Steps in Differential Geometry, Proceedings of the Colloquium on Differential Geometry, 25–30 July, 2000, Debrecen, Hungary

SECOND ORDER VARIATIONS IN VARIATIONAL SEQUENCES

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Abstract. In this note we provide a geometrical characterization of the second order variation of a generalized Lagrangian in the framework of variational sequences.

We define the *variational vertical derivative* as an operator on the sheaves of the variational sequence and stress its link with the classical concept of *variation.* The main result is the intrinsic formulation of a theorem which states the relation between the variational vertical derivative of the Euler–Lagrange morphism of a generalized Lagrangian and the Euler–Lagrange morphism of the variational vertical derivative of the Lagrangian itself.

1. Introduction

Our framework is the calculus of variations on finite order jets of a fibered manifold. More precisely, we consider the geometrical formulation of this framework in terms of *variational sequences* introduced by Krupka [12, 13]. As it is well known, in this formulation the variational sequence is defined as a quotient of the de Rham sequence on a finite order jet of a fibered manifold with respect to an intrinsically defined subsequence, the "contact" subsequence. Standard objects of the calculus of variations can be interpreted as sheaf sections and morphisms in the variational sequence, which turns out to be an exact resolution of the constant sheaf $\mathbb{R}$ over the relevant fibered manifold.

We provide a geometrical characterization of the second order variation (see *e.g.* [1, 2, 3, 4, 6, 18]) of a Lagrangian in the framework of finite order variational sequences. We introduce the notion of iterated variation of a section as an i–parameter 'deformation' of the section by means of vertical flows and thus define the i–th variation of a morphism which is very simply related with the iterated Lie derivative of the morphism itself. Relying on previous results of us [5] on the representation of the Lie derivative operator in the variational sequence we can then define an operator on the quotient sheaves of the sequence, the *variational vertical derivative*. We stress some linearity properties of this operator and show

1991 *Mathematics Subject Classification.* 55R10, 58A12, 58A20, 58E30.

Key words and phrases. fibered manifold, jet space, variational sequence, variation.

Supported by GNFM of INDAM, MURST and University of Torino.

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that it is a functor on the category of variational sequences. Making use of suitable representations of the variational vertical derivative we relate the second order variation of a generalized Lagrangian with the variational Lie derivative of generalized Euler–Lagrange operators associated with the Lagrangian itself.

The Lagrangian characterization of the second variation of a Lagrangian in the framework of jet bundles has been considered in [2, 3, 4, 6]. In particular, in [2, 3, 4] it was shown how to recast (up to divergencies) the system formed by the Euler–Lagrange equations together with the Jacobi equations for a given Lagrangian as the Euler Lagrange equations for a *'deformed'* Lagrangian.

As an outcome of the intrinsic representation of the second order variation of a generalized Lagrangian, the above mentioned invariant decomposition of the second variation is here provided at any jet order and geometrically interpreted as a simple and direct application of the representation of the variational Lie derivative of variational morphisms provided in [5] together with a suitable version of a global decomposition formula of vertical morphisms due to Kolář [7, 9]. A new geometric object, the generalized Jacobi morphism, is finally represented in the variational sequence.

2. Variational sequences on jets of fibered manifolds

Our framework is a fibered manifold $\pi : \boldsymbol{Y} \to \boldsymbol{X}$, with $\dim \boldsymbol{X} = n$ and $\dim \boldsymbol{Y} = n + m$ (see *e.g.* [17]).

For $r \geq 0$ we are concerned with the r–jet space $J_r\boldsymbol{Y}$; in particular, we set $J_0\boldsymbol{Y} \equiv \boldsymbol{Y}$. We recall the natural fiberings $\pi^r_s : J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$, $r \geq s$, $\pi^r : J_r\boldsymbol{Y} \to \boldsymbol{X}$, and, among these, the *affine* fiberings π^r_{r-1}. We denote by $V\boldsymbol{Y}$ the vector subbundle of the tangent bundle $T\boldsymbol{Y}$ of vectors on $\boldsymbol{Y}$ which are vertical with respect to the fibering π.

Charts on $\boldsymbol{Y}$ adapted to π are denoted by (x^λ, y^i). Greek indices $\lambda, \mu, \dots$ run from 1 to n and they label base coordinates, while Latin indices $i, j, \dots$ run from 1 to m and label fibre coordinates, unless otherwise specified. We denote by $(\partial_\lambda, \partial_i)$ and (d^λ, d^i) the local bases of vector fields and 1–forms on $\boldsymbol{Y}$ induced by an adapted chart, respectively.

We denote multi–indices of dimension n by boldface Greek letters such as $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, with $0 \leq \alpha_\mu$, $\mu = 1, \dots, n$; by an abuse of notation, we denote with λ the multi–index such that $\alpha_\mu = 0$, if $\mu \neq \lambda$, $\alpha_\mu = 1$, if $\mu = \lambda$. We also set $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_n$ and $\boldsymbol{\alpha}! := \alpha_1! \dots \alpha_n!$. The charts induced on $J_r\boldsymbol{Y}$ are denoted by $(x^\lambda, y^i_{\boldsymbol{\alpha}})$, with $0 \leq |\boldsymbol{\alpha}| \leq r$; in particular, we set $y^i_{\boldsymbol{0}} \equiv y^i$. The local vector fields and forms of $J_r\boldsymbol{Y}$ induced by the above coordinates are denoted by $(\partial^{\boldsymbol{\alpha}}_i)$ and $(d^i_{\boldsymbol{\alpha}})$, respectively.

For $r \geq 1$, we consider the natural complementary fibered morphisms over the affine fibering $J_r\boldsymbol{Y} \to J_{r-1}\boldsymbol{Y}$ induced by *contact maps* on jet spaces

$$Д : J_r\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X} \to TJ_{r-1}\boldsymbol{Y}\,, \qquad \vartheta : J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} TJ_{r-1}\boldsymbol{Y} \to VJ_{r-1}\boldsymbol{Y}\,,$$

with coordinate expressions, for $0 \leq |\boldsymbol{\alpha}| \leq r-1$, given by

$$\mathit{\Pi} \;= d^\lambda \otimes \mathit{\Pi}_\lambda = d^\lambda \otimes (\partial_\lambda + y^j_{\boldsymbol{\alpha}+\lambda} \partial^{\boldsymbol{\alpha}}_j)\,, \qquad \vartheta \;= \vartheta^j_{\boldsymbol{\alpha}} \otimes \partial^{\boldsymbol{\alpha}}_j = (d^j_{\boldsymbol{\alpha}} - y^j_{\boldsymbol{\alpha}+\lambda} d^\lambda) \otimes \partial^{\boldsymbol{\alpha}}_j\,.$$

We have the following natural fibered splitting

$$J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*J_{r-1}\boldsymbol{Y} = \left(J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*\boldsymbol{X} \right) \oplus \overset{*}{\mathcal{C}}_{r-1}[\boldsymbol{Y}]\,, \tag{1}$$

where $\overset{*}{\mathcal{C}}_{r-1}[\boldsymbol{Y}] := \operatorname{im} \vartheta^*_r$ and the canonical isomorphism $\overset{*}{\mathcal{C}}_{r-1}[\boldsymbol{Y}] \simeq J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} V^*J_{r-1}\boldsymbol{Y}$ holds true (see [12, 13, 15, 17]).

The above splitting induces also a decomposition of the exterior differential on $\boldsymbol{Y}$, $(\pi^{r+1}_r)^* \circ d = d_H + d_V$, where d_H and d_V are called the *horizontal* and *vertical differential*, respectively. The action of d_H and d_V on functions and 1–forms on $J_r\boldsymbol{Y}$ uniquely characterizes d_H and d_V (see, *e.g.*, [17, 21] for more details).

If $f : J_r\boldsymbol{Y} \to I\!R$ is a function, then we set $D_\lambda f := \mathit{\Pi}_\lambda f$, $D_{\boldsymbol{\alpha}+\lambda} f := D_\lambda D_{\boldsymbol{\alpha}} f$, where the operator D_λ is the standard *formal derivative*.

A *projectable vector field* on $\boldsymbol{Y}$ is defined to be a pair (Ξ, ξ), where $\Xi : \boldsymbol{Y} \to T\boldsymbol{Y}$ and $\xi : \boldsymbol{X} \to T\boldsymbol{X}$ are vector fields and Ξ is a fibered morphism over ξ. A projectable vector field (Ξ, ξ) can be conveniently prolonged to a projectable vector field $(j_r\Xi, \xi)$, whose coordinate expression can be found *e.g.* in [5] and [11, 15, 17]. A *vertical vector field* on $\boldsymbol{Y}$ is a projectable vector field on $\boldsymbol{Y}$ such that $\xi = 0$.

i. For $r \geq 0$, we consider the standard sheaves $\overset{p}{\Lambda}_r$ of p–forms on $J_r\boldsymbol{Y}$.

ii. For $0 \leq s \leq r$, we consider the sheaves $\overset{p}{\mathcal{H}}_{(r,s)}$ and $\overset{p}{\mathcal{H}}_r$ of *horizontal forms*, i.e. of local fibered morphisms over π^r_s and π^r of the type $\alpha : J_r\boldsymbol{Y} \to \overset{p}{\wedge} T^*J_s\boldsymbol{Y}$ and $\beta : J_r\boldsymbol{Y} \to \overset{p}{\wedge} T^*\boldsymbol{X}$, respectively.

iii. For $0 \leq s < r$, we consider the subsheaf $\overset{p}{\mathcal{C}}_{(r,s)} \subset \overset{p}{\mathcal{H}}_{(r,s)}$ of *contact forms*, i.e. of sections $\alpha \in \overset{p}{\mathcal{H}}_{(r,s)}$ with values into $\overset{p}{\wedge}(\overset{*}{\mathcal{C}}_s[\boldsymbol{Y}])$. There is a distinguished subsheaf $\overset{p}{\mathcal{C}}_r \subset \overset{p}{\mathcal{C}}_{(r+1,r)}$ of local fibered morphisms $\alpha \in \overset{p}{\mathcal{C}}_{(r+1,r)}$ such that $\alpha = \overset{p}{\wedge} \operatorname{im} \vartheta^*_{r+1}[\boldsymbol{Y}] \circ \tilde{\alpha}$, where $\tilde{\alpha}$ is a section of the fibration $J_{r+1}\boldsymbol{Y} \underset{J_r\boldsymbol{Y}}{\times} \overset{p}{\wedge} V^*J_r\boldsymbol{Y} \to J_{r+1}\boldsymbol{Y}$ which projects down onto $J_r\boldsymbol{Y}$.

According to [21], the fibered splitting (1) yields naturally the sheaf splitting $\overset{p}{\mathcal{H}}_{(r+1,r)} = \bigoplus_{t=0}^{p} \overset{p-t}{\mathcal{C}}_{(r+1,r)} \wedge \overset{t}{\mathcal{H}}_{r+1}$, which restricts to the inclusion $\overset{p}{\Lambda}_r \subset \bigoplus_{t=0}^{p} \overset{p-t}{\mathcal{C}}_r \wedge \overset{t}{\mathcal{H}}{}^h_{r+1}$, where $\overset{p}{\mathcal{H}}{}^h_{r+1} := h(\overset{p}{\Lambda}_r)$ for $0 < p \leq n$ and h is defined to be the restriction to $\overset{p}{\Lambda}_r$ of the projection of the above splitting onto the non–trivial summand with the highest value of t. We define also the map $v := \operatorname{id} - h$.

Let $\alpha \in \overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}$. Then there is a unique pair of sheaf morphisms

$$E_\alpha \in \overset{1}{\mathcal{C}}_{(2r,0)} \wedge \overset{n}{\mathcal{H}}{}^h_{2r+1}\,, \qquad F_\alpha \in \overset{1}{\mathcal{C}}_{(2r,r)} \wedge \overset{n}{\mathcal{H}}{}^h_{2r+1}\,, \tag{2}$$

such that $(\pi^{2r+1}_{r+1})^*\alpha = E_\alpha - F_\alpha$, and F_α is locally of the form $F_\alpha = d_H p_\alpha$, with $p_\alpha \in \overset{1}{\mathcal{C}}_{(2r-1,r-1)} \wedge \overset{n-1}{\mathcal{H}}{}_{2r}$ (see *e.g.* [9, 11, 21]).

Recall (see [21]) that if $\beta \in \overset{1}{\mathcal{C}}_s \wedge \overset{1}{\mathcal{C}}_{(s,0)} \wedge \overset{n}{\mathcal{H}}_s$, then, there is a unique $\tilde{H}_\beta \in \overset{1}{\mathcal{C}}_{(2s,s)} \otimes \overset{1}{\mathcal{C}}_{(2s,0)} \wedge \overset{n}{\mathcal{H}}_{2s}$ such that, for all $\Xi : \boldsymbol{Y} \to V\boldsymbol{Y}$, $E_{\hat{\beta}} = C^1_1\left(j_{2s}\Xi \otimes \tilde{H}_\beta\right)$, where $\hat{\beta} := j_s\Xi \,\lrcorner\, \beta$, C^1_1 stands for tensor contraction and $\lrcorner$ denotes inner product. Then there is a unique pair of sheaf morphisms

$$H_\beta \in \overset{1}{\mathcal{C}}_{(2s,s)} \wedge \overset{1}{\mathcal{C}}_{(2s,0)} \wedge \overset{n}{\mathcal{H}}_{2s}\,, \quad G_\beta \in \overset{2}{\mathcal{C}}_{(2s,s)} \wedge \overset{n}{\mathcal{H}}_{2s}\,, \tag{3}$$

such that $\pi^{2s\,*}_s\beta = H_\beta - G_\beta$ and $H_\beta = \frac{1}{2}\,A(\tilde{H}_\beta)$, where A stands for antisymmetrisation. Moreover, G_β is locally of the type $G_\beta = d_H q_\beta$, where $q_\beta \in \overset{2}{\mathcal{C}}_{2s-1} \wedge \overset{n-1}{\mathcal{H}}{}_{2s-1}$, hence $[\beta] = [H_\beta]$. Coordinate expressions of the morphisms E_α and H_β can be found in [21].

2.1. **Variational sequences.** We recall now the theory of variational sequences on finite order jet spaces, as it was developed by Krupka in [12]. By an abuse of notation, denote by $d\ker h$ the sheaf generated by the presheaf $d\ker h$. Set $\overset{*}{\Theta}_r := \ker h + d\ker h$. The following diagram is commutative and its rows and columns are exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \overset{1}{\Theta}_r \arrow[r, "d"] \arrow[d] & \overset{2}{\Theta}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & \overset{I}{\Theta}_r \arrow[r, "d"] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \arrow[r] & 0 \\
0 \arrow[r] & I\!\!R \arrow[r] \arrow[d] & \overset{0}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \overset{1}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \overset{2}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & \overset{I}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \overset{I+1}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & 0 \\
0 \arrow[r] & I\!\!R \arrow[r] \arrow[d] & \overset{0}{\Lambda}_r \arrow[r, "\mathcal{E}_0"] \arrow[d] & \overset{1}{\Lambda}_r/\overset{1}{\Theta}_r \arrow[r, "\mathcal{E}_1"] \arrow[d] & \overset{2}{\Lambda}_r/\overset{2}{\Theta}_r \arrow[r, "\mathcal{E}_2"] \arrow[d] & \dots \arrow[r, "\mathcal{E}_{I-1}"] & \overset{I}{\Lambda}_r/\overset{I}{\Theta}_r \arrow[r, "\mathcal{E}_I"] \arrow[d] & \overset{I+1}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & 0 \\
 & 0 & 0 & 0 & 0 & & 0 & 0 & &
\end{tikzcd}$$

Definition 2.1. The bottom row of the above diagram is called the r–th order *variational sequence* associated with the fibered manifold $\boldsymbol{Y} \to \boldsymbol{X}$. $\square$

The quotient sheaves in the variational sequence can be conveniently represented as follows [21].

Let $k \leq n$. Then, the sheaf morphism h yields the natural isomorphism

$$I_k : \overset{k}{\Lambda}_r / \overset{k}{\Theta}_r \to \overset{k}{\mathcal{H}}{}^h_{r+1} := \overset{k}{\mathcal{V}}_r : [\alpha] \mapsto h(\alpha) .$$

Let $k > n$. Then, the projection h induces a natural sheaf isomorphism

$$I_k : \left(\overset{k}{\Lambda}_r / \overset{k}{\Theta}_r \right) \to \left(\overset{k-n}{\mathcal{C}}{}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1} \right) / h(\overline{d \ker h}) := \overset{k}{\mathcal{V}}_r : [\alpha] \mapsto [h(\alpha)] .$$

We remark that a section $\lambda \in \overset{n}{\mathcal{V}}_r$ is just a Lagrangian of order $(r+1)$ of the standard literature. Furthermore, $\mathcal{E}_n(\lambda) \in \overset{n+1}{\mathcal{V}}{}_r$ coincides with the standard higher order Euler–Lagrange morphism $\mathcal{E}(\lambda)$ associated with λ.

Making use of the above sheaf isomorphisms and of decomposition formulae (2) and (3), in [5] we proved that the Lie derivative operator with respect to the r-th order prolongation $j_r\Xi$ of a projectable vector field (Ξ, ξ) can be conveniently represented on the quotient sheaves of the variational sequence in terms of an operator, the *variational Lie derivative* $\mathcal{L}_{j_r\Xi}$, as follows:

if $0 \leq p \leq n-1$ and $\mu \in \overset{p}{\mathcal{V}}_r$, then

$$\mathcal{L}_{j_r\Xi}\mu = \xi \rfloor d_H\mu + j_{r+2}\Xi_V \rfloor d_V\mu + d_H(\xi \rfloor \mu) ; \tag{4}$$

if $p = n$ and $\lambda \in \overset{n}{\mathcal{V}}_r$, then

$$\mathcal{L}_{j_r\Xi}\lambda = \Xi_V \rfloor \mathcal{E}(\lambda) + d_H(j_r\Xi_V \rfloor p_{d_V\lambda} + \xi \rfloor \lambda) ; \tag{5}$$

if $p = n+1$ and $\eta \in \overset{n+1}{\mathcal{V}}{}_r$, then

$$\mathcal{L}_{j_r\Xi}\eta = \mathcal{E}(j_r\Xi_V \rfloor \eta) + \tilde{H}_{d\eta}(j_{2r+1}\Xi_V) . \tag{6}$$

3. Variations

We shall here introduce *variations* of a morphism as *multiparameter deformations* showing that this is equivalent to take iterated *variational Lie derivatives with respect to vertical vector fields.*

We define the i–th variation of a section and introduce the i–th variation of a morphism of the kind $\alpha : J_r\boldsymbol{Y} \to \overset{k}{\wedge} T^*J_r\boldsymbol{Y}$ along the section σ in terms of the pull–back by means of the r–th prolongation of the i–th variation of a section $\sigma : \boldsymbol{X} \to \boldsymbol{Y}$.

Definition 3.1. Let $\sigma : \boldsymbol{X} \to \boldsymbol{Y}$ be a section and i any integer. An i–th variation of σ is a smooth section $\Gamma_i : \boldsymbol{I} \times \boldsymbol{X} \to \boldsymbol{Y}$, $\mathbf{0} \in \boldsymbol{I} \subset \mathbf{R}^{\mathbf{i}}$, such that $\Gamma_i(\mathbf{0}) = \sigma$. □

In other words, Γ_i is a i–parameter smooth deformation of σ.

Let $\Xi_1, \dots, \Xi_i$ be vertical vector fields on $\boldsymbol{Y}$ and let $\Gamma_i(t_1, \dots, t_i)$ be an i–th variation of the section σ such that

$$\begin{aligned}
&\frac{\partial \Gamma_i}{\partial t_1}(t_1, 0, \dots, 0)\big|_{t_1=0} = \Xi_1 \circ \sigma \,, \\
&\frac{\partial \Gamma_i}{\partial t_2}(t_1, t_2, 0, \dots, 0)\big|_{t_2=0} = \Xi_2 \circ \Gamma_i(t_1, 0, \dots, 0) \,, \\
&\dots , \\
&\frac{\partial \Gamma_i}{\partial t_i}(t_1, t_2, \dots, t_{i-1}, t_i)\big|_{t_i=0} = \Xi_i \circ \Gamma_i(t_1, t_2, \dots, t_{i-1}, 0) \,.
\end{aligned}$$

In this case we say that Γ_i is generated by the i–tuple $(\Xi_1, \dots, \Xi_i)$.

We have thus the following characterization of Γ_i as the variation of σ by means of vertical flows.

Proposition 3.2. *Let $\psi^k_{t_k}$, with $1 \leq k \leq i$, be the flows generated by the vertical vector fields Ξ_k. Then for the Γ_i generated by $(\Xi_1, \dots, \Xi_i)$ we have:*

$$\Gamma_i(t_1, \dots, t_i) = \psi^i_{t_i} \circ \dots \circ \psi^1_{t_1} \circ \sigma \,. \quad \square \tag{7}$$

Definition 3.3. Vertical vector fields on any fiber bundle which 'deform' sections as above are called *variation vector fields*.

$\square$

3.1. **Variations of morphisms and Lie derivative.** We shall define the variation of a generic fibered morphism of the kind $\alpha : J_r\boldsymbol{Y} \to \overset{k}{\wedge} T^* J_r\boldsymbol{Y}$ along the section σ.

Definition 3.4. Let $\alpha : J_r\boldsymbol{Y} \to \overset{k}{\wedge} T^* J_r\boldsymbol{Y}$ and let Γ_i be an i–th variation of the section σ. We define the i–th variation of the morphism α to be

$$\delta^i \alpha := \frac{\partial^i}{\partial t_1 \dots \partial t_i}\big|_{t_1, \dots, t_i = 0} (\alpha \circ j_r \Gamma_i(t_1, \dots, t_i)) \,. \quad \square \tag{8}$$

If just one variation Γ_1, generated by a single vector field Ξ_1, is involved we write for simplicity δ instead of δ^1. The following Lemma states the relation between the i–th variation of a morphism and its iterated Lie derivative.

Lemma 3.5. *Let $\alpha : J_r\boldsymbol{Y} \to \overset{k}{\wedge} T^* J_r\boldsymbol{Y}$ and $L_{j_r\Xi_k}$ be the Lie derivative operator with respect to $j_r\Xi_k$.*

Let Γ_i be the i–th variation of the section σ by means of the variation vector fields $\Xi_1, \dots, \Xi_i$ on $\boldsymbol{Y}$. Then we have

$$\delta^i \alpha = (j_r\sigma)^* L_{j_r\Xi_1} \dots L_{j_r\Xi_i} \alpha \,. \tag{9}$$

PROOF. By applying Definition 3.4 and Proposition 3.2 we have

$$\begin{aligned}\delta^i\alpha = \frac{\partial^i}{\partial t_1 \dots \partial t_i}\big|_{t_1\dots t_i=0}[(j_r(\psi^i_{t_i}\circ\dots\circ\psi^1_{t_1}\circ\sigma))^*\alpha] = \\ \frac{\partial^i}{\partial t_1 \dots \partial t_i}\big|_{t_1\dots t_i=0}[(j_r\sigma)^*\circ(j_r\psi^1_{t_1})^*\circ\dots\circ(j_r\psi^i_{t_i})^*\alpha] = \\ (j_r\sigma)^*\frac{\partial^i}{\partial t_1 \dots \partial t_i}\big|_{t_1\dots t_i=0}[(j_r\psi^1_{t_1})^*\circ\dots\circ(j_r\psi^i_{t_i})^*]\alpha = \\ (j_r\sigma)^*[\frac{\partial}{\partial t_1}(j_r\psi^1_{t_1})^*\big|_{t_1=0}\circ\dots\circ\frac{\partial}{\partial t_i}(j_r\psi^i_{t_i})^*\big|_{t_i=0}]\alpha = \\ (j_r\sigma)^*L_{j_r\Xi_1}\dots L_{j_r\Xi_i}\alpha\,,\end{aligned}$$

where $\psi^k_{t_k}$ are the vertical flows respectively generated by Ξ_k, for $k=1,\dots,i$. $\boxed{QED}$

From now on, we will restrict to the case of the second order variation of a morphism $\lambda\in\overset{n}{\Lambda}_r$. Higher order variations will be deeply investigated elsewhere.

In the following Lemma we show that the first order variation $\delta^1\lambda:=\delta\lambda$ of λ is simply related to the vertical differential of λ.

Lemma 3.6. *Let σ be a section of $\boldsymbol{Y}$, Ξ a variation vector field on $\boldsymbol{Y}$ and $\lambda\in\overset{n}{\Lambda}_r$. Then we have*

$$\delta\lambda = j_r\sigma^*(j_r\Xi\rfloor d_V\lambda)\,. \tag{10}$$

PROOF. In fact we have

$$\begin{aligned}\delta\lambda = j_r\sigma^*\mathcal{L}_{j_r\Xi}\lambda = j_r\sigma^*(\Xi\rfloor\mathcal{E}(\lambda)+d_H(j_r\Xi\rfloor p_{d_V\lambda})) = \\ j_r\sigma^*(\Xi\rfloor\mathcal{E}(\lambda)+(j_r\Xi\rfloor d_H p_{d_V\lambda})) = j_r\sigma^*(j_r\Xi\rfloor d_V\lambda)\,,\end{aligned}$$

since $\Xi\rfloor d_H = d_H\Xi\rfloor$, for any vertical vector field Ξ on $\boldsymbol{Y}$. $\boxed{QED}$

For notational convenience, in the sequel we shall denote with a superimposed bar all objects defined on a vertical prolongation $V\boldsymbol{Y}$ and with two bars those defined on the iterated vertical prolongation $V(V\boldsymbol{Y})$; e.g. $\bar{\Xi}$ will denote a variational vector field on $V\boldsymbol{Y}$ and $\bar{\delta}$ the variation operator on $V\boldsymbol{Y}$.

By iterating this result we can characterize the second order variation of λ as follows.

Proposition 3.7. *Let $\lambda\in(\overset{n}{\Lambda}_r)_{\boldsymbol{Y}}$ and σ be a section of $\boldsymbol{Y}\to\boldsymbol{X}$. Let Ξ_1, Ξ_2 be two variation vector fields on $\boldsymbol{Y}$ generating a second order variation Γ_2 of σ, and let $\bar{\Xi}_2$ be a variation vector field on $V\boldsymbol{Y}$, which projects down onto Ξ_2 and generates a first order variation $\bar{\Gamma}$ of σ. Moreover, let $\bar{d}$, $\bar{\mathcal{E}}$ and $\bar{p}$ be the exterior* *differential, the Euler–Lagrange morphism and the momentum morphism on $V\boldsymbol{Y}$, respectively. Then we have*

$$\delta^2\lambda = \bar{\Xi}_2 \rfloor \bar{\mathcal{E}}(\delta\lambda) + \bar{d}_H(j_r\bar{\Xi}_2 \rfloor \bar{p}_{\bar{d}_V\delta\lambda}) \tag{11}$$

$$= \bar{\delta}(\Xi_1 \rfloor \mathcal{E}(\lambda) + d_H(j_r\Xi_1 \rfloor p_{d_V\lambda}))\,. \tag{12}$$

Proof. We apply the above Lemma and the fact that $d_H\delta = \delta d_H$, which follows directly from the analogous naturality property of the Lie derivative operator.

QED

Remark 3.8. Owing to the linearity properties of $d_V\lambda$, by an abuse of notation, we can think of the operator δ as a *linear* morphism with respect to the vector bundle structure $J_rV\boldsymbol{Y} \to \boldsymbol{Y}$ of the kind

$$\delta : J_r\boldsymbol{Y} \underset{\boldsymbol{Y}}{\times} J_rV\boldsymbol{Y} \to \overset{n}{\wedge} T^*\boldsymbol{X}\,. \tag{13}$$

This property can be obviously iterated for each integer i, so that one can analogously define an i–linear morphism δ^i.

□

4. Variational vertical derivatives

In this Section we restrict our attention to morphisms which are sections of sheaves in the variational sequence. We define the i–th order *variational vertical derivative* of morphisms, by showing that the i–th variation operator passes to the quotient in the variational sequence.

Lemma 4.1. *Let $\alpha \in (\overset{n}{\mathcal{V}}_r)_{\boldsymbol{Y}}$. We have*

$$[\delta^i\alpha] = \hat{\delta}^i[\alpha]\,,$$

where we set $\hat{\delta}^i := \mathcal{L}_{\Xi_i} \dots \mathcal{L}_{\Xi_1}$.

Proof. In fact, we have

$$[\delta^i\alpha] = [L_{\Xi_i} \dots L_{\Xi_1}\alpha] = \mathcal{L}_{\Xi_i} \dots \mathcal{L}_{\Xi_1}[\alpha] := \hat{\delta}^i[\alpha]\,,$$

since δ^i, as well as L_{Ξ_i} (see [5]), preserve the contact structure. QED

Definition 4.2. We call the operator $\hat{\delta}^i$ the *i–th variational vertical derivative operator.* □

Theorem 4.3. *The functor $\hat{\delta}$ is defined on the category of variational sequences and sends the sequence associated with the fibration $\boldsymbol{Y} \to \boldsymbol{X}$ into the sequence associated with the fibration $V\boldsymbol{Y} \to \boldsymbol{X}$.*

Proof. It follows from Proposition 3.7, Equation (11), Remark 3.8 and Lemma 4.1. QED

The above can be summarized in the following diagram.

$$
\begin{tikzcd}
0 \arrow[r] & \dots\dots \arrow[r, "d_H"] & (\overset{n}{\mathcal{V}}_r)_{\boldsymbol{Y}} \arrow[r, "\mathcal{E}_n"] \arrow[d, "\hat{\delta}"'] & (\overset{n+1}{\mathcal{V}}_r)_{\boldsymbol{Y}} \arrow[r, "\mathcal{E}_{n+1}"] \arrow[d, "\hat{\delta}"'] & \mathcal{E}_{n+1}(\overset{n+1}{\mathcal{V}}_r)_{\boldsymbol{Y}} \arrow[r, "\mathcal{E}_{n+2}"] \arrow[d, "\hat{\delta}"'] & 0 \\
0 \arrow[r] & \dots\dots \arrow[r, "\bar{d}_H"] & (\overset{n}{\mathcal{V}}_r)_{V\boldsymbol{Y}} \arrow[r, "\bar{\mathcal{E}}_n"] \arrow[d, "\bar{\hat{\delta}}"'] & (\overset{n+1}{\mathcal{V}}_r)_{V\boldsymbol{Y}} \arrow[r, "\bar{\mathcal{E}}_{n+1}"] \arrow[d, "\bar{\hat{\delta}}"'] & \bar{\mathcal{E}}_{n+1}(\overset{n+1}{\mathcal{V}}_r)_{V\boldsymbol{Y}} \arrow[r, "\bar{\mathcal{E}}_{n+2}"] \arrow[d, "\bar{\hat{\delta}}"'] & 0 \\
0 \arrow[r] & \dots\dots \arrow[r, "\bar{\bar{d}}_H"] & (\overset{n}{\mathcal{V}}_r)_{V(V\boldsymbol{Y})} \arrow[r, "\bar{\bar{\mathcal{E}}}_n"] & (\overset{n+1}{\mathcal{V}}_r)_{V(V\boldsymbol{Y})} \arrow[r, "\bar{\bar{\mathcal{E}}}_{n+1}"] & \bar{\bar{\mathcal{E}}}_{n+1}(\overset{n+1}{\mathcal{V}}_r)_{V(V\boldsymbol{Y})} \arrow[r, "\bar{\bar{\mathcal{E}}}_{n+2}"] & 0
\end{tikzcd}
$$

This enables us to represent variations of morphisms in the variational sequence. Here we shall investigate in detail the case of the second order variation of morphisms in the variational sequence. We show that classical results concerning the second variation can be restated in a very simple way in terms of variational Lie derivatives and global decomposition formulae.

To this aim, let us consider how the representations (5) and (6) of the variational Lie derivative specialize in the case of variation vector fields.

Lemma 4.4. *Let Ξ_1, Ξ_2 be two variation vector fields on $\boldsymbol{Y}$ and $\bar{\Xi}_2$ be a variation vector field on $V\boldsymbol{Y}$, which projects down onto Ξ_2. Let $\bar{\mathcal{E}}$ and $\bar{p}$ be, respectively, the Euler–Lagrange morphism and the momentum morphism associated with the Lagrangian $\hat{\delta}\lambda$. Then we have*

$$\mathcal{L}_{j_r\Xi_1}\lambda = \Xi_1 \rfloor \mathcal{E}(\lambda) + d_H(j_r\Xi_1 \rfloor p_{d_V\lambda})\,; \tag{14}$$

$$\mathcal{L}_{j_r\bar{\Xi}_2}\mathcal{L}_{j_r\Xi_1}\lambda = \bar{\Xi}_2 \rfloor \bar{\mathcal{E}}(\hat{\delta}\lambda) + \bar{d}_H[j_r\bar{\Xi}_2 \rfloor \bar{p}_{\bar{d}_V(\hat{\delta}\lambda)} - \hat{\delta}(j_r\Xi_1 \rfloor p_{d_V\lambda})]\,. \tag{15}$$

Proof. It follows from Equation (5) and Lemma 4.1. $\boxed{QED}$

The above Lemma together with Proposition 3.7 and Equation (11) gives us the following characterization of the second order variation of a generalized Lagrangian in the variational sequence.

Proposition 4.5. *Let $\lambda \in (\overset{n}{\mathcal{V}}_r)_{\boldsymbol{Y}}$, $\hat{\delta}\lambda \in (\overset{n}{\mathcal{V}}_r)_{V\boldsymbol{Y}}$. We have*

$$\hat{\delta}^2\lambda = \bar{\mathcal{E}}(\bar{\Xi}_2 \rfloor \hat{\delta}\lambda) + \tilde{H}_{d\hat{\delta}\lambda}(\bar{\Xi}_2)\,, \tag{16}$$

where $\tilde{H}_{d\hat{\delta}\lambda}$ is the unique morphism belonging to $\overset{1}{\mathcal{C}}_{(2r,r)} \otimes \overset{1}{\mathcal{C}}_{(2r,0)} \wedge \overset{n}{\mathcal{H}}_{2r}$ such that, for all $\Xi_1 : \boldsymbol{Y} \to V\boldsymbol{Y}$, $E_{j_r\Xi \rfloor d\hat{\delta}\lambda} = C^1_1(j_{2r}\Xi_1 \otimes \tilde{H}_{d\hat{\delta}\lambda})$, and C^1_1 stands for tensor contraction.

Proof. In fact we have

$$\hat{\delta}^2\lambda = \mathcal{L}_{j_r\bar{\Xi}_2}\hat{\delta}\lambda\,, \tag{17}$$

so that the assertion follows from a straightforward application of the representation provided by Equation (6). $\boxed{QED}$

This result provides an equivalent intrinsic interpretation of the second variation of the action functional evaluated in [4], equation (2.14), by just setting $\bar{\Xi}_2 = \rho$ and $\Xi_1 = \eta$.

The following is an application of an abstract result due to Kolář [10], concerning a global decomposition formula for vertical morphisms.

Lemma 4.6. *Let $\hat{\mu} : J_s\boldsymbol{Y} \to \overset{*}{\mathcal{C}}_k[V\boldsymbol{Y}] \wedge \overset{p}{\wedge} T^*\boldsymbol{X}$, with $0 \leq p \leq n$ and let $\bar{d}_H\hat{\mu} = 0$. Then we have $\hat{\mu} = E_{\hat{\mu}} + F_{\hat{\mu}}$, where*

$$E_{\hat{\mu}} : J_{2s+k}V\boldsymbol{Y} \to \overset{*}{\mathcal{C}}_0[V\boldsymbol{Y}] \wedge \overset{p}{\wedge} T^*\boldsymbol{X}\,, \tag{18}$$

and locally, $F_{\hat{\mu}} = \bar{d}_H M_{\hat{\mu}}$, with $M_{\hat{\mu}} : J_{2s+k-1}V\boldsymbol{Y} \to \overset{*}{\mathcal{C}}_{k-1}[V\boldsymbol{Y}] \wedge \overset{p-1}{\wedge} T^*\boldsymbol{X}$.

Proof. Following *e.g.* [9, 10, 21], the global morphisms $E_{\hat{\mu}}$ and $D_H M_{\hat{\mu}}$ can be evaluated by means of a backwards procedure. Hereafter the canonical isomorphism $J_rV\boldsymbol{Y} \simeq VJ_r\boldsymbol{Y}$ is obviously understood. $\boxed{QED}$

Now, it is very easy to see from Remark 3.8 and by means of a simple calculation that the following holds true.

Lemma 4.7. *Let $\chi(\lambda) := \tilde{H}_{d\hat{\delta}\lambda}$. We have $\chi(\lambda) : J_{2r}\boldsymbol{Y} \to \overset{*}{\mathcal{C}}_r[V\boldsymbol{Y}] \wedge (\overset{n}{\wedge} T^*\boldsymbol{X})$ and $\bar{d}_H\chi(\lambda) = 0$.*

Thus, as a straightforward application of Lemma 4.6, we obtain our main result which consists in a suitable geometrical interpretation of the second variation of a generalized Lagrangian and provides a new characterization of the Jacobi morphism in the framework of variational sequences.

Theorem 4.8. *Let $\chi(\lambda)$ be as in the above Lemma. Then we have*

$$\chi(\lambda) = E_{\chi(\lambda)} + F_{\chi(\lambda)}\,,$$

where

$$E_{\chi(\lambda)} : J_{4r}\boldsymbol{Y} \to \overset{*}{\mathcal{C}}_0[V\boldsymbol{Y}] \wedge (\overset{n}{\wedge} T^*\boldsymbol{X})\,, \tag{19}$$

and locally, $F_{\chi(\lambda)} = \bar{d}_H M_{\chi(\lambda)}$, with $M_{\chi(\lambda)} : J_{4r-1}\boldsymbol{Y} \to \overset{*}{\mathcal{C}}_{r-1}[V\boldsymbol{Y}] \wedge \overset{n-1}{\wedge} T^*\boldsymbol{X}$.

Remark 4.9. If the coordinate expression of $\chi(\lambda)$ is given by

$$\chi(\lambda) = \chi^{\boldsymbol{\alpha}}_i \boldsymbol{\vartheta}^i_{\boldsymbol{\alpha}} \wedge \omega\,,$$

where $\boldsymbol{\vartheta}^i_{\boldsymbol{\alpha}}$ are contact forms on $J_k V\boldsymbol{Y}$, the corresponding coordinate expressions of $E_{\chi(\lambda)}$ and $M_{\chi(\lambda)}$ are respectively given by

$$E_{\chi(\lambda)} = E_i \boldsymbol{\vartheta}^i \wedge \omega \,,$$
$$M_{\chi(\lambda)} = M_i^{\boldsymbol{\alpha}+\lambda} \boldsymbol{\vartheta}^i_{\boldsymbol{\alpha}} \wedge \omega_\lambda \,.$$

We have, in particular

$$E_{\chi(\lambda)} = (-1)^{|\boldsymbol{\beta}|} D_{\boldsymbol{\beta}} \chi_i^{\boldsymbol{\beta}} \boldsymbol{\vartheta}^i \wedge \omega \,,$$

with $0 \leq |\boldsymbol{\beta}| \leq k$. □

Definition 4.10. We call the morphism $\mathcal{J}(\lambda) := E_{\chi(\lambda)}$ the *generalized Jacobi morphism* associated with the Lagrangian λ. □

Again, this result can be compared with equation (2.19) of [4].

Acknowledgments. Thanks are due to R. Vitolo for many valuable and fruitful discussions and suggestions. The second author (M. P.) is very grateful to D. Krupka for useful comments and for the kind hospitality at the Mathematical Institute of the Silesian University in Opava, where most part of this paper has been written.
Commutative diagrams have been drawn by Paul Taylor's `diagrams` macro package.

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