G}_- : M \to S^2$, which is defined by* (2.4)*, satisfies the Dirac-type equation*

$$\begin{pmatrix} 0 & \partial_z \\ -\partial_{\overline{z}} & 0 \end{pmatrix} \begin{pmatrix} S_1 \\ \overline{S_2} \end{pmatrix} = \begin{pmatrix} \overline{U_-} & V \\ -\overline{V} & U_- \end{pmatrix} \begin{pmatrix} S_1 \\ \overline{S_2} \end{pmatrix}. \tag{2.5}$$

Remark. For a Lagrangian conformal immersed surface in $\mathbb{C}^2$ with the Lagrangian angle β, we obtain that $V \equiv 0$ and $U_\pm = \pm\beta_z/2$. Hence the Dirac-type equations (2.1) and (2.5) are the same as the Davey-Stewartson linear problem appeared in the Konopelchenko's representation for surfaces in $\mathbb{R}^4$ ([KL]). For the explicit representation formula of Lagrangian immersed surfaces in $\mathbb{C}^2$, see also [A]. The Hélein-Romon's representation formula for Lagrangian surfaces in $\mathbb{C}^2$ ([HR]) corresponds to the method of constructing a surface by integrating a combination of the components of a solution S of the Dirac-type equation (2.5).

Here we give a simple example.

Example (Clifford torus)*.* The rectangular torus $T = \mathbb{C}/(a_1\mathbb{Z} \oplus \mathbf{i}a_2\mathbb{Z})$ is conformally embedded in $\mathbb{C}^2$ as the product of circles by the map

$$f(x + \mathbf{i}y) = \big(\mathbf{i}a_1 e^{2\pi\mathbf{i}x/a_1}, \mathbf{i}a_2 e^{2\pi\mathbf{i}y/a_2}\big).$$

(When $a_1 = a_2 = 1$, it is called the Clifford torus.) It is well known that this torus in $\mathbb{R}^4$ is flat and has parallel mean curvature vector. Moreover, it is a Lagrangian surface with the Lagrangian angle $\beta = 2\pi(x/a_1 + y/a_2)$, and hence the Maslov class $(1,1) \in H^1(T;\mathbb{Z}) \cong \mathbb{Z}^2$. So this immersion f corresponds to the solution

$$F = (F_1, F_2) = (1/\sqrt{2})(a_2 + \mathbf{i}a_1)\big(e^{\pi\mathbf{i}(x/a_2 - y/a_1)}, \mathbf{i}e^{-\pi\mathbf{i}(x/a_2 - y/a_1)}\big)$$

of the Dirac-type equation

$$\begin{pmatrix} 0 & \partial_z \\ -\partial_{\overline{z}} & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ \overline{F_2} \end{pmatrix} = \begin{pmatrix} \frac{\pi}{2}(\frac{1}{a_1} - \mathbf{i}\frac{1}{a_2}) & 0 \\ 0 & \frac{\pi}{2}(\frac{1}{a_1} + \mathbf{i}\frac{1}{a_2}) \end{pmatrix} \begin{pmatrix} F_1 \\ \overline{F_2} \end{pmatrix}.$$

3. Curvatures of totally real surfaces in $\mathbb{C}^2$

Let $f : M \to \mathbb{C}^2$ be a totally real conformal immersion with the Kähler angle α and the Lagrangian angle β, and let $\mathcal{E} = (f, (\mathcal{E}_-, \mathcal{E}_+)) : M \to G = \mathbb{R}^4 \rtimes (SU(2) \times SU(2))$ be the adapted framing of f as in (1.2) and (1.3). The Gauss-Weingarten equation of the immersed surface in $\mathbb{R}^4$ is given by the pull-back of the Maurer-Cartan form on the Lie group G by $\mathcal{E}$, and hence described as follows:

$$\mathcal{E}_-^{-1} d\mathcal{E}_- = \frac{1}{2} \begin{pmatrix} \mathbf{i}(\omega_1^3 - \omega_2^4) & -(\omega_3^4 + \omega_1^2) + \mathbf{i}(\omega_2^3 + \omega_1^4) \\ (\omega_3^4 + \omega_1^2) + \mathbf{i}(\omega_2^3 + \omega_1^4) & -\mathbf{i}(\omega_1^3 - \omega_2^4) \end{pmatrix},$$

$$\mathcal{E}_+^{-1} d\mathcal{E}_+ = \frac{1}{2} \begin{pmatrix} -\mathbf{i}(\omega_1^3 + \omega_2^4) & -(\omega_3^4 - \omega_1^2) + \mathbf{i}(\omega_2^3 - \omega_1^4) \\ (\omega_3^4 - \omega_1^2) + \mathbf{i}(\omega_2^3 - \omega_1^4) & \mathbf{i}(\omega_1^3 + \omega_2^4) \end{pmatrix},$$

where ω_b^a are the connection forms on M defined by $\omega_b^a = \langle e_a, \overline{\nabla} e_b \rangle$ for the Levi-Civita connection of $(\mathbb{R}^4, \langle\ ,\ \rangle)$. Moreover, from (1.1), we obtain that

$$\omega_3^4 = \omega_1^2 - \cot\alpha(\omega_1^3 + \omega_2^4), \qquad \omega_2^3 = \omega_1^4 - d\alpha. \tag{3.1}$$

Hence the Gauss-Weingarten equation of the totally real immersed surface in $\mathbb{C}^2$ is written as the following matrix equation

$$\mathcal{E}_-^{-1}d\mathcal{E}_- = T^* \begin{pmatrix} \mathbf{i}(\rho - \cot\alpha\eta) & -\overline{\psi} \\ \psi & -\mathbf{i}(\rho - \cot\alpha\eta) \end{pmatrix} T, \tag{3.2}$$

$$\mathcal{E}_+^{-1}d\mathcal{E}_+ = T^* \begin{pmatrix} -\mathbf{i}\cot\alpha\eta & -\eta - (\mathbf{i}/2)d\alpha \\ \eta - (\mathbf{i}/2)d\alpha & \mathbf{i}\cot\alpha\eta \end{pmatrix} T, \tag{3.3}$$

where $\rho = \omega_1^2$, $\psi = (1/2)\{(\omega_2^4 - \omega_1^3) + \mathbf{i}(\omega_1^4 + \omega_2^3)\}$ and $\eta = (1/2)(\omega_1^3 + \omega_2^4)$. Combining (2.3) with (3.3), we obtain

$$\eta = \frac{1}{2}\sin\alpha d\beta. \tag{3.4}$$

The second fundamental form of f is given by

$$\Pi = h_{ij}^3\omega^i \otimes \omega^j \otimes e_3 + h_{ij}^4\omega^i \otimes \omega^j \otimes e_4,$$

where $h_{ij}^3 = \omega_i^3(e_j) = \omega_j^3(e_i)$ and $h_{ij}^4 = \omega_i^4(e_j) = \omega_j^4(e_i)$ $(i, j = 1, 2)$. Moreover, from the second equation in (3.1), these components satisfy

$$h_{11}^4 = h_{12}^3 + d\alpha(e_1), \quad h_{12}^4 = h_{22}^3 + d\alpha(e_2).$$

Put $h^3 = (1/2)(h_{11}^3 + h_{22}^3)$ and $h^4 = (1/2)(h_{11}^4 + h_{22}^4)$. The mean curvature vector $\overrightarrow{H}$ of f is given by

$$\overrightarrow{H} = h^3e_3 + h^4e_4 = \frac{1}{2}(h^3 + \mathbf{i}h^4)(e_3 - \mathbf{i}e_4) + \frac{1}{2}(h^3 - \mathbf{i}h^4)(e_3 + \mathbf{i}e_4).$$

The 1-form η is also given by

$$\begin{aligned} \eta &= \frac{1}{2}(h_{11}^3 + h_{21}^4)\omega^1 + \frac{1}{2}(h_{12}^3 + h_{22}^4)\omega^2 \\ &= h^3\omega^1 + h^4\omega^2 + \frac{1}{2}\{d\alpha(e_2)\omega^1 - d\alpha(e_1)\omega^2\} \\ &= \frac{1}{2}\{(h^3 - \mathbf{i}h^4)\phi + (h^3 + \mathbf{i}h^4)\overline{\phi}\} + \frac{\mathbf{i}}{2}\{\alpha_z dz - \alpha_{\overline{z}}d\overline{z}\}. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we obtain that $h^3 - \mathbf{i}h^4 = -2e^{-\lambda}U_-$. Namely, the mean curvature vector $\overrightarrow{H}$ has the representation of

$$\overrightarrow{H} = \mathbf{i}e^{-2\lambda}\{-\overline{U_-}\cot(\alpha/2)f_z + U_-\tan(\alpha/2)f_{\overline{z}}\},$$

and the mean curvature $H = |\overrightarrow{H}|$ is given by

$$H = 2e^{-\lambda}|U_-|.$$

It follows from (3.2) combined with

$$\mathcal{E}_+T^* = e^{-\lambda}\begin{pmatrix} -S_2 & -\overline{S_1} \\ S_1 & -\overline{S_2} \end{pmatrix}$$

that

$$\psi = e^{-2\lambda}(S_1dS_2 - S_2dS_1),$$

$$\rho = \frac{1}{2}\{(\cos\alpha)d\beta - \mathbf{i}e^{-2\lambda}(\overline{S_1}dS_1 + \overline{S_2}dS_2 - S_1 d\overline{S_1} - S_2 d\overline{S_2})\},$$
$$d\rho = -\frac{1}{2}(\sin\alpha)d\alpha \wedge d\beta + \mathbf{i}\psi \wedge \overline{\psi}.$$

We note that the $(0,1)$-part $\psi'' d\bar{z}$ of $\psi = \psi' dz + \psi'' d\bar{z}$ coincides with $-(1/2)(h^3 - \mathbf{i}h^4)\overline{\phi} = U_- d\bar{z}$. The Gauss curvature K of f is given by $d\rho = -(\mathbf{i}/2)K\phi \wedge \overline{\phi}$, and hence

$$K = -e^{-2\lambda}\{\mathbf{i}(\alpha_z \beta_{\bar{z}} - \alpha_{\bar{z}}\beta_z)\sin\alpha + 2(|\psi'|^2 - |\psi''|^2)\}.$$

Proposition 3. *If a totally real immersed oriented surface M in $\mathbb{C}^2$ is minimal, the Kähler angle α and Lagrangian angle β satisfies the partial differential equation*

$$\mathbf{i}\alpha_z - \beta_z \sin\alpha = 0,$$

and hence the Gauss curvature is given by

$$K = -2e^{-2\lambda}(|\alpha_z|^2 + |\psi'|^2).$$

Corollary. *If a totally real immersed oriented surface M in $\mathbb{C}^2$ with either constant Kähler angle or constant Lagrangian angle is minimal, then the other angle is also constant and the map $F = (F_1, F_2) : M \to \mathbb{C}^2$ defined as in (2.2) is holomorphic. Namely, such a surface can be represented as a holomorphic curve by exchanging the orthogonal complex structure on $\mathbb{R}^4$.*

So, this corollary implies the known result for minimal Lagrangian surfaces in $\mathbb{C}^2$ mentioned as in Introduction (Chen-Morvan [CM1]).

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