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ON CERTAIN PROPERTY OF A CLOSED 2-FORM

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The goal of this paper is to show that in $\mathbb{R}^n$ or $\mathbb{C}^n$, if $\omega = \omega_1 \wedge \omega_2$ is a 2-form such that $d\omega = 0$, then there exist 1-forms α and β such that $\omega = \alpha \wedge \beta$ and $d\alpha \wedge \alpha = d\beta \wedge \beta = 0$.

In order to fix the ideas, we will work initially with the n-dimensional space $\mathbb{R}^n$. Let $p \in \mathbb{R}^n$, $\mathbb{R}^n_p$ the tangent space of $\mathbb{R}^n$ at p and $(\mathbb{R}^n_p)^*$ its dual spaces. Let $\bigwedge^k(\mathbb{R}^n_p)^*$ be the set of all k-linear alternating maps

$$\varphi : \underbrace{\mathbb{R}^n_p \times \cdots \times \mathbb{R}^n_p}_{k \text{ times}} \to \mathbb{R}.$$

With the usual operations, $\bigwedge^k(\mathbb{R}^n_p)^*$ is a vector space. Given $\varphi_1, \ldots, \varphi_k \in (\mathbb{R}^n_p)^*$, we can obtain an element $\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_k$ of $\bigwedge^k(\mathbb{R}^n_p)^*$ by setting

$$(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_k)(v_1, v_2, \ldots, v_k) = \det(\varphi_i(v_j)),$$

where $v_1, \ldots, v_k \in \mathbb{R}^n_p$, $i, j = 1, \ldots, k$. It follows from the properties of determinates that $\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_k$ is in fact k-linear and alternate.

The set

$$\{(dx^{i_1} \wedge \cdots \wedge dx^{i_k})_p, i_1 < i_2 < \cdots < i_k, i, j \in \{1, \ldots, n\}\}$$

is a basics for $\bigwedge^k(\mathbb{R}^n_p)^*$. An exterior k-form in $\mathbb{R}^n$ is a map φ that associates to each $p \in \mathbb{R}^n$ an element $\varphi(p) \in \bigwedge^k(\mathbb{R}^n_p)^*$ and φ can be written as

$$\varphi(p) = \sum_{i_1 < \cdots < i_k} a_{i_1} \ldots a_{i_k}(p)(dx^{i_1} \wedge \cdots \wedge dx^{i_k})_p,$$

$i_j \in \{1, \ldots, n\}$, where $a_{i_1 \ldots i_k}$ are differentiable functions, φ is called differential k-form.

Proposition 1. *If $\varphi_1 \wedge \cdots \wedge \varphi_k = \beta_1 \wedge \cdots \wedge \beta_k = \varphi$, where $\beta_j \in (\mathbb{R}^n_p)^*$, $j = 1, \ldots, k$, are two representations of φ, then $\varphi_i = \sum_j a_{ij}\beta_j$, $i = 1, \ldots, k$, with* $\det(a_{ij}) = 1$.

Proof. Extend the β_j into a basis $\beta_1, \ldots, \beta_k$, $\beta_{k+1}, \ldots, \beta_n$ of $(\mathbb{R}^n_p)^*$ and write

$$\varphi_i = \sum_j a_{ij}\beta_j + \sum_r b_{ir}\beta_r, r = k+1, \ldots, n.$$

Notice that $\beta_1 \wedge \dots \beta_k \wedge \varphi_i = \varphi_1 \wedge \dots \wedge \varphi_k \wedge \varphi_i = 0$. This implies that

$$\sum_r b_{ir} \beta_1 \wedge \dots \wedge \beta_k \wedge \beta_r = 0,$$

and since $\beta_1 \wedge \dots \wedge \beta_k \wedge \beta_r$ are linearly independent, so $b_{ir} = 0$.

Now let $v_1, \dots, v_k$ be the vectors of $\mathbb{R}^n_p$. From definition it clearly follows that

$$\begin{aligned}\langle \varphi_1 \wedge \dots \wedge \varphi_k ; v_1, \dots, v_k \rangle &= \det(\langle \varphi_i, v_r \rangle) \\ &= \det\Big(\Big\langle \sum_{j=1}^k a_{ij}\beta_j, v_r \Big\rangle\Big) \\ &= \det(a_{ij}) \sum_{r_1=1}^k \cdots \sum_{r_k=1}^k \epsilon_{r_1 \dots r_k} \langle \beta_1, v_{r_1} \rangle \dots \langle \beta_k, v_{r_k} \rangle \\ &= \det(a_{ij}) \det(\langle \beta_j, v_r \rangle) \\ &= \det(a_{ij}) \langle \beta_1 \wedge \dots \wedge \beta_k ; v_1, \dots, v_k \rangle.\end{aligned}$$

The symbol

$$\epsilon^{i_1 \dots i_n} \text{ (or } \epsilon_{i_1 \dots i_n})$$

is zero unless $i_1, \dots, i_n$ is some derangement of the first n natural numbers. If the derangement is an even permutation,

$$\epsilon^{i_1 \dots i_n} = \epsilon_{i_1 \dots i_n} = 1.$$

If the derangement is an odd permutation,

$$\epsilon^{i_1 \dots i_n} = \epsilon_{i_1 \dots i_n} = -1.$$

Thus,

$$\det(a_{ij}) = 1,$$

as we wished to prove.

Proposition 2. *Let P, Q, R denote three differential functions in an open set U of $\mathbb{R}^3$, which do not vanish simultaneously; put*

$$\omega = P dx^2 \wedge dx^3 + Q dx^3 \wedge dx^1 + R dx^1 \wedge dx^2.$$

Then in the neighbourhood of each point of U there exist pairs of functions u, v satisfying

$$du \wedge \omega = 0, \quad dv \wedge \omega = 0, \quad du \wedge dv \neq 0,$$

and if u, v is such a pair, there exists a function λ such that

$$\omega = \lambda du \wedge dv.$$

PROOF. Consider the forms

$$\begin{aligned} du\wedge\omega &= \left(\frac{\partial u}{\partial x^1}dx^1+\frac{\partial u}{\partial x^2}dx^2+\frac{\partial u}{\partial x^3}dx^3\right)\wedge \\ &\wedge\left(Pdx^2\wedge dx^3+Qdx^3\wedge dx^1+Rdx^1\wedge dx^2\right) \\ &= \left(P\frac{\partial u}{\partial x^1}+Q\frac{\partial u}{\partial x^2}+R\frac{\partial u}{\partial x^3}\right)dx^1\wedge dx^2\wedge dx^3, \\ dv\wedge\omega &= \left(\frac{\partial v}{\partial x^1}dx^1+\frac{\partial v}{\partial x^2}dx^2+\frac{\partial v}{\partial x^3}dx^3\right)\wedge \\ &\wedge\left(Pdx^2\wedge dx^3+Qdx^3\wedge dx^1+Rdx^1\wedge dx^2\right) \\ &= \left(P\frac{\partial v}{\partial x^1}+Q\frac{\partial v}{\partial x^2}+R\frac{\partial v}{\partial x^3}\right)dx^1\wedge dx^2\wedge dx^3 \end{aligned}$$

and

$$\begin{aligned} du\wedge dv &= \left(\frac{\partial u}{\partial x^1}dx^1+\frac{\partial u}{\partial x^2}dx^2+\frac{\partial u}{\partial x^3}dx^3\right)\wedge\left(\frac{\partial v}{\partial x^1}dx^1+\frac{\partial v}{\partial x^2}dx^2+\frac{\partial v}{\partial x^3}dx^3\right) \\ &= \left(\frac{\partial u}{\partial x^2}\frac{\partial v}{\partial x^3}-\frac{\partial u}{\partial x^3}\frac{\partial v}{\partial x^2}\right)dx^2\wedge dx^3+\left(\frac{\partial u}{\partial x^3}\frac{\partial v}{\partial x^1}-\frac{\partial u}{\partial x^1}\frac{\partial v}{\partial x^3}\right)dx^2\wedge dx^1+ \\ &+\left(\frac{\partial u}{\partial x^1}\frac{\partial v}{\partial x^2}-\frac{\partial u}{\partial x^2}\frac{\partial v}{\partial x^1}\right)dx^1\wedge dx^2. \end{aligned}$$

From the conditions $du\wedge\omega=0$ and $dv\wedge\omega=0$ it follows that

$$P\frac{\partial u}{\partial x^1}+Q\frac{\partial u}{\partial x^2}+R\frac{\partial u}{\partial x^3}=0 \text{ and } P\frac{\partial v}{\partial x^1}+Q\frac{\partial v}{\partial x^2}+R\frac{\partial v}{\partial x^3}=0.$$

And hence, in the neighbourhood of each point of U the vector fields $(\frac{\partial u}{\partial x^1},\frac{\partial u}{\partial x^2},\frac{\partial u}{\partial x^3})$ and $(\frac{\partial v}{\partial x^1},\frac{\partial v}{\partial x^2},\frac{\partial v}{\partial x^3})$ are orthogonal to the vector field (P,Q,R). Thus

$$\begin{aligned} (P,Q,R) \,\Big\|\, &\left(\frac{\partial u}{\partial x^1},\frac{\partial u}{\partial x^2},\frac{\partial u}{\partial x^3}\right)\times\left(\frac{\partial v}{\partial x^1},\frac{\partial v}{\partial x^2},\frac{\partial v}{\partial x^3}\right)= \\ &=\left(\left(\frac{\partial u}{\partial x^2}\frac{\partial v}{\partial x^3}-\frac{\partial u}{\partial x^3}\frac{\partial v}{\partial x^2}\right),\left(\frac{\partial u}{\partial x^3}\frac{\partial v}{\partial x^1}-\frac{\partial u}{\partial x^1}\frac{\partial v}{\partial x^3}\right),\left(\frac{\partial u}{\partial x^1}\frac{\partial v}{\partial x^2}-\frac{\partial u}{\partial x^2}\frac{\partial v}{\partial x^1}\right)\right), \end{aligned}$$

since from the condition $du\wedge dv\neq 0$ it follows that not all the coordinates of this vector field, in the neighbourhood of each point of U, vanish simultaneously. Therefore, from parallely it follows that there exists a function λ such that

$$\omega=\lambda du\wedge dv. \blacksquare$$

Now we consider a differential 3-form $d\omega$:

$$\begin{aligned} d\omega &= d(\lambda du\wedge dv) \\ &= d\lambda\wedge du\wedge dv+\lambda d(du\wedge dv) \\ &= d\lambda\wedge du\wedge dv. \end{aligned}$$

If $d\omega = 0$, then

$$\frac{\partial\lambda}{\partial x^1}\left(\frac{\partial u}{\partial x^2}\frac{\partial v}{\partial x^3}-\frac{\partial u}{\partial x^3}\frac{\partial v}{\partial x^2}\right)+\frac{\partial\lambda}{\partial x^2}\left(\frac{\partial u}{\partial x^2}\frac{\partial v}{\partial x^1}-\frac{\partial u}{\partial x^1}\frac{\partial v}{\partial x^2}\right)+\frac{\partial\lambda}{\partial x^3}\left(\frac{\partial u}{\partial x^1}\frac{\partial v}{\partial x^2}-\frac{\partial u}{\partial x^2}\frac{\partial v}{\partial x^1}\right) = 0,$$

this means that in the neighbourhood of each point of U the vector field $\left(\frac{\partial\lambda}{\partial x^1}, \frac{\partial\lambda}{\partial x^2}, \frac{\partial\lambda}{\partial x^3}\right)$ is orthogonal to the vector field

$$\left(\left(\frac{\partial u}{\partial x^2}\frac{\partial v}{\partial x^3}-\frac{\partial u}{\partial x^3}\frac{\partial v}{\partial x^2}\right),\left(\frac{\partial u}{\partial x^2}\frac{\partial v}{\partial x^1}-\frac{\partial u}{\partial x^1}\frac{\partial v}{\partial x^2}\right),\left(\frac{\partial u}{\partial x^1}\frac{\partial v}{\partial x^2}-\frac{\partial u}{\partial x^2}\frac{\partial v}{\partial x^1}\right)\right).$$

Hence and from Proposition 2 it follows that the vector field $\left(\frac{\partial\lambda}{\partial x^1}, \frac{\partial\lambda}{\partial x^2}, \frac{\partial\lambda}{\partial x^3}\right)$ is a linear combination vector fields $\left(\frac{\partial u}{\partial x^1}, \frac{\partial u}{\partial x^2}, \frac{\partial u}{\partial x^3}\right)$ and $\left(\frac{\partial v}{\partial x^1}, \frac{\partial v}{\partial x^2}, \frac{\partial v}{\partial x^3}\right)$. Thus $\lambda = au + bv$, where a, b are constants. Inversely, if $\lambda = au + bv$, where a, b are constats, then

$$\begin{aligned} d\omega &= d\lambda \wedge du \wedge dv \\ &= d(au + bv) \wedge du \wedge dv \\ &= (adu + bdv) \wedge du \wedge dv \\ &= adu \wedge du \wedge dv + bdv \wedge du \wedge dv \\ &= 0. \end{aligned}$$

Now notice that if $\lambda = au + bv$, where a, b are constants, then

$$\omega = \lambda du \wedge dv = d\left(\frac{1}{2}au^2 + buv\right) \wedge dv.$$

Setting $f = \frac{1}{2}au^2 + buv$, $g = v$ we have

$$\omega = df \wedge dg.$$

Hence the following:

Proposition 3. *In the neighbourhood of each point of U there exist two functions f, g such that $\omega = df \wedge dg$ if and only if $d\omega = 0$.*

Notice furthermore that if α, β are 1-forms in $U \subset \mathbb{R}^3$, $\omega = \alpha \wedge \beta$ and $d\omega = 0$, then $\alpha = df$ and $\beta = dg$. From

$$\begin{aligned} d(\alpha \wedge \beta) &= d\alpha \wedge \beta - \alpha \wedge d\beta \\ &= \left[\left(\frac{\partial a_3}{\partial x^2} - \frac{\partial a_2}{\partial x^3}\right)b_1 - \left(\frac{\partial a_3}{\partial x^1} - \frac{\partial a_1}{\partial x^3}\right)b_2 + \left(\frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2}\right)b_3\right] + \\ &\quad - \left[a_1\left(\frac{\partial b_3}{\partial x^2} - \frac{\partial b_2}{\partial x^3}\right) - a_2\left(\frac{\partial b_3}{\partial x^1} - \frac{\partial b_1}{\partial x^3}\right) + a_3\left(\frac{\partial b_2}{\partial x^1} - \frac{\partial b_1}{\partial x^2}\right)\right] \\ &\quad \cdot dx^1 \wedge dx^2 \wedge dx^3 \\ &= [\mathrm{rot}(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) - (a_1, a_2, a_3)\,\mathrm{rot}(b_1, b_2, b_3)] \\ &\quad \cdot dx^1 \wedge dx^2 \wedge dx^3, \end{aligned}$$

where $\alpha = \sum_{i=1}^{3} a_i(x)dx^i$ and $\beta = \sum_{i=1}^{3} b_i(x)dx^i$, and from Proposition 3 it follows that $d(\alpha \wedge \beta) = 0$ if and only if $\operatorname{rot}(a_1, a_2, a_3) = 0$ and $\operatorname{rot}(b_1, b_2, b_3) = 0$ in the neighbourhood of each point of U.

Now let $\alpha = \sum_{i=1}^{n} a_i(x)dx^i$ and $\beta = \sum_{i=1}^{n} b_i(x)dx^i$ be two differential forms of degree one defined in an open set $U \subset \mathbb{R}^n$. The exterior differential $d(\alpha \wedge \beta)$ of $\alpha \wedge \beta$ is defined by

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta - \alpha \wedge d\beta.$$

Therefore

$$\begin{aligned}
d(\alpha \wedge \beta) &= d\Big(\sum_i a_i(x)dx^i\Big) \wedge \sum_j b_j(x)dx^j - \sum_i a_i(x)dx^i \wedge d\Big(\sum_j b_j(x)dx^j\Big) \\
&= \sum_i da_i(x) \wedge dx^i \wedge \sum_j b_j(x)dx^j - \sum_i a_i(x)dx^i \wedge \sum_j db_j(x) \wedge dx^j \\
&= \sum_{i_1<i_2<i_3} \Bigg[\Big(\frac{\partial a_{i_3}(x)}{\partial x^{i_2}} - \frac{\partial a_{i_2}(x)}{\partial x^{i_3}}\Big) b_{i_1} - \Big(\frac{\partial a_{i_3}(x)}{\partial x^{i_1}} - \frac{\partial a_{i_1}(x)}{\partial x^{i_3}}\Big) b_{i_2} + \\
&\quad + \Big(\frac{\partial a_{i_2}(x)}{\partial x^{i_1}} - \frac{\partial a_{i_1}(x)}{\partial x^{i_2}}\Big) b_{i_3}\Bigg] dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3} + \\
&\quad - \sum_{i_1<i_2<i_3} \Bigg[a_{i_1}\Big(\frac{\partial b_{i_3}(x)}{\partial x^{i_2}} - \frac{\partial b_{i_2}(x)}{\partial x^{i_3}}\Big) - a_{i_2}\Big(\frac{\partial b_{i_3}(x)}{\partial x^{i_1}} - \frac{\partial b_{i_1}(x)}{\partial x^{i_3}}\Big) + \\
&\quad + a_{i_3}\Big(\frac{\partial b_{i_2}(x)}{\partial x^{i_1}} - \frac{\partial b_{i_1}(x)}{\partial x^{i_2}}\Big)\Bigg] dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3} \\
&= \sum_{i_1<i_2<i_3} [\operatorname{rot}(a_{i_1}, a_{i_2}, a_{i_3}) \cdot (b_{i_1}, b_{i_2}, b_{i_3}) + \\
&\quad - (a_{i_1}, a_{i_2}, a_{i_3}) \cdot \operatorname{rot}(b_{i_1}, b_{i_2}, b_{i_3})] dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3}
\end{aligned}$$

for all $1 \leq i_1 < i_2 < i_3 \leq n$. That is in the neighbourhood of each point U, so $d(\alpha \wedge \beta) = 0$ if and only if

$$\operatorname{rot}(a_{i_1}, a_{i_2}, a_{i_3}) = 0 \text{ and } \operatorname{rot}(b_{i_1}, b_{i_2}, b_{i_3}) = 0$$

for all $1 \leq i_1 < i_2 < i_3 \leq n$. This implies that

$$\operatorname{rot} a = 0 \text{ and } \operatorname{rot} b = 0, \tag{1}$$

if $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$, differential vector fields, are correspondence 1-forms α and β induceed by the inner product.

We would like to remind that if v is a differential vector field in $\mathbb{R}^n$, then the rotational $\operatorname{rot} v$ is the $(n-2)$-form defined by

$$v \mapsto \omega \mapsto d\omega \mapsto *(d\omega) = \operatorname{rot} v,$$

where $v \mapsto \omega$ is the correspondence between 1-forms and vector fields induceed by the inner product.

Notice that the condition (1) is equivalent to:

$$d\alpha = 0 \text{ and } d\beta = 0. \tag{2}$$

Hence and from Poincaré's Lemma for 1-forms, it follows that the forms α and β are locally exact, i.e., for each $p \in U$ there is a neighbourhood $V \subset U$ of p and differentiable functions $f, g : V \to \mathbb{R}$ such that $df = \alpha$ and $dg = \beta$.

We will say that a connected open set $U \subset \mathbb{R}^n$ is simply-connected if every continuous closed curve in U is freely homotopic to a point in U.

PROPOSITION 4. *Let ω be a closed form defined in a simply-connected domain. Then ω is exact (proof, see [2]).*

From the above it follows that if the set U is simply-connected, then α and β are exact in U. Thus, the condition $d\alpha \wedge \alpha = d\beta \wedge \beta = 0$ are satisfied.

If we change all real field $\mathbb{R}^n$ into complex field $\mathbb{C}^n$ and each mapping a_i and b_i, $i = 1, \ldots, n$, into complex holomorphic function in the previous definition, the form α and β are called complex holomorphic forms.

Let $\alpha = \sum_{j=1}^n a_j(z)dz^j$ and $\beta = \sum_{j=1}^n b_j(z)dz^j$ be two differential forms of degree one defined in an open set $U \subset \mathbb{C}^n$. Here the complex space $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$ by setting $z^j = x^j + iy^j$, $z^j \in \mathbb{C}$, $(x^j, y^j) \in \mathbb{R}^2$ for all $j = 1, \ldots, n$. It is convenient to introduce the complex differential form $dz^j = dx^j + idy^j$ and to write

$$\begin{aligned} a_j(z) &= u_j(x, y) + iv_j(x, y), \\ b_j(z) &= w_j(x, y) + i\omega_j(x, y), \end{aligned}$$

where $x = (x^1, \ldots, x^n)$, $y = (y^1, \ldots, y^n)$. Then the complex form

$$\begin{aligned} a_j(z)dz^j &= (u_j + iv_j)(dx^j + idy^j) \\ &= (u_j dx^j - v_j dy^j) + i(u_j dy^j + v_j dx^j) \end{aligned}$$

has $u_j dx^j - v_j dy^j$ as its real part and $u_j dy^j + v_j dx^j$ as its imaginary part, for all $j = 1, \ldots, n$. Similarly the complex form

$$\begin{aligned} b_j(z)dz^j &= (w_j + i\omega_j)(dx^j + idy^j) \\ &= (w_j dx^j - \omega_j dy^j) + i(w_j dy^j + \omega_j dx^j) \end{aligned}$$

has $w_j dx^j - \omega_j dy^j$ as its real part and $w_j dy^j + \omega_j dx^j$ as its imaginary part, for all $j = 1, \ldots, n$.

Now, a computation shows that

$$\begin{aligned} d\alpha \wedge \beta &= d\Big(\sum_{j=1}^n a_j(z)dz^j\Big) \wedge \sum_{k=1}^n b_k(z)dz^k \\ &= \Big[\Big(\sum_{j=1}^n du_j \wedge dx^j - \sum_{j=1}^n dv_j \wedge dy^j\Big) + i\Big(\sum_{j=1}^n du_j \wedge dy^j + \sum_{j=1}^n dv_j \wedge dx^j\Big)\Big] \\ &\quad \wedge \Big[\Big(\sum_{k=1}^n w_k dx^k - \sum_{k=1}^n \omega_k dy^k\Big) + i\Big(\sum_{k=1}^n w_k dy^k + \sum_{k=1}^n \omega_k dx^k\Big)\Big] \end{aligned}$$

$$= \sum_{j=1}^{n} du_j \wedge dx^j \wedge \sum_{k=1}^{n} w_k dx^k + \sum_{j=1}^{n} dv_j \wedge dy^j \wedge \sum_{k=1}^{n} \omega_k dy^k +$$
$$- \sum_{j=1}^{n} dv_j \wedge dy^j \wedge i \sum_{k=1}^{n} w_k dy^k + \sum_{j=1}^{n} du_j \wedge dx^j \wedge i \sum_{k=1}^{n} \omega_k dx^k +$$
$$+ i \sum_{j=1}^{n} dv_j \wedge dx^j \wedge \sum_{k=1}^{n} w_k dx^k - i \sum_{j=1}^{n} du_j \wedge dy^j \wedge \sum_{k=1}^{n} \omega_k dy^k +$$
$$- \sum_{j=1}^{n} du_j \wedge dy^j \wedge \sum_{k=1}^{n} w_k dy^k - \sum_{j=1}^{n} dv_j \wedge dx^j \wedge \sum_{k=1}^{n} \omega_k dx^k,$$

and

$$\alpha \wedge d\beta = \sum_{j=1}^{n} a_j(z) dz^j \wedge d\Big(\sum_{k=1}^{n} b_k(z) dz^k\Big)$$
$$= \Big[\Big(\sum_{j=1}^{n} du_j dx^j - \sum_{j=1}^{n} v_j dy^j\Big) + i\Big(\sum_{j=1}^{n} u_j dy^j + \sum_{j=1}^{n} v_j dx^j\Big)\Big]$$
$$\wedge \Big[\Big(\sum_{k=1}^{n} dw_k \wedge dx^k - \sum_{k=1}^{n} d\omega_k \wedge dy^k\Big) +$$
$$+ i\Big(\sum_{k=1}^{n} dw_k \wedge dy^k + \sum_{k=1}^{n} d\omega_k \wedge dx^k\Big)\Big]$$
$$= \sum_{j=1}^{n} u_j dx^j \wedge \sum_{k=1}^{n} dw_k \wedge dx^k + \sum_{j=1}^{n} v_j dy^j \wedge \sum_{k=1}^{n} d\omega_k \wedge dy^k +$$
$$- \sum_{j=1}^{n} v_j dy^j \wedge i \sum_{k=1}^{n} dw_k \wedge dy^k + \sum_{j=1}^{n} u_j dx^j \wedge i \sum_{k=1}^{n} d\omega_k \wedge dx^k +$$
$$+ i \sum_{j=1}^{n} v_j dx^j \sum_{k=1}^{n} dw_k \wedge dx^k - i \sum_{j=1}^{n} u_j dx^j \wedge \sum_{k=1}^{n} d\omega_k \wedge dy^k +$$
$$- \sum_{j=1}^{n} u_j dy^j \wedge \sum_{k=1}^{n} dw_k \wedge dy^k - \sum_{j=1}^{n} v_j dx^j \wedge \sum_{k=1}^{n} d\omega_k \wedge dx^k.$$

In a similar way as for the real case, we can prove that

$$\begin{aligned}
d(\alpha\wedge\beta) = \sum_{j_1<j_2<j_3} &\{[(\mathrm{rot}_x(u_{j_1},u_{j_2},u_{j_3}) + i\,\mathrm{rot}_x(v_{j_1},v_{j_2},v_{j_3}))\cdot(w_{j_1},w_{j_2},w_{j_3})+\\
&-(\mathrm{rot}_x(v_{j_1},v_{j_2},v_{j_3}) + i\,\mathrm{rot}_x(u_{j_1},u_{j_2},u_{j_3}))\cdot(\omega_{j_1},\omega_{j_2},\omega_{j_3})]+\\
&-[(u_{j_1},u_{j_2},u_{j_3})\cdot(\mathrm{rot}_x(w_{j_1},w_{j_2},w_{j_3}) + i\,\mathrm{rot}_x(\omega_{j_1},\omega_{j_2},\omega_{j_3}))+\\
&-(v_{j_1},v_{j_2},v_{j_3})\cdot(\mathrm{rot}_x(\omega_{j_1},\omega_{j_2},\omega_{j_3}) + i\,\mathrm{rot}_x(w_{j_1},w_{j_2},w_{j_3}))]\}\\
&\cdot dx^{j_1}\wedge dx^{j_2}\wedge dx^{j_3}+\\
&+\sum_{j_1<j_2<j_3}\{[(\mathrm{rot}_y(v_{j_1},v_{j_2},v_{j_3}) - i\,\mathrm{rot}_y(u_{j_1},u_{j_2},u_{j_3}))\cdot(\omega_{j_1},\omega_{j_2},\omega_{j_3})+\\
&-(\mathrm{rot}_y(u_{j_1},u_{j_2},u_{j_3}) - i\,\mathrm{rot}_y(v_{j_1},v_{j_2},v_{j_3}))\cdot(w_{j_1},w_{j_2},w_{j_3})]+\\
&-[(v_{j_1},v_{j_2},v_{j_3})\cdot(\mathrm{rot}_y(\omega_{j_1},\omega_{j_2},\omega_{j_3}) - i\,\mathrm{rot}_y(w_{j_1},w_{j_2},w_{j_3}))+\\
&-(u_{j_1},u_{j_2},u_{j_3})\cdot(\mathrm{rot}_y(w_{j_1},w_{j_2},w_{j_3}) - i\,\mathrm{rot}_y(\omega_{j_1},\omega_{j_2},\omega_{j_3}))]\}\\
&\cdot dy^{j_1}\wedge dy^{j_2}\wedge dy^{j_3}
\end{aligned}$$

for all $1 \le j_1 < j_2 < j_3 \le n$. That is in the neighbourhood of each point U, so $d(\alpha\wedge\beta) = 0$ if and only if

$$\mathrm{rot}_x(u_{j_1},u_{j_2},u_{j_3}) = 0, \mathrm{rot}_x(v_{j_1},v_{j_2},v_{j_3}) = 0,$$

$$\mathrm{rot}_y(u_{j_1},u_{j_2},u_{j_3}) = 0, rot_y(v_{j_1},v_{j_2},v_{j_3}) = 0$$

and

$$\mathrm{rot}_x(w_{j_1},w_{j_2},w_{j_3}) = 0, \mathrm{rot}_x(\omega_{j_1},\omega_{j_2},\omega_{j_3}) = 0,$$

$$\mathrm{rot}_y(w_{j_1},w_{j_2},w_{j_3}) = 0, \mathrm{rot}_y(\omega_{j_1},\omega_{j_2},\omega_{j_3}) = 0$$

for all $1 \le j_1 < j_2 < j_3 \le n$. Hence, it follows that the forms

$$\sum_{j=1}^n (u_j dx^j - v_j dy^j) \text{ and } \sum_{j=1}^n (u_j dy^j + v_j dx^j)$$

are closed. Similarly the forms

$$\sum_{j=1}^n (w_j dx^j - \omega_j dy^j) \text{ and } \sum_{j=1}^n (w_j dy^j + \omega_j dx^j)$$

are closed too. By Poincaré's Lemma for 1-forms it follows that these forms are locally exact. Therefore, if $U \subset \mathbb{C}^n$ is a simply-connected domain of these forms, then there are differential functions $f, g : U \to \mathbb{C}$ such that $df = \alpha$ and $dg = \beta$. Thus the condition $d\alpha\wedge\alpha = d\beta\wedge\beta = 0$ is satisfied.

References

[1] Bo-Yu Hou and Bo-Yuan Hou, *Differential Geometry for Physicists*, Advanced Series on Theoretical Physical Science, 1997.

[2] Henri Cartan, *Differential Forms*, Hermann, Paris, 1971.

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