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A CONTRIBUTION TO LOCAL BOL LOOPS

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Abstract. On an n-sphere, $n \geq 2$ a geodesic local loop introduced in [Ki] is a Bol loop, has $SO(n+1)$ as the group topologically generated by left translations of the loop, and is called here an n-dimensional spherical local Bol loop. Our aim is to prove that all smooth n-dimensional local Bol loops which are locally isotopic to an n-dimensional spherical local Bol loop are locally isomorphic to it.

1. Introduction

A *smooth* ($= C^\infty$-differentiable) *(local) loop* $(L, \cdot, /, e)$, $e \in L$ is a pointed smooth manifold with a triple of smooth (local) mappings $\cdot$, $\backslash$, $/$ from open domains of $L \times L$ to L such that for x, y, $z \in L$ the identities

$$(x/y) \cdot y \approx x, \quad y \cdot (y \backslash x) \approx x, \quad (x \cdot y)/y \approx x, \quad y \backslash (y \cdot x) \approx x,$$
$$x \cdot e \approx e \cdot x \approx x/e \approx e \backslash x \approx x$$

hold (whenever the left side of the identity is defined). A *germ* of smooth local loops with unit e can be introduced as an equivalence class in a usual way, [P], p. 67.

Due to smoothness, the conditions on the accompanying operations $\backslash$, $/$ can be substituted by the assumption that both families of left translations $\lambda_a : x \mapsto a \cdot x$ and right translations $\varrho_a : x \mapsto x \cdot a$ are (local) diffeomorphisms (L.V. Sabinin in [K&N I], p. 298, [Ki]). Then a (local) *isotopism* of a smooth (local) loop $(L, \cdot)$ onto a smooth (local) loop $(M, \circ)$ can be introduced as a triple of (local) diffeomorphisms α,β, $\gamma : L \to M$ such that $\gamma(x \cdot y) = \alpha(x) \circ \beta(y)$ for such x, y from L for which one side of the identity is defined. Two isotopic (local) loops determine the same web, [A&S]. *Isomorphisms* are obtained for $\alpha = \beta = \gamma$.

Example 1. Given a smooth manifold (M, ∇) with an affine connection, or especially a Riemannian manifold (M, g) with the canonical connection, then in a restricted normal neighbourhood U of a distinguished point $e \in M$ the so called *geodesic local loop at the point* e can be introduced with multiplication on U given by $x \cdot y = \exp_x \tau_{(e,x)} \exp_e^{-1}(y)$, [Ki]. Here $\exp_x tX$, $0 < t < \delta$ denotes the geodesic through x in the direction of a tangent vector $X \in T_xM$, and $\tau_{(e,x)} : T_eM \to T_xM$

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intermediates the parallel translation of the tangent spaces along the geodesic segment.

A smooth (local) loop L is called a (local) *left Bol* loop if the identity $(x\cdot(y\cdot(x\cdot z)))\approx(x\cdot(y\cdot x))\cdot z$ is satisfied (on some neighbourhood of the unit e). In the following, "left" will be omitted. A (local) loop isotopic to a (local) Bol loop is also a (local) Bol loop.

Let $G(L)$ denote the (local) group topologically generated by the family of left translations of a smooth (local) loop L, $G(L)=\langle\Lambda\rangle$, $\Lambda=\{\lambda_x : y\mapsto x\cdot y;\ x\in L\}$, let 1 denote the unit in G. Let H denote the isotropic subgroup of a point e under the (partial) action of $G(L)$ on L. If L is a smooth connected (local) Bol loop then $G(L)$ is a connected (local) Lie group (by similar arguments as in [M&S1], Prop. XII.2.14.), and H is its closed subgroup. Let $\mathfrak{g}=T_1G$ ($\mathfrak{h}$, respectively) be the Lie algebra of $G(L)$ (of H, respectively) and let $\mathfrak{m}:=T_1\Lambda$ denote the tangent space of Λ at the unit $1\in G$. Then $\mathfrak{m}$ is a vector complement of $\mathfrak{h}$ in $\mathfrak{g}$, $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$, $\mathfrak{m}$ generates $\mathfrak{g}$ as Lie algebra and the following relation holds [M&S1], Prop. XII.8.23:

$$[\mathfrak{m},[\mathfrak{m},\mathfrak{m}]]\subset\mathfrak{m}. \tag{1}$$

Vice versa, given a Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h}$ containing no non-trivial ideal of $\mathfrak{g}$ then a vector complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$ determines a unique local Bol loop L if and only if $\mathfrak{g}=\mathfrak{m}+[\mathfrak{m},\mathfrak{m}]$ ($\mathfrak{m}$ generates $\mathfrak{g}$ as Lie algebra), and the relation (1) holds, [M&S1] p. 428. The local Bol loop L associated with the triple $(\mathfrak{g},\mathfrak{h},\mathfrak{m})$ has the property that the group $G=\exp\mathfrak{g}$ with unit $1\in G$ is the group topologically generated by the family of left translations of L, the group $H=\exp\mathfrak{h}$ is the stabilizer of the unit $e\in L$, and $\Lambda=\exp\mathfrak{m}$ is the set of left translation of L ([M&S2], p. 62-65).

Example 2. Example 2 If M is a symmetric (locally symmetric, respectively) space equipped with the canonical connection then a geodesic loop at any point $e\in M$ is a smooth local Bol loop, [M&S2], p. 12, 13. If M is a compact symmetric space then the group G topologically generated by the left translations of $(M,\cdot)$ coincides with the compact connected Lie group of displacements of the symmetric space L. Since the group G acts transitively on the symmetric space L the geodesic loops for different points as units are isomorphic.

To distinguish isotopic (respectively isomorphic) smooth local Bol loops we can use a local version of the result proved by K. Strambach and P.T. Nagy which can be be formulated as follows, [Va]:

Lemma 1. *Let L_1 and L_2 be smooth connected local Bol loops realized on the same manifold and having the same group $G=\langle\Lambda_1\rangle=\langle\Lambda_2\rangle$ topologically generated by the family of the left translations of L_1, or L_2, respectively. Consider the tangent subspaces $T_1\Lambda_1=\mathfrak{m}_1$ and $T_1\Lambda_2=\mathfrak{m}_2$ of the Lie algebra $\mathfrak{g}=T_1G$ of G. The local loops L_1 and L_2 are isotopic if and only if there exists an element $g\in G$ such that $\mathrm{Ad}(g)(\mathfrak{m}_1)=\mathfrak{m}_2$ where Ad is the adjoint action of G on $\mathfrak{g}$. The local loops L_1 and L_2 are isomorphic if and only if there exists an automorphism $\alpha\in\mathrm{Aut}\,G$ of the* *group such that for the induced automorphism* α_* *of* $\mathfrak{g}$ *the relations* $\alpha_*(\mathfrak{h}_1) = \mathfrak{h}_2$ *and* $\alpha_*(\mathfrak{m}_1) = \mathfrak{m}_2$ *hold.*

2. Spherical geometry

The unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ is a compact Riemannian manifold of constant curvature (equal 1, [W], p. 66) endowed with a Riemannian metric induced by the standard scalar product on $\mathbb{R}^{n+1}$. The compact orthogonal group $O(n+1)$ plays the role of the full group of isometries of the n-sphere, and $G = SO(n+1)$ is the connected component of unit. An *n-dimensional spherical geometry* $\mathcal{S}_n$ has elements of $\mathbb{S}^n$ as its points and maximal geodesics as lines; maximal geodesics are sections of $\mathbb{S}^n$ with 2-planes of $\mathbb{R}^{n+1}$ containing the origin. *Collineations* of the spherical geometry arise as restrictions to $\mathbb{S}^n$ of actions of elements $A \in G$ on $\mathbb{R}^{n+1}$.

On G, an involutive automorphism σ is given by $\sigma(A) = SAS^{-1}$, $A \in G$ where $S = \operatorname{diag}(-1, 1, \dots, 1)$. The component of unit H^0 of the subgroup H consisting of all elements invariant under σ is of the form

$$H^0 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} ; B \in SO(n) \right\}. \tag{2}$$

So H^0 may be identified with $SO(n)$. The triple (G, H^0, σ) determines a symmetric Riemannian space ([K&N II], p. 208, 225), and there is a diffeomorphism between the homogeneous symmetric space $SO(n+1)/SO(n)$ and $\mathbb{S}^n$ such that the canonical connection of the symmetric space coincides with the Riemannian connection on $\mathbb{S}^n$, [K&N I], p. 277–228. Given a point e of $\mathcal{S}_n$ we can choose an orthonormal basis $\langle e, e_1, \dots, e_n \rangle$ in $\mathbb{R}^{n+1}$ with respect to which the isotropic subgroup in G of the point e is exactly H^0.

The scalar product determines an orthogonality relation on $\mathbb{R}^{n+1}$ (and in $\mathcal{S}_n$) which is denoted by $\perp$. A *reflection* $\sigma_{(x,-x)}$, $x \in \mathbb{S}^n$ at the point pair $\{x, -x\}$ in $\mathcal{S}_n$ is a map induced by the orthogonal transformation of $\mathbb{R}^{n+1}$ which fixes elementwise the 1-dimensional subspace X of $\mathbb{R}^{n+1}$ containing both points x, $-x$ and induces the inversion $y \mapsto -y$, $y \in X^\perp$ on the hyperplane $X^\perp$ orthogonal to X. Hence the matrix of $\sigma_{(x,-x)}$ is conjugate with the matrix $\operatorname{diag}(1, -1, \dots, -1)$.

Let U be a neighbourhood of e in $\mathbb{S}^n$ such that for every point x in U there exists exactly one geodesic in U incident with e and x, [K&N I], Th. 8.7., p. 146. Let $x \in U$, $x \neq e$. In the geodesic segment $[e, x]$ contained in U there is a unique middle point $\frac{x}{2}$ such that the reflection $\sigma_{(\frac{x}{2}, -\frac{x}{2})}$ at $\{\frac{x}{2}, -\frac{x}{2}\}$ maps e onto x. The product $\lambda_x := \sigma_{(\frac{x}{2}, -\frac{x}{2})}\sigma_{(e,-e)}$ called *a local transvection at* x, [K&N II], p. 219, [W], p. 232, maps also e onto x, and is contained in the connected group $SO(n+1)$, [K&N II], Lemma 1, p. 218. (Local) transvections are (local) isometries the tangent maps of which induce parallel translation of tangent spaces along geodesics, $T\lambda_x : T_e\mathbb{S}^n \to T_x\mathbb{S}^n$, [W], L.8.1.2. p. 232. If we denote by $U \cap \mathbb{S}^n$ the line of $\mathcal{S}_n$ containing the segment $[e, x]$ then the local transvection λ_x can be characterized as a map induced by the orthogonal transformation of $\mathbb{R}^{n+1}$ which fixes an $(n-1)$-dimensional subspace $U^{\perp}$ of U in $\mathbb{R}^{n+1}$ elementwise and acts on U as a rotation. To any point x in the neighbourhood U there exists precisely one transvection mapping e onto x. All transvections λ_y for points y of the geodesic segment $[e,x] \subset U$ form a local 1-parameter group. On U the local geodesic loop multiplication is given as in the Example 1. If V is a normal neighborhood of e contained in U such that for any two points $x, y \in V$ the image $\lambda_x(y)$ is contained in U then the multiplication $(x,y) \mapsto \lambda_x(y)$, $V \times V \to U$ coincides with the geodesic multiplication $(x,y) \mapsto x \cdot y$ on V since the formula is the same. That is, $(x,y) \mapsto \lambda_x(y)$ belongs to the germ of geodesic multiplication of a locally symmetric space at e, and hence defines on V a smooth local Bol loop with identity e. It will be called *an n-dimensional spherical local Bol loop* $(L(\mathcal{S}_n, e))$; by the Example 2, it is independent of the choice of the point e up to isomorphism.

3. The structure of the groups $SO(n+1)$

Let $n \geqq 2$. In the Lie algebra $\mathfrak{so}(n+1) = \{A \in M(\mathbb{R}, n+1);\ A + A^t = 0\}$ we can choose an $\mathbb{R}$-basis consisting of the family of $\frac{n(n+1)}{2}$ matrices $M_{i,j} = E_{ij} - E_{ji}$ where $i < j$ and E_{ij} is a matrix with 1 on the position (i,j) and 0 otherwise. The Lie multiplication $[M_{r,s}, M_{u,v}] = M_{r,s}M_{u,v} - M_{u,v}M_{r,s}$ satisfies the relations

$$\begin{aligned} &= M_{k,i}, \quad \text{for } k < i, \qquad & [M_{i,j}, M_{i,l}] &= M_{l,j}, \quad \text{for } l < j,\\ &= -M_{i,k}, \quad \text{for } i < k, & &= -M_{j,l}, \quad \text{for } j < l,\\ [M_{i,j}, M_{j,l}] &= M_{i,l}, & [M_{i,j}, M_{k,i}] &= -M_{k,j} \end{aligned}$$

and is equal 0 otherwise. The matrices $M_{i,j}$ with $2 \leq i < j \leq n+1$ and $2 \leq i \leq n$ form a basis for the Lie algebra $\mathfrak{h}$ of the isotropic subgroup H^0 of e in G,

$$\mathfrak{h} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} ;\ b \in \mathfrak{so}(n) \right\}.$$

The matrices $M_{1,j}$, $2 \leq j \leq n+1$ form a basis of a vector subspace $\mathfrak{m}$ which is complementary to $\mathfrak{h}$ in the Lie algebra $\mathfrak{so}(n+1)$, $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. The inclusions $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ hold. Hence the space $\mathfrak{m}$ determines together with the Lie algebra $\mathfrak{h}$ a symmetric space and if we denote by Λ the family of transvections $\lambda_x = \sigma_{(\frac{x}{2}, -\frac{x}{2})}\sigma_{(e,-e)}$ then $\Lambda = \exp \mathfrak{m}$. The matrix group

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} ;\ A \in O(n) \right\} \tag{3}$$

leaves the vector subspace $U_n = \left\{ \sum_{i=2}^{n+1} a_i M_{1,i} \,|\, (a_2, \dots, a_{n+1}) \in \mathbb{R}^n \right\}$ invariant with respect to the conjugation and acts on U_n as the full orthogonal group $O(n)$ on the euclidean space $\mathbb{R}^n$. Hence the vector space $\mathfrak{so}(n+1)$ can be decomposed as a direct sum $U_n \oplus U_{n-1} \oplus \dots \oplus U_1$ of subspaces U_i which are orthogonal to each other and the matrices $M_{i,j}$, $i+1 \leq j \leq n+1$ form an orthogonal basis of U_i. The subgroup $\left\{ \begin{pmatrix} I_r & 0 \\ 0 & C \end{pmatrix} ;\ C \in O(n+1-r) \right\}$ of the matrix group (3) where $2 \leq r \leq n$ and I_r is the $(r \times r)-$identity matrix fixes each of the subspaces U_i for $1 \le i \le r-1$. Now let us choose a special canonical basis in each vector complement to the Lie algebra of the stabilizer. Namely, let $\mathfrak{m}$ be an n-dimensional complement of the subalgebra $\mathfrak{h}$ in $\mathfrak{so}(n+1)$ and let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be an orthogonal basis of $\mathfrak{m}$. If we denote by π_k the projection of $\mathfrak{so}(n+1)$ onto U_k then the images $\pi_k(\mathfrak{a}_i)$ of the vectors $\mathfrak{a}_i$ are for different i orthogonal to each other. Using the action of the group (3) on the orthogonal subspaces U_i we can suitably transform the original basis vectors; in fact we can assume that the k-th vector $\mathfrak{a}_k$ is of the form

$$\mathfrak{a}_k = M_{1,k+1} + \sum_{j=2}^{n+1-k} \beta_k^j M_{j,\,k+j}, \qquad 1 \le k \le n$$

where $\beta_k^j \ge 0$ are non-negative reals (otherwise we can conjugate $\mathfrak{a}_k$ by a suitable diagonal matrix having as entries 1 and -1). A canonical basis $\mathcal{B}(\beta_j^k)$, $1 \le k \le n$, $2 \le j \le n+1-k$ of this type will be called a *normalized basis* of a complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$. Two complements having different normalized basis cannot determine isomorphic local Bol loops.

4. Isotopisms and isomorphisms

Now we are interested how many isomorphism subclasses can be distinguished in the class of smooth (local) Bol loops isotopic with an n-dimensional spherical local Bol loop $(L(\mathcal{S}_n, e))$. We shall show that in this case, isotopism is equivalent with isomorphism. The following technical lemma shows that there is the only isomorphism class since there is in fact a unique complement $\mathfrak{m}$ of $\mathfrak{h}$ satisfying (1), namely a subspace spanned by the normalized basis $\langle M_{1,2}, M_{1,3}, \dots, M_{1,n+1}\rangle$.

Lemma 2. *Let $\mathfrak{m} = \langle \mathfrak{a}_1, \dots, \mathfrak{a}_n\rangle$ be an n-dimensional complement of the subalgebra $\mathfrak{h}$ in the Lie algebra $\mathfrak{so}(n+1)$ spanned by normalized basis vectors*

$$\mathfrak{a}_{n-k} = M_{1,n-k+1} + \beta_{n-k}^2 M_{2,n-k+2} + + \cdots + \beta_{n-k}^{k+1} M_{k+1,n+1} \tag{4}$$

with $k = 0, \dots, n-1$. The subspace $\mathfrak{m}$ satisfies the condition (1), if and only if $\beta_{n-k}^t = 0$ for all $t \in \{2, \dots, k+1\}$, $k \in \{0, \dots, n-1\}$, i.e. if and only if

$$\mathfrak{a}_{n-k} = M_{1,n-k+1} \quad \textit{for} \;\; k = 0, \dots, n-1. \tag{5}$$

Proof. If the basis vectors are of the form (5) then (1) holds. Vice versa, let us verify that if a vector subspace $\mathfrak{m}$ satisfies (1) then all coefficients β_j^t are equal zero. Let $k = 1$. Then $[\mathfrak{a}_{n-1}, \mathfrak{a}_n] = -M_{n,n+1} + \beta_{n-1}^2 M_{1,2}$ and

$$\mathfrak{u}(n-1,n,n-1) = [[\mathfrak{a}_{n-1}, \mathfrak{a}_n], \mathfrak{a}_{n-1}] = (1 + (\beta_{n-1}^2)^2) M_{1,n+1} - 2\beta_{n-1}^2 M_{2,n}. \tag{6}$$

If (1) holds then $\mathfrak{u}(n-1,n,n-1) \in \mathfrak{m}$ which means that the element can be written in the form $\mathfrak{u}(n-1,n,n-1) = \varrho_1^{(n-1,n,n-1)}\mathfrak{a}_1 + \cdots + \varrho_n^{(n-1,n,n-1)}\mathfrak{a}_n$. Comparing both expressions we deduce that this is true if and only if $\varrho_n^{(n-1,n,n-1)} = 1 + (\beta_{n-1}^2)^2$ and $\varrho_p^{(n-1,n,n-1)} = 0$ for $p < n$ since no multiple of $M_{1,k}$ appears in the formula (6) for $k < n+1$. Consequently, $\mathfrak{u}(n-1,n,n-1) \in \mathfrak{m}$ holds if and only if $\beta_{n-1}^2 = 0$,

$\varrho_n^{(n-1,n,n-1)} = 0$, and $\mathfrak{a}_{n-1} = M_{1,n}$. We can proceed step by step. In the k-th step, assume that the statement holds for some fixed $k-1 \in \{2,\dots,n-1\}$, that is, we know that $\beta_{n-s}^t = 0$ for all $s \in \{0,\dots,k-1\}$ and all $t \in \{2,\dots,s+1\}$, and $\mathfrak{a}_n = M_{1,n+1}$, $\mathfrak{a}_{n-1} = M_{1,n}$, $\dots$, $\mathfrak{a}_{n-k+1} = M_{1,n-k+2}$. Let us check $\mathfrak{a}_{n-k} = M_{1,n-k+1}$ by proving that $\beta_{n-k}^2 = \dots = \beta_{n-k}^{k+1} = 0$. For any $j \in \{n-k+1,\dots,n-1\}$,

$$= -M_{n-k+1,j} + \beta_{n-k}^{k-n+j+1} M_{1,k-n+j+1},$$
$$u(n-k,j,n) = [[\mathfrak{a}_{n-k},\mathfrak{a}_j],\mathfrak{a}_n] = -\beta_{n-k}^{k-n+j+1} M_{k-n+j+1,n+1}.$$

An element $u(n-k,j,n) \in \mathfrak{m}$ if and only if $u(n-k,j,n) = \sum_p \varrho_p^{(n-k,j,n)} \mathfrak{a}_p$. Comparing both expressions we obtain that all coefficients in the combination vanish, $\varrho_p^{(n-k,j,n)} = 0$, $p = 1,\dots,n$, and $u(n-k,j,n)$ must be a zero vector. Equivalently, $\beta_{n-k}^{k-n+j+1} = 0$ for all $j \in \{n-k+1,\dots,n-1\}$. It remains to verify $\beta_{n-k}^{k+1} = 0$. By similar arguments as above, the product $\mathfrak{u}(n-k,n,n-1) = [-M_{n-k+1,n+1} + \beta_{n-k}^{k+1} M_{1,k+1}, M_{1,n}] = -\beta_{n-k}^{k+1} M_{k+1,n} \in \mathfrak{m}$ if and only if $\beta_{n-k}^{k+1} = 0$. Hence $\mathfrak{a}_{n-k} = M_{1,n-k+1}$ under the assumption (1), and the statement is true also for k. Consequently the complementary subspace $\mathfrak{m}$ satisfies (1) if and only if it is spanned by the normalized basis $\langle M_{1,2}, M_{1,3},\dots,M_{1,n+1}\rangle$. □

Theorem 1. *All smooth n-dimensional local Bol loops which are locally isotopic to an n-dimensional spherical local Bol loop $(L(\mathcal{S}_n,e))$ are locally isomorphic to it.*

Proof. Let L be a smooth local Bol loop locally isotopic to an n-dimensional spherical geodesic loop $(L(\mathcal{S}_n,e))$. Then the left translation group of L is locally isomorphic to $SO(n+1)$, the stabilizer of a unit is locally isomorphic to the Lie group of the shape (2), and its Lie algebra is $\mathfrak{h}$. Using the above considerations and notation we can pass to the tangent objects and say that two vector complements $\mathfrak{m}$, $\mathfrak{m}'$ to $\mathfrak{h}$ in $\mathfrak{so}(n+1)$ determine isomorphic local Bol loops if and only if they are provided with the same normalized basis $\mathcal{B}(\beta_j^k)$. But they also satisfy the condition (1), and the only normalized basis for which the products of basis vectors $[[\mathfrak{a}_i,\mathfrak{a}_j],\mathfrak{a}_k]$ belong to $\mathfrak{m}$ is the basis $\langle M_{1,2},\dots,M_{1,n+1}\rangle$ presented in the above Lemma 2. □

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