PERCEPTUAL AND SYMBOLIC REPRESENTATIONS AS A STARTING POINT OF THE ACQUISITION OF THE DERIVATIVE

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In this paper we study how a student begins to acquire the concept of the derivative, what kind of representations he acquires and how he connects these representations. A teaching period, in which different perceptual and symbolic representation were emphasized, was carried out and task based interviews conducted to five students. The students used several different kinds of representations to process the derivative. They were all able to consider the derivative as an object by using perceptual representations at an early stage of the acquisition process. Also they all could calculate the derivative at a point by using symbolic representations. They all formed perceptual representations of the limiting process of the difference quotient but they would still need guidance to connect them to symbolic representations.

INTRODUCTION

Within the mathematics education research community there has been a lot of discussion about students' conceptions of mathematical concepts and about the development of these conceptions. There has been discussion about process and object conceptions (Asiala & al. 1997, Gray & Tall 2001, Sfard 1991) and about different representations (Goldin 2001 & 1998, Gray & Tall 2001). The concept acquisition process can start from perceptions of objects or from actions on objects (Gray & Tall 2001). Asiala et al. (1997) have presented a genetic decomposition corresponding to the APOS-theory in which they described students' analytical and graphical ways to construct the concept of the derivative. The students whose course was based on this analysis may have had more success than students of traditional courses (Asiala & al. 1997). The same result was found in Repo's research (1996), in which she implemented a calculus course planned on the basis of the APOS-theory and in which different representations of the derivative were emphasized. According to the research of Kendal and Stacey (2000) teacher's emphasis on certain representations of the derivative influence on how students can deal with representations. According to Watson's and Tall's research (2002) students attending teaching based on perceptual representations and process-object development had more success than students attending standard teaching in the subject of the vectors.

This research is part of the author's ongoing work on his PhD thesis, in which students' acquisition process of the derivative is studied. This paper is focused on the beginning of that process. Data was collected by conducting task based interviews after a five-hour teaching period to get information on with which kind of representations a student can start to acquire the derivative. Especially student's perceptual and symbolic representations and connections between them is studied.

THE CONCEPT ACQUISITION PROCESS

A student can make internal representations of a mathematical concept which is presented to him by using external representations. According to Goldin (2001) the internal representation systems can be a) verbal/syntactic, b) imagistic, c) formal notational, d) strategic and heuristic, and e) affective. According to him the study of student's conception and understanding of a concept should focus on studying student's internal representations. This is done by interpreting student's interaction with, discourse about, or production of external representations (ibid. 5-6). A concept is learned when a variety of appropriate internal representations have been developed with functioning relationships among them (ibid. 6).

According to the APOS-theory the student constructs a mathematical concept so that an action performed to an object is interiorized to a process which then encapsulates to an object. A schema is a collection of processes, objects and other schemas. (Asiala & al. 1997.) According to Gray and Tall (2001) the concept acquisition can start by an action performed on an object, but also by making a perception of an object. Gray and Tall call this kind of perceived objects embodied objects. The embodied objects are mental constructs of perceived reality, and through reflection and discourse they can become more abstract constructs, which do not anymore refer to specific objects in the real world (Gray & Tall 2001). Hence student's conception can start to develop from perceptual or from symbolic representations, and it is important to connect these representations. The conception of a mathematical object, formed by encapsulation, already has a primitive existence as an embodied object (Gray & Tall 2001).

In Goldin's classification verbal/syntactic and formal notational representations are symbolic representations and imagistic representations are perceptual representations. According to Goldin (1998, 156) representation systems are proposed to develop through three stages, so that first, new signs are taken to symbolize aspects of a previously established system of representation. Then the structure of the new representation system develops in the old system and finally the new system becomes autonomous.

THE TEACHING PERIOD OF THE DERIVATIVE

In the five-hour teaching period, planned according to theoretical framework, the derivative was introduced by using different representations and open approach. At first, the rate of change of the function was perceived from the graph. Moving a hand along the curve, placing a pencil as a tangent, looking how steep the graph was and the local straightness of the graph were used as perceptual representations. Then, the average rate of change was calculated by difference quotient and as the slope of the secant. After that the students were given the following problem: How to determine the instantaneous rate of change at a certain point? Finally, the derivative was defined as the limit of the difference quotient.

THE RESEARCH METHODS AND DATA COLLECTION

The teaching period was carried out by the author in the autumn of 2003 as a part of a Finnish high school course "Differentiaalilaskenta 1". There were 14 about 17-year old students in the course. The data was collected by a pretest, by videotaping the lessons and by conducting videotaped task based interviews to five students. The students were directed to think their solutions aloud. In five tasks they were asked to tell in their own words what the derivative is, make observations of the derivative of the function from its graph (Fig. 1), estimate the derivative of the function 2^x at the point $x = 1$, interpret the form (new to them) of the difference quotient and the limit of it from the graph of an unknown function, sketch the graph of the distance and acceleration from the graph of the velocity, and determine the average and instantaneous accelerations from the graph of the velocity.

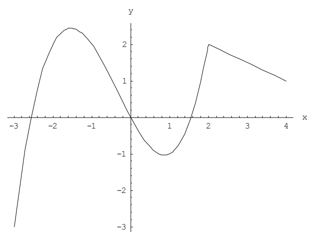

Figure 1: The given graph of function

The interviews were analyzed by using the constant comparative method to find the representations that the students used. After that it was analyzed how each student had connected the representations together and how strong each representation was. Special attention was paid to perceptual and symbolic representations. When appropriate it was analyzed whether the representation used was an action, a process or an object.

Interviews of the subjects Tommi and Niina were carried out right after the teaching period. Samuel was interviewed one, Susanna three and Daniel five lessons after the teaching period. Under that time the teacher of the course continued with the concept of the derivative function and with differentiation rules.

Based on their success on mathematics before the course the students could be classified so that Niina and Susanna are weak (w), Tommi and Samuel are average (a) and Daniel is good (g). In the pretest all these students could determine the average velocity from the graph of the distance, but only Daniel estimated the instantaneous velocity. They all, except Susanna, determined correctly also the sign of the velocity. Niina and Susanna had some difficulties with functions and they could not draw a tangent. The other three could also draw a tangent of slope zero and, except Samuel, determine the slope of a general tangent. Only Daniel could interpret the difference quotient as the slope of a secant and estimate how it changes when the base interval decreases.

THE STUDENT'S REPRESENTATIONS OF THE DERIVATIVE

All interviewed students were able to consider the derivative qualitatively as an object by using perceptual representations and quantitatively by using symbolic representations. For example they could determine the sign of the derivative from the graph of the function and, except Niina, estimate the derivative of the function 2^x at the point $x = 1$. Table 1 summarizes the most common representations used by the student and classifies them as actions, processes or objects.

Representations		**Niina** (w)	**Susanna** (w)	**Tommi** (a)	**Samuel** (a)	**Daniel** (g)
Perceptual	Tangent	Object	Object	Object	Object	Object
	Rate of change	Process	Process / Object	Object and symbolic process	Verbal	
	Steepness	Object	Object		Object	Object
	Concrete objects or processes		Pencil as a tangent	Pencil as a tangent	Ruler as a tangent	Tangents in the air
	Limiting process	Local straight-ness	Secants approach tangent	Average rate of change over a small interval	Secants approach tangent	Average derivative and secants
Symbolic	Limit of the difference quotient	Action	Action	Process	Process	Action
	Slope of the tangent	Process	Process	Process	Verbal	Process
	Differentiation rule		Action			Process

Table 1: The student's most common representations of the derivative

The perceptual representations of the derivative

All the students had an imagistic representation of the tangent as an object. This was especially strong for Tommi, Samuel and Daniel. Niina, Susanna and Tommi had also the rate of change as an important representation. Tommi explained that when the rate of change is three, it means instantaneously that "if we move one step forward, we must go three steps upward". It seems that Tommi has understood the rate of change as a perceptual object and as a symbolic process which is closely connected to the process of defining the slope of the tangent. He understood also that acceleration is "the rate of change of the velocity". Niina's and Susanna's representations were a bit weaker and Samuel's only verbal and not connected to the other representations.

As the most concrete representations Susanna, Tommi and Samuel used pencil or ruler as a tangent to embody the derivative. In addition, Daniel used tangents freely drawn in the air. They used these concrete representations only in individual cases when arguing or examining problematic points. So obviously these representations were connected closely to other perceptual representations which are more abstract and with them the derivative can be examined mentally. Only in some cases they needed to use concrete representations. For example Samuel argued his observations of the derivative made from the graph of the function by placing the ruler as a tangent: “because the tangent goes like this”. Tommi’s pencil-representation seemed to be connected to the rate of change. When trying to sketch the acceleration from the graph of the velocity, he moved pencil as a tangent along the graph and when asked explained: “If it points up, then accelerating.”

All the students, except Tommi, used the imagistic representation of the steepness of the graph when making observations of the derivative of the function from its graph (Fig. 1). They used this representation especially when defining the point where the derivative is at the largest and where at the smallest. Probably the extreme values are easy to examine with this representation. For example Susanna explained, that the derivative is at greatest, when “the graph rises most steeply” and she used the pencil as a tangent to find this point. On the other hand, the point where the derivative is at the smallest was more difficult to find:

> “Where it goes most steeply downward, hmm (places pencil as a tangent). Somewhere hmm. Its a bit difficult to look, but maybe somewhere there (points the graph approximately at a point 0,8).”

Also Niina failed to determine this point and proposed the same point as Susanna. It seems that the negative rate of change as a more abstract concept causes difficulties. Niina and Susanna seemed to consider the rate of change more as a process of change than as an object of one point. Niina was the only student who demonstrated the representation of the local straightness when arguing why the derivative is not zero at the point where there is an angle in the graph (Fig. 1):

> “If you would zoom in on here (x = 0,8) for example, it would be straight for a while (draws a line with a finger), but not there (points at the angle, x = 2)”

However, all the interviewed students determined correctly from the graph of the function (Fig. 1) the points where the derivative is zero, the sign of the derivative, the points where the derivative is at the greatest and the interval where the derivative is constant. Tommi, Daniel and Samuel determined correctly also the point where the derivative is not defined and the point where it is at the smallest. In addition, Tommi and Daniel made observations correctly about the rate of change of the derivative (that is the sign of the second derivative which was not taught). Daniel made his observations of the derivative from right to left:

> “When we start to go forward down here (from x = 2 to x = 0,8), it (the derivative) is all the time actually increasing, since it’s steepest there (x = 2). No, it decreases, because it’s positive there. It goes here (x = 0,8), down here it’s zero. Then here it becomes negative, we are going upward. It starts to increase again somewhere in the middle (x = -0,4) and there (x = 2,5) it’s zero again.”

All the students except Susanna sketched almost correctly the graph of the distance from the graph of the velocity. Incorrectly Niina’s graph was formed from line segments and Daniel’s graph was steepest at a wrong point. Tommi, Samuel and Daniel sketched also the graph of the acceleration correctly.

The symbolic representations of the derivative

All the students except Samuel had interiorized the process of determining the slope of the tangent to a symbolic process and this was connected to the tangent as a perceived object. For example Niina explained that the value three of the derivative means the slope of the tangent which is calculated so that "the change in y is divided by the change in x, and you'll get three". On the other hand, Samuel had only a verbal representation for the slope of the tangent, since he could determine the slope only by using the derivative. Apparently it would be easy to guide Samuel to acquire the missing process, since he already could determine the slope by the limit of the difference quotient, connect difference quotient to the secant and could determine the average acceleration from the graph of the velocity.

Only Tommi and Samuel had interiorized the limit of the difference quotient as a process and were able to describe this process without performing it. When trying to estimate the derivative of the function 2^x at the point $x = 1$ Samuel first tried to determine the limit of the difference quotient, but did not figure out how to simplify the quotient. Since he could not determine the exact value, he estimated it by calculating the difference quotients over the intervals [0,9; 1], [0,99; 1] and [0,999; 1]. Samuel has connected this symbolic process to the perceptual process of secants approaching the tangent:

Interviewer: What do these (difference quotients) tell about the function? If this is the derivative and these aren't quite the derivative, what do they mean?

Samuel: (Draws a graph and a tangent.) It would really be that. (Draws the secants approaching the tangent.) They approach constantly the correct derivative.

Interviewer: Ok. Ok. Do you have more to say about that?

Samuel: No, or well, that this is because you can't substitute one here, because it would be zero here, but you can put it however close to... mm close to one, but not however one, then there will be no zero and you can calculate this and this is why it approaches.

The other students were only at the action level in determining the limit of the difference quotient and they only tried to remember the formula. However, Daniel was able to interpret the form (new to him) of the difference quotient and the limit of it from the graph of the unknown function by using his representations of the slope of the line and the "average derivative": "It would be the slope of that line (secant), that is average, how to say it, average derivative at that interval." Also Niina and Susanna tried to interpret, but did not proceed very well. In order to really understand the limit of the difference quotient, one should also have some other representations than only the symbolic representation (like Samuel and Daniel have). Tommi instead had interiorized the limit of the difference quotient to a process, but he had not connected it to the perceptual representations. When trying to estimate the derivative of the function 2^x at the point $x = 1$, he had the following representation of the limiting process:

"You could calculate the average rate of change of the function for example at points 1,1 and 0,9 and continue to approach 1. Finally it would become very close to that correct one. – – I don't remember at all how it's calculated."

Tommi could make the missing connection by combining the perceptual representations of the average rate of change and the slope of the secant to the symbolic representations of them and to the difference quotient. He almost did this when trying to interpret the form of the difference quotient (new to him):

"Could it then be the rate of change over that interval. (Draws a secant.) No, it is like the average rate of change over that interval."

Daniel should instead practice to determine the limit of the difference quotient, so that this representation would become stronger and interiorize to a process. Niina and Susanna should both connect the limit of the difference quotient to the other representations and try to interiorize it to a process. Susanna had a perceptual representation of the secants approaching the tangent and Niina the local straightness of the graph which could be connected to the limit of the difference quotient.

Susanna and Daniel were the only students to whom the differentiation rule of the function x^n had been taught. They had a very strong representation of it and used it as the first method to determine the derivative even when it was not appropriate. Susanna was easily guided to use her perceptual representations to notice that she had not applied the differentiation rule correctly to the function 2^x:

Susanna: Actually the derivative of 2 would be zero. Would this be then zero?

Interviewer: How could you figure this out?

(Susanna draws under the guidance the graph on paper and also with a calculator.)

Susanna: Actually it can't be zero, since it's after all increasing at point one.

It seems that students like Niina and Susanna could base their learning of the derivative to the perceptual representations if they do not succeed in connecting the limit of the difference quotient to the perceptual representations.

CONCLUSIONS

All the interviewed students seemed to be able to begin their concept acquisition of the derivative by developing different perceptual representations. By using these perceptual representations they could understand the derivative as an object. At the beginning of the course they examined the derivative qualitatively at the level which is the goal in terms of differentiation rules and sign considerations at the end of the course. Especially well the perceptual representations seem to suit for beginning to develop understanding of the relation between function and its derivative function.

Students may have very different representations and as Kendal and Stacey (2000) stated, representations which are emphasized in the teaching influence on the construction of students' internal representations. Usually the most important perceptual representations are considered to be the slope of the tangent and the rate of change, but one should consider also emphasizing some other representations. For example, steepness of the graph could be used to examine the derivative as an object and local straightness to demonstrate the limit of the difference quotient.

Perceptual representation of the tangent was usually connected to the symbolic process of determining the slope of the tangent. Instead, it was difficult to understand the limit of the difference quotient by using other representations though all the students had some kind of perceptual representation which could be used for this. Apparently students would need some individual guidance in this acquisition process. As Gray and Tall (2001) underline it's very important to connect perceptual representations to symbolic representations. Because it seems to be easy to deal with the derivative with perceptual representations, one should consider in how these could be used along the course for example to intuitively derive differentiation rules.

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