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Optimized Krylov-Ventcell method. Application to convection-diffusion problems

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1 Introduction

In this paper a domain decomposition method with non-overlapping subdomains is presented, applied to the convection-diffusion problem:

$$\mathcal{L}(u) = cu + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} - \nu\Delta u = f \text{ in } \Omega \tag{1.1}$$
$$\mathcal{C}(u) = g, \text{ on } \partial\Omega$$

where Ω is a bounded open set of $\mathcal{R}^2$, $\boldsymbol{a} = (a,b)$ is the velocity field, ν is the viscosity, $\mathcal{C}$ is a linear operator, c is a constant which could be $c = \frac{1}{\Delta t}$ with Δt a time step of a backward-Euler scheme for solving the time dependent convection-diffusion problem. The strategy could be applied to other PDE's.

Substructuring formulation

Let $\bar{\Omega} = \cup_{i=1}^{N} \bar{\Omega}_i$, $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$. We denote by $\Gamma_{i,j}$ the common interface to Ω_i and Ω_j, $i \neq j$. The outward normal from Ω_i to Ω_j is $\boldsymbol{n}_{i,j}$ and $\boldsymbol{\tau}_{i,j}$ is a tangential unit vector.
The additive Schwarz algorithm [Lio89] is:

$$\begin{aligned} \mathcal{L}(u_i^{n+1}) &= f, \quad \text{in } \Omega_i \\ \mathcal{B}_{i,j}(u_i^{n+1}) &= \mathcal{B}_{i,j}(u_j^{n}), \quad \text{on } \Gamma_{i,j},\ i \neq j \\ \mathcal{C}(u_i^{n+1}) &= g, \quad \text{on } \partial\Omega_i \cap \partial\Omega \end{aligned}$$

Where $\mathcal{B}_{i,j}$ is an interface operator.
In [NRdS95], this algorithm is interpreted as a Jacobi algorithm applied to the interface problem

$$(Id - \mathcal{T})(H) = G \tag{1.2}$$

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where $\mathcal{T}$ is an interface operator, G a second member only depending on f and g. To accelerate convergence, the Jacobi algorithm is replaced by a BICGSTAB [Van92] or GMRES [SS86] algorithm.
In this paper, to accelerate convergence again, the interface conditions $\mathcal{B}_{i,j}$ between subdomains are chosen as partial differential operators of order 2 in the tangential direction to the interface, which minimize the rate of convergence of the Schwarz algorithm. We denote them by Optimized Order 2 conditions (OO2). To introduce them, other interface conditions are first recalled.

Interface conditions
General interface operators of order 2 in the tangential direction,

$$\mathcal{B}_{i,j} = \frac{\partial}{\partial \boldsymbol{n}_{i,j}} + c_1 + c_2 \frac{\partial}{\partial \boldsymbol{\tau}_{i,j}} + c_3 \frac{\partial^2}{\partial \boldsymbol{\tau}_{i,j}^2}, \quad \mathcal{B}_{j,i} = \frac{\partial}{\partial \boldsymbol{n}_{j,i}} + c_4 + c_5 \frac{\partial}{\partial \boldsymbol{\tau}_{j,i}} + c_6 \frac{\partial^2}{\partial \boldsymbol{\tau}_{j,i}^2} \quad (1.3)$$

where c_m are constants, $1 \leq m \leq 6$.

In [Des90],[CQ95], the interface conditions are of order 0 :
<u>Taylor order 0 interface conditions (T0)</u>

$$\mathcal{B}_{i,j} = \frac{\partial}{\partial \boldsymbol{n}_{i,j}} - \frac{\boldsymbol{a}.\boldsymbol{n}_{i,j} - \sqrt{(\boldsymbol{a}.\boldsymbol{n}_{i,j})^2 + 4c\nu}}{2\nu} \quad (1.4)$$

and $\mathcal{B}_{j,i}$ is defined as is $\mathcal{B}_{i,j}$, replacing $\boldsymbol{n}_{i,j}$ by $\boldsymbol{n}_{j,i}$.
In [NRdS95], [NR95], the interface conditions are of order 2 :
<u>Taylor order 2 interface conditions (T2)</u>

$$\begin{aligned}\mathcal{B}_{i,j} = \frac{\partial}{\partial \boldsymbol{n}_{i,j}} - \frac{\boldsymbol{a}.\boldsymbol{n}_{i,j} - \sqrt{(\boldsymbol{a}.\boldsymbol{n}_{i,j})^2 + 4c\nu}}{2\nu} + \frac{\boldsymbol{a}.\boldsymbol{\tau}_{i,j}}{\sqrt{(\boldsymbol{a}.\boldsymbol{n}_{i,j})^2 + 4c\nu}} \frac{\partial}{\partial \boldsymbol{\tau}_{i,j}} \\ - \frac{\nu}{\sqrt{(\boldsymbol{a}.\boldsymbol{n}_{i,j})^2 + 4c\nu}} \left(1 + \frac{(\boldsymbol{a}.\boldsymbol{\tau}_{i,j})^2}{(\boldsymbol{a}.\boldsymbol{n}_{i,j})^2 + 4c\nu}\right) \frac{\partial^2}{\partial \boldsymbol{\tau}_{i,j}^2}\end{aligned} \quad (1.5)$$

and $\mathcal{B}_{j,i}$ is defined as is $\mathcal{B}_{i,j}$, replacing $\boldsymbol{n}_{i,j}$ by $\boldsymbol{n}_{j,i}$ and $\boldsymbol{\tau}_{i,j}$ by $\boldsymbol{\tau}_{j,i}$.

Conditions (1.4), (1.5) can be seen as Taylor approximations of order 0 and order 2, for low wave numbers, of the artificial boundary conditions [EM77], [Hal86]: If $\Omega_i = \mathcal{R}^- \times \mathcal{R}$, $\Omega_j = \mathcal{R}^+ \times \mathcal{R}$, and $\Gamma_{i,j}$ is the axis $x = 0$, the artificial boundary conditions are $\partial_x - \Lambda^-$, $\partial_x - \Lambda^+$, with Λ^- the Dirichlet to Neumann operator of the right half plane defined as $\Lambda^- : u_0 \longrightarrow \frac{\partial w}{\partial x}(0, y)$ with w such as

$\mathcal{L}(w) = 0,\ x > 0,\ w(0, y) = u_0(y)$ at $x = 0$, and w bounded at infinity

The Dirichlet to Neumann operator of the left half plane Λ^+ is defined in the same way. When the coefficients of $\mathcal{L}$ are constant, if we denote by Λ^+_{ap} and Λ^-_{ap} the Taylor approximations of order 0 or 2, for low wave numbers, of Λ^+ and Λ^-, they satisfy:

$$\Lambda^+_{ap} + \Lambda^-_{ap} = \Lambda^+ + \Lambda^- = \frac{a}{\nu} \quad (1.6)$$

Then, $\mathcal{B}_{i,j} = \partial_x - \Lambda^-_{ap}$, $\mathcal{B}_{j,i} = \partial_x - \Lambda^+_{ap}$, and $\mathcal{B}_{j,i}$ can be obtained from $\mathcal{B}_{i,j}$, using (1.6). So in (1.3) the coefficients c_4, c_5, c_6 are obtained from c_1, c_2, c_3 (or reciprocally).

In [TB94], interface operators of order 1 are used : $c_3 = c_6 = 0$. The coefficients c_1, c_2, c_4, c_5 are chosen in order to minimize the convergence rate. As the minimization problem on the four parameters is very costly, an approximate minimization problem is solved, but it may lead to non convergence in some cases. The link between $\mathcal{B}_{i,j}$ and $\mathcal{B}_{j,i}$ as in (1.6) has not been done.

2 OO2 interface conditions

In this paper, the interface conditions are of order 2 as in (1.3) and are chosen as follows:

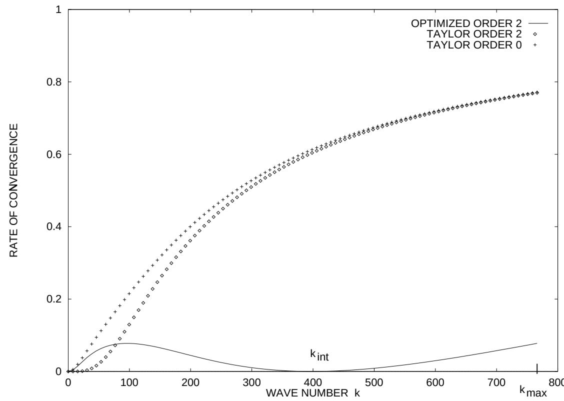
- First we link $\mathcal{B}_{i,j}$ and $\mathcal{B}_{j,i}$ as in (1.6). This means that c_4, c_5, c_6 are obtained from c_1, c_2, c_3: $c_1 = c_1(\boldsymbol{a}.\boldsymbol{n}_{i,j}, \boldsymbol{a}.\boldsymbol{\tau}_{i,j})$, $c_2 = c_2(\boldsymbol{a}.\boldsymbol{n}_{i,j}, \boldsymbol{a}.\boldsymbol{\tau}_{i,j})$, $c_3 = c_3(\boldsymbol{a}.\boldsymbol{n}_{i,j}, \boldsymbol{a}.\boldsymbol{\tau}_{i,j})$, and $c_4 = c_1 + \frac{\boldsymbol{a}.\boldsymbol{n}_{i,j}}{\nu}$, $c_5 = c_2(\boldsymbol{a}.\boldsymbol{n}_{j,i}, \boldsymbol{a}.\boldsymbol{\tau}_{j,i})$, $c_6 = c_3(\boldsymbol{a}.\boldsymbol{n}_{j,i}, \boldsymbol{a}.\boldsymbol{\tau}_{j,i})$. So we only have to determine c_1, c_2, c_3.
- Then, we choose $c_1 = -\frac{\boldsymbol{a}.\boldsymbol{n}_{i,j} - \sqrt{(\boldsymbol{a}.\boldsymbol{n}_{i,j})^2 + 4c\nu}}{2\nu}$ so that the interface condition is exact for the lowest wave number.
- Finally, we compute c_2 and c_3 by minimizing the convergence rate of the Schwarz algorithm in the case of 2 subdomains and constant coefficients.

Advantages (N subdomains case): The minimization problem on c_2 and c_3 is on a set of conditions which verify (1.6), i.e. on a set of conditions which ensures (adding a condition on the sign of c_2) the convergence of the Schwarz algorithm (see [NN94]). So an approximate minimization problem on the same set of conditions will also ensure the convergence. In the case of 2 subdomains, the convergence is proved by computing explicitely the convergence rate. When the domain is decomposed in N subdomains (strips) the convergence rate is estimated in function of the convergence rate of the 2 subdomain case and the decomposition geometry. The convergence is proved by using techniques issued from formal language theory (see [NN94]).

The minimization problem on c_2 and c_3 is sought in term of wave numbers k: we minimize the maximum of the convergence rate function $k \to \rho(k, c_2, c_3)$ on the interval $|k| \leq k_{max}$ where k_{max} is a given constant, $k_{max} > 0$ (in the discrete case, $k_{max} = \frac{constant}{h}$ where h is the mesh size in y) (see [Jap96]).
The study of the function ρ leads us to determine only one parameter, which is a low wave number k_{int} (see Figure 1). This parameter is computed with a dichotomy algorithm, which is not costly. With k_{int} we can compute $c_2 = c_2(k_{int})$ and $c_3 = c_3(k_{int})$.

Theorem 2.1 *Let Ω be decomposed in 2 subdomains, with $a \in \mathcal{R}$, $a \neq 0$, $b = 0$ and $c \geq 0$ in (1.1). Let $k_{max} = \frac{\pi}{h}$ where h is the mesh size, and let $(\rho_{max})_{IC}$ be the maximum of ρ on $0 \leq k \leq k_{max}$ with the interface condition IC. Let $\alpha = 1 + \frac{4\nu c}{a^2}$. Then, when $h \to 0$:*
$(\rho_{max})_{T0} \approx 1 - \frac{2}{\pi}\alpha^{\frac{1}{2}}(\frac{|a|h}{\nu})$, $(\rho_{max})_{T2} \approx 1 - \frac{4}{\pi}\alpha^{\frac{1}{2}}(\frac{|a|h}{\nu})$,
$(\rho_{max})_{OO2} \approx 1 - 8\alpha^{\frac{1}{6}}(\frac{1}{4\pi}\frac{|a|h}{\nu})^{\frac{1}{3}}$

Figure 1 Rate of convergence versus wave numbers k, $\mathbf{0 \leq k \leq k_{max}} = \frac{\boldsymbol{\pi}}{\mathbf{h}}$
$a = 1,\ b = 1,\ \nu = 0.01,\ c = 0,\ h = \frac{1}{240}$



So the condition number is asymptotically much better for OO2 than for Taylor T0 or T2 interface conditions.

3 Numerical results

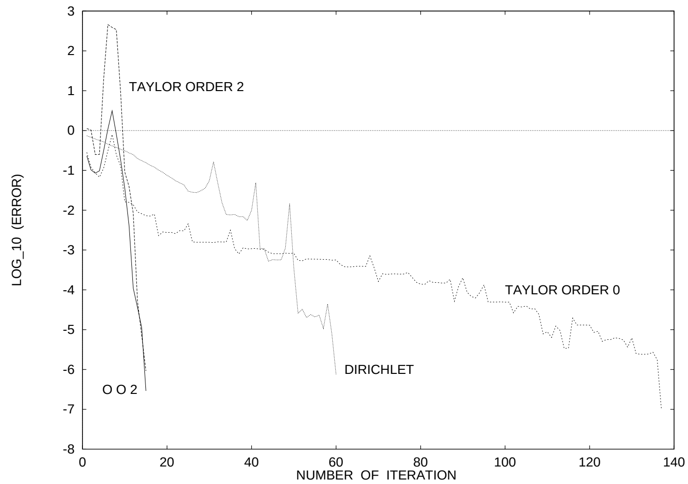
Let the problem be: $\mathcal{L}(u) = f, \quad 0 \leq x \leq 1,\ 0 \leq y \leq 1$
with $u(0, y) = \frac{\partial u}{\partial x}(1, y) = 0,\ 0 \leq y \leq 1,\ \frac{\partial u}{\partial y}(x, 1) = 0,\ u(x, 0) = 1,\ 0 \leq x \leq 1.$
We consider a rectangular finite difference grid with a mesh size h. The operator $\mathcal{L}$ and $\mathcal{B}_{i,j}$ are discretized by a standard upwind difference scheme (see [Fle90]). The unit square is decomposed into N rectangles with one overlapping mesh cell.
Remark: another discretization could be used, with a non-overlapping decomposition.
Algorithm :
The interface problem (1.2) is solved by a Bicgstab algorithm. This involves solving N independant subproblems which can be done in parallel. Each subproblem is solved by a direct method:
- First we compute the LU factorization of the matrix corresponding to the discretization of the subproblems. This is a parallel task.
- Then at each iteration of Bicgstab, we solve in parallel the subproblems using this LU factorization.
Important point: Each iteration has the same cost for all the interface conditions OO2,(1.4),(1.5),and Dirichlet, because the use of order 2 conditions does not increase the bandwidth of the local matrix.

We are more interested in the stationary case, so we take $c = 10^{-6}$ in the following results. The convergence is also significantly better when $c >> 1$.

The OO2 interface conditions give a significantly better convergence which is independant of the convection velocity angle to the interfaces (see Figure 2 and

Figure 3).

Figure 2 Error versus the number of iterations. Normal velocity to the interfaces, $\mathbf{16} \times \mathbf{1}$ subdomains, $a = y,\ b = 0,\ \nu = 0.01,\ c = 10^{-6},\ h = \frac{1}{240}$.



One of the advantages to have a convergence independant of the convection velocity angle to the interfaces is that for a given number of subdomains, the decomposition of the domain (in strips or in rectangles) doesn't affect the convergence (see Figures 2 and 4).

Figure 5 shows that the convergence with OO2 interface conditions is significantly better for a more general convection velocity (a rotating velocity) with a decomposition in 8×4 rectangles. The OO2 interface conditions are easy to implement and not cost increasing at each iteration. We observed numerically that the convergence with the OO2 interface conditions is also practically independant of the viscosity ν.
Moreover the OO2 interface conditions can be seen as a preconditioner for iterative methods, and the convergence for the studied numerical cases is independent of the mesh size (see Figures 6 and 7).

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Figure 3 Error versus the number of iterations. Tangential velocity to the interfaces, **16** $\times$ **1** subdomains, $a = y,\ b = 0,\ \nu = 0.01,\ c = 10^{-6},\ h = \frac{1}{240}$.

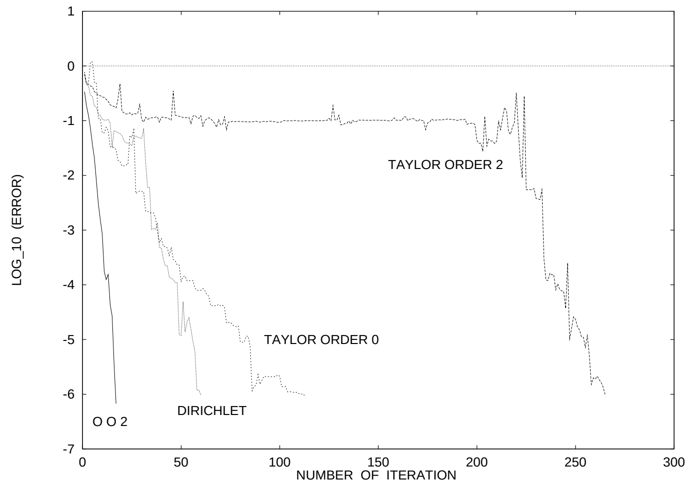

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Figure 4 Error versus the number of iterations.
$\mathbf{4 \times 4}$ subdomains, $a = y,\ b = 0,\ \nu = 0.01,\ c = 10^{-6},\ h = \frac{1}{240}$, with conditions of order 0 at cross points.

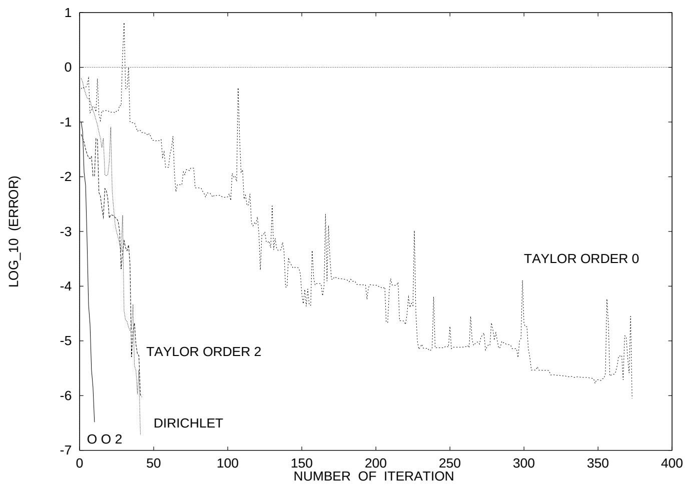

Figure 5 Error versus the number of iteration.
$\mathbf{8 \times 4}$ subdomains, $\nu = 0.01,\ c = 0,\ h = \frac{1}{240}$,
$a = -\sin(\pi(y - \frac{1}{2}))\cos(\pi(x - \frac{1}{2})),\ b = \cos(\pi(y - \frac{1}{2}))\sin(\pi(x - \frac{1}{2}))$, with conditions of order 0 at cross points.

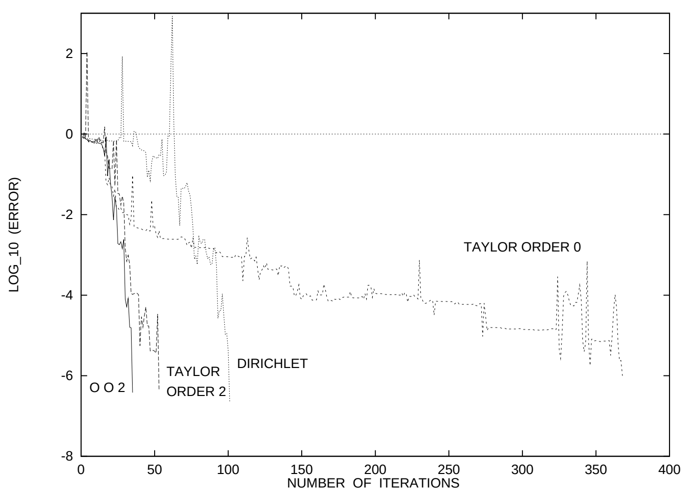

Figure 6 Number of iterations versus the mesh size.
16 × 1 subdomains, $a = y,\ b = 0,\ \nu = 0.01,\ c = 10^{-6},\ \max(Error) < 10^{-6}$

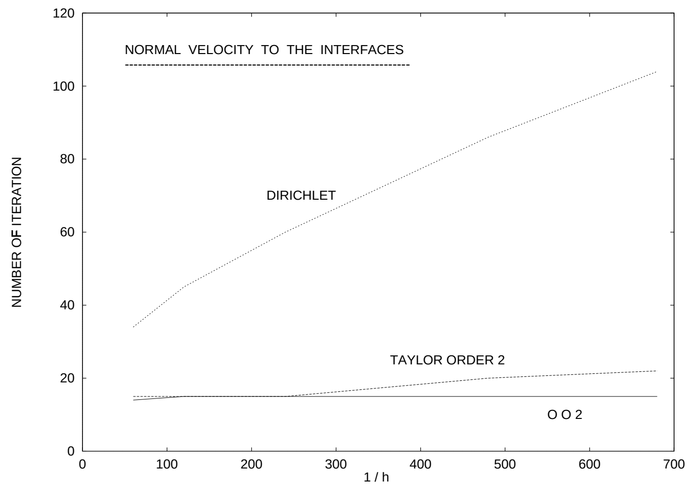

Figure 7 Number of iterations versus the mesh size.
16 × 1 subdomains, $\nu = 0.01,\ c = 10^{-6},\ \max(Error) < 10^{-6}$
$a = -\sin(\pi(y - \frac{1}{2}))\cos(\pi(x - \frac{1}{2})),\ b = \cos(\pi(y - \frac{1}{2}))\sin(\pi(x - \frac{1}{2}))$