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ON LOCALLY LAGRANGIAN SYMPLECTIC STRUCTURES

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Abstract. Some results on global symplectic forms defined by local Lagrangians of a tangent manifold, studied earlier by the author, are summarized without proofs.

This is a summary of some of our results on locally Lagrange symplectic and Poisson manifolds [3, 4].

The symplectic forms used in Lagrangian dynamics are defined on tangent bundles TN, and they are of the type

$$\omega_{\mathcal{L}} = \sum_{i,j=1}^{n} \left(\frac{\partial^2 \mathcal{L}}{\partial x^i \partial \xi^j} \, \mathrm{d}x^i \wedge \mathrm{d}x^j + \frac{\partial^2 \mathcal{L}}{\partial \xi^i \partial \xi^j} \, \mathrm{d}\xi^i \wedge \mathrm{d}x^j \right) \tag{1}$$

where $(x^i)_{i=1}^n$ $(n = \dim N)$ are local coordinates on N, (ξ^i) are the corresponding natural coordinates on the fibers of TN, and $\mathcal{L} \in C^\infty(TN)$ is a non degenerate Lagrangian.

An **almost tangent structure** on a differentiable manifold M^{2n} is a tensor field $S \in \Gamma \operatorname{End}(TM)$ (necessarily of rank n) such that

$$S^2 = 0, \qquad \operatorname{Im} S = \operatorname{Ker} S \,. \tag{2}$$

If the Nijenhuis tensor vanishes, i. e. $\forall X, Y \in \Gamma TM$,

$$\mathcal{N}_S(X,Y) = [SX, SY] - S[SX, Y] - S[X, SY] + S^2[X, Y] = 0, \tag{3}$$

S is a **tangent structure**. Then, $V = \operatorname{Im} S$, is an integrable subbundle, and we call its tangent foliation the **vertical foliation** $\mathcal{V}$. Furthermore, M has local coordinates $(x^i, \xi^i)_{i=1}^n$ such that

$$S\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial \xi^i}, \qquad S\left(\frac{\partial}{\partial \xi^i}\right) = 0\,. \tag{4}$$

A manifold M endowed with a tangent structure is called a **tangent manifold**. A **locally Lagrangian symplectic (l.L.s.) structure** on a tangent manifold (M, S) is a symplectic form ω which is locally of the form (1) with respect to local Lagrangians $\mathcal{L}_\alpha \in C^\infty(U_\alpha)$, where $M = \cup_\alpha U_\alpha$ is an open covering of the manifold M. A tangent manifold (M, S) endowed with a l.L.s. structure ω is called a l.L.s. manifold.

Theorem 1. *Let (M, S) be a tangent manifold and ω a symplectic form on M. Then ω is locally Lagrangian with respect to S iff ω and S are compatible in the sense that*

$$\omega(X, SY) = \omega(Y, SX)\,, \qquad \forall X, Y \in \Gamma TM\,. \tag{5}$$

In particular, the compatibility condition implies that the vertical foliation $\mathcal{V}$ of S is a Lagrangian foliation for ω.

Put

$$\Theta([X]_V, [Y]_V) = \omega(SX, Y) \tag{6}$$

where the arguments are cross sections of the transversal bundle $\nu\mathcal{V} = TM/V$ of the foliation $\mathcal{V}$. Θ is a well defined pseudo-Euclidean metric with the local components $(\partial^2 \mathcal{L}_\alpha / \partial\xi^i \partial\xi^j)$. If this metric is positive definite, we say that the manifold (M, S, ω) is of the **elliptic type**.

Theorem 2. *Let (M, ω) be a symplectic manifold endowed with a Lagrangian foliation $\mathcal{V}$ ($T\mathcal{V} = V$), and a $\mathcal{V}$-projectable pseudo-Euclidean metric Θ on $\nu\mathcal{V} = TM/V$. Then, there exists a unique ω-compatible tangent structure S on M for which Θ is the metric (6).*

Examples of l.L.s. manifolds include tori, compact quotients of products of generalized Heisenberg groups, Iwasawa manifolds, all tangent bundles of symplectic manifolds etc.

Theorem 3. *Let (M, S, ω) be a l.L.s. manifold. Then ω is also given by the expression (1) with a global Lagrangian $\mathcal{L} \in C^\infty(M)$ iff $\omega = \mathrm{d}\epsilon$ for some global 1-form ϵ on M such that:*

i) *ϵ vanishes on the vertical leaves of S;*

ii) *if η is the cross section of V^* which satisfies $\eta \circ S = \epsilon$, then $\eta = \mathrm{d}_\mathcal{V}\mathcal{L}$, where $\mathrm{d}_\mathcal{V}$ is the differential along the leaves of $\mathcal{V}$ and $\mathcal{L} \in C^\infty(M)$.*

It is possible to describe all the l.L.s. forms on a tangent bundle TN with its canonical tangent structure defined by formula (4), where the local coordinates are those of (1). In particular, one has

Theorem 4. *The symplectic form ω on (TN, S) is l.L.s. iff: (i) the foliation of TN by fibers is Lagrangian with respect to ω; (ii) there exist global ω-Hamiltonian vector fields locally defined by systems of autonomous second order differential equations on N.*

Another result that is worth mentioning is the following symplectic reduction theorem

Theorem 5. *Let N be a coisotropic submanifold of the l.L.s. manifold (M, S, ω), with the kernel foliation $C = (TN)^{\perp_\omega}$. Suppose that the following conditions hold:*

i) *the leaves of C are the fibers of a submersion $\sigma\colon N \to Q$;*
ii) $S(TN) \subseteq TN, V \cap TN \subseteq S(TN) + C\,,\ V = \operatorname{Im} S$;
iii) *the restriction of S to TN sends C-projectable vector fields to C-projectable vector fields. Then S projects to a tangent structure S' of Q such that (Q, S', ω'), where ω' is the symplectic reduction of ω, is a l.L.s. manifold.*

In a different direction, in [4], we computed representative differential forms of the Maslov classes of Lagrangian submanifolds of elliptic l.L.s. manifolds (M, S, ω), with respect to the vertical Lagrangian foliation, by using the general method of [1].

The following definition provides a generalization of the notion of a l.L.s. structure to Poisson geometry [2].

Definition 1. *A **locally Lagrangian Poisson (l.L.P.) structure** on a differentiable manifold M is a pair (P, S) where P is a Poisson bivector field on M, and $S \in \Gamma\operatorname{End} TM$ and satisfies the properties:*

$$P(\alpha, \beta \circ S) = P(\beta, \alpha \circ S) \tag{7}$$

$$P(\alpha \circ S, \beta \circ S) = 0 \tag{8}$$

$$\operatorname{rank}_x S/_{\operatorname{Im}\sharp_P} = \frac{1}{2}\operatorname{rank}_x P \tag{9}$$

$$\mathcal{N}_S(X, Y) = 0\,, \qquad \forall X, Y \in \Gamma(\operatorname{Im}\sharp_P) \tag{10}$$

where $\alpha, \beta \in \Gamma T^*M$, $x \in M$, $\sharp_P\colon T^*M \to TM$ is defined by $\langle \sharp_P\alpha, \beta\rangle = P(\alpha, \beta)$, and $\mathcal{N}_S$ is the Nijenhuis tensor (3).

From this definition we get

Proposition 1. *The symplectic leaves of a l.L.P. manifold are locally Lagrangian symplectic manifolds.*

If we start with a Poisson structure w on the manifold N, the complete lift w^C of w to TN, together with the canonical tangent structure S of TN, is a l.L.P. structure on TN. The complete lift is the lift of multivector fields from N to TN, which is induced by the lift of the flow of the tangent vector fields of N. If w is non degenerate this construction yields the example of a l.L.s. structure of the tangent bundle of a symplectic manifold that we have mentioned earlier.

References

[1] Vaisman I., *Symplectic Geometry and Secondary Characteristic Classes*, Progress in Math. Series 72, Birkhäuser, Boston 1987.

[2] Vaisman I., *Lectures on the Geometry of Poisson Manifolds*, Progress in Math. Series 118, Birkhäuser, Basel 1994.

[3] Vaisman I., *Locally Lagrange-symplectic Manifolds*, Geom. Dedicata **74** (1999) 79–89.

[4] Vaisman I., *Locally Lagrangian Symplectic and Poisson Manifolds*, Rendiconti Sem. Mat. Torino (to appear) and arXivmath.SG/0008097.