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EXAMPLES OF PSEUDO-RIEMANNIAN G.O. MANIFOLDS

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Abstract. We modify the metrics on six-dimensional and seven-dimensional Riemannian g.o. manifolds constructed in previous published papers and we obtain pseudo-Riemannian g.o. manifolds. We describe geodesic graphs of corresponding g.o. spaces. We show that if these geodesic graphs are nonlinear, they are discontinuous on an nonempty set but they are continuous at the origin.

1. Introduction

Let M be a pseudo-Riemannian manifold. If there is a connected Lie group $G \subset I_0(M)$ which acts transitively on M as a group of isometries, then M is called a **homogeneous pseudo-Riemannian manifold**. Let $p \in M$ be a fixed point. If we denote by H the isotropy group at p, then M can be identified with the *homogeneous space* G/H. In general, there may exist more than one such group $G \subset I_0(M)$. For any fixed choice $M = G/H$, G acts effectively on G/H on the left. The pseudo-Riemannian metric g on M can be considered as a G-invariant metric on G/H. The pair $(G/H, g)$ is then called a **pseudo-Riemannian homogeneous space**.

If the metric g is a positive definite, then $(G/H, g)$ is always a *reductive homogeneous space*: We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively and consider the adjoint representation $\mathrm{Ad} : H \times \mathfrak{g} \to \mathfrak{g}$ of H on $\mathfrak{g}$. There exists a direct sum decomposition (*reductive decomposition*) of the form $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ where $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$. If the metric g is indefinite, the reductive decomposition may not exist (see [6] for an example of nonreductive pseudo-Riemannian homogeneous space). For a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ there is a natural identification of $\mathfrak{m} \subset \mathfrak{g} = T_eG$ with the tangent space T_pM via the projection $\pi : G \to G/H = M$. Using this natural identification and the scalar product g_p on T_pM we obtain a scalar product $\langle\, , \rangle$ on $\mathfrak{m}$. This scalar product is obviously $\mathrm{Ad}(H)$-invariant.

The definition of a homogeneous geodesic is well-known in the Riemannian case (see e.g., [11]). In the pseudo-Riemannian case, the necessary generalized version was given in [4]

Definition 1. *Let $M = G/H$ be a homogeneous pseudo-Riemannian space, $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ a reductive decomposition and p the basic point of G/H. The geodesic $\gamma(s)$ through the point p defined in an open interval J (where s is an affine parameter) is said to be homogeneous if there exists*

1) *a diffeomorphism $s = \varphi(t)$ between the real line and the open interval J*
2) *a vector $X \in \mathfrak{g}$ such that $\gamma(\varphi(t)) = \exp(tX)(p)$ for all $t \in (-\infty, +\infty)$.*

The vector X is then called a ***geodesic vector****.*

The basic formula characterizing geodesic vectors in the pseudo-Riemannian case appeared in [6] and [13], but without a proof. The correct mathematical formulation with the proof was given in [4]

Lemma 1. *Let $M = G/H$ be a homogeneous pseudo-Riemannian space, $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ a reductive decomposition and p the basic point of G/H. Let $X \in \mathfrak{g}$. Then the curve $\gamma(t) = \exp(tX)(p)$ (the orbit of a one-parameter group of isometries) is a geodesic curve with respect to some parameter s if and only if*

$$\langle [X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = k \langle X_{\mathfrak{m}}, Z \rangle \tag{1}$$

for all $Z \in \mathfrak{m}$, where $k \in \mathbb{R}$ is some constant. Further, if $k = 0$, then t is an affine parameter for this geodesic. If $k \neq 0$, then $s = \mathrm{e}^{-kt}$ is an affine parameter for the geodesic. The second case can occur only if the curve $\gamma(t)$ is a light-like curve in a (properly) pseudo-Riemannian space.

For the results on homogeneous geodesics on homogeneous Riemannian manifolds we refer for example to [10, 12]. Further references can be found also in [4]. First results for pseudo-Riemannian manifolds were obtained in [1, 4].

Definition 2. *A pseudo-Riemannian homogeneous space $(G/H, g)$ is called a g.o. space (or pseudo-Riemannian manifold (M, g) is called a g.o. manifold, respectively) if every geodesic of $(G/H, g)$ (or of (M, g)) is homogeneous. Here "g.o." means "geodesics are orbits".*

Our technique used for the characterization of g.o. spaces and g.o. manifolds is based on the concept of "geodesic graph". The original idea (not using any explicit name) comes from J. Szenthe [14].

Definition 3. *Let* $(G/H, g)$ *be a g.o. space and* $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ *an* $\mathrm{Ad}(H)$*-invariant decomposition of the Lie algebra* $\mathfrak{g}$*. A geodesic graph is an* $\mathrm{Ad}(H)$*-equivariant map* $\eta : \mathfrak{m} \to \mathfrak{h}$ *which is rational on an open dense subset* U *of* $\mathfrak{m}$ *and such that* $X + \eta(X)$ *is a geodesic vector for each* $X \in \mathfrak{m}$.

On every g.o. space $(G/H, g)$, there exists at least one geodesic graph. The construction of a *canonical geodesic graph* and *general geodesic graphs* (on open dense subsets) through rational maps is described in details in [3, 9]. For the vectors $X \in \mathfrak{m} \setminus \{U\}$, the map η must be constructed part by part using again some rational maps and the geodesic graph may be discontinuous on some subset $V \subset \mathfrak{m} \setminus \{U\}$.

On the subset U, the components η_i of a geodesic graph are always rational functions in the form $\eta_i = P_i/P$, where P_i and P are homogeneous polynomials (of the coordinates on $T_p(M)$) and $\deg(P_j) = \deg(P) + 1$. The *degree of a geodesic graph* is defined as the degree of the denominator P in the situation when P_i and P are relatively prime.

Definition 4. *If* (M, g) *is a g.o. manifold then the degree of* (M, g) *is the minimum of degrees of all geodesic graphs (either canonical or general) constructed for all possible g.o. spaces* $(G/H, g)$ *where* $G \subset I_0(M)$ *and* $M = G/H$.

According to the results of Szenthe, the degree of (M, g) is zero if and only if (M, g) can be made a naturally reductive space $(G/H, g)$ for a suitable choice $G \subset I_0(M)$.

For the examples of geodesic graphs of various degrees we refer to [2, 3, 5, 9]. The systematic description of temporary results about Riemannian g.o. manifolds was given in [5].

In this paper we are going to consider the six and sevendimensional manifolds which were described with Riemannian metrics in [5, 7, 9] and geodesic graphs of these Riemannian g.o. manifolds that were described in [5, 9]. We now modify these metrics and obtain invariant pseudo-Riemannian metrics. We show that the corresponding pseudo-Riemannian manifolds are g.o. manifolds and we describe corresponding geodesic graphs.

We remark already here that we will use Lemma 1 with $k = 0$. This is the consequence of the fact that our metrics are the simplest modifications of Riemannian g.o. metrics where the compact isotropy group remains unchanged.

2. Examples

2.1. Five Dimensional Examples of Type $\mathbf{U(3)/U(2)}$

Let us consider the five-dimensional vector space $\mathfrak{m}$ with the pseudo-orthonormal basis $\{E_1, \ldots, E_4, Z_1\}$ with the signature $(1, 1, 1, 1, \varepsilon)$, where $\varepsilon = \pm 1$. We denote by A_{ij} (for $1 \leq i < j \leq 4$) the elements of $\mathfrak{so}(\mathfrak{m})$, with the corresponding action given by the formulas

$$A_{ij}(E_k) = \delta_{ik}E_j - \delta_{jk}E_i \quad \text{for} \quad k = 1, \ldots, 4. \tag{2}$$

Further, we denote

$$\tilde{A} = A_{34} + A_{12}, \quad A = A_{34} - A_{12}, \quad B = A_{13} + A_{24}, \quad C = A_{14} - A_{23}. \tag{3}$$

We notice the Lie bracket relations

$$[A, B] = 2C, \qquad [B, C] = 2A, \qquad [C, A] = 2B \tag{4}$$

and the isomorphism $\mathrm{span}(A, B, C) \simeq \mathfrak{su}(2)$. We consider the algebra $\mathfrak{h} = \mathrm{span}(\tilde{A}, A, B, C) \simeq \mathfrak{u}(2)$ of the operators on $\mathfrak{m}$. Put $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ and define the Lie bracket on $\mathfrak{g}$ by the additional relations

$$\begin{aligned} &[E_1, E_2] = \alpha Z_1 + \lambda A, && [E_2, E_3] = \lambda C \\ &[E_1, E_3] = -\lambda B, && [E_2, E_4] = -\lambda B \\ &[E_1, E_4] = -\lambda C, && [E_3, E_4] = \alpha Z_1 - \lambda A \\ &[Z_1, E_1] = pE_2, && [Z_1, E_3] = pE_4 \\ &[Z_1, E_2] = -pE_1, && [Z_1, E_4] = -pE_3 \end{aligned} \tag{5}$$

for the parameters p, α, λ, such that $p\alpha + 3\lambda = 0$.

The algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{u}(3)$ for $\alpha > 0$ and to $\mathfrak{u}(1, 2)$ for $\alpha < 0$. The scalar product on $\mathfrak{m}$ is $\mathrm{ad}(H)$-invariant and hence the relations above define a two-parameter family of pseudo-Riemannian metrics on $\mathrm{U}(3)/\mathrm{U}(2)$, or $\mathrm{U}(1, 2)/\mathrm{U}(2)$, respectively.

We are going to compute a geodesic graph. For that purpose we shall apply Lemma 1 to the given decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ and will present each vector $X \in \mathfrak{m}$ in the form $X = x_1E_1 + \cdots + x_4E_4 + z_1Z_1$ and each vector $\xi(X) \in \mathfrak{h}$ in the form $\xi(X) = \xi_1\tilde{A} + \xi_2A + \xi_3B + \xi_4C$. In this way, we identify these vectors with the arithmetic vectors $X = (x_1, \ldots, x_4, z_1)$ and $\xi(X) = (\xi_1, \ldots, \xi_4)$ of their components with respect to the basis $\{E_1, \ldots, E_4, Z_1\}$ of $\mathfrak{m}$ and the basis $\{A, B, C, D\}$ of $\mathfrak{h}$. Let us now consider the equation

$$\langle [X + \xi(X), Y]_{\mathfrak{m}}, X\rangle = 0 \tag{6}$$

where Y runs over all $\mathfrak{m}$. This gives the condition for $X + \xi(X)$ to be a geodesic vector. We have to determine the corresponding $\xi(X)$ to the given X. Here for $Y \in \mathfrak{m}$ we substitute, step by step, all five elements $E_1, \ldots, E_4, Z_1$ of the given pseudo-orthonormal basis into the formula (6). In this way we obtain a system of five linear equations for the parameters $\xi_1, \ldots, \xi_4$ (satisfying the Frobenius' criterion of compatibility). Now, for a *generic* vector X, the rank of this system is three. We select, in a convenient way, a subsystem of three linearly independent equations. The matrix $\mathbf{A}$ of the coefficients of the corresponding homogeneous system and the vector of the right-hand sides are given by

$$\mathbf{A} = \begin{pmatrix} x_2 & -x_2 & x_3 & x_4 \\ -x_1 & x_1 & x_4 & -x_3 \\ x_4 & x_4 & -x_1 & x_2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} (\varepsilon\alpha - p)x_2 z_1 \\ -(\varepsilon\alpha - p)x_1 z_1 \\ (\varepsilon\alpha - p)x_4 z_1 \end{pmatrix}. \tag{7}$$

Now we choose an invariant scalar product on $\mathfrak{h}$ and add the condition $\xi(X) \perp \mathfrak{q}_X$. Here $\mathfrak{q}_X$ is the subalgebra of $\mathfrak{h}$ defined by the condition

$$\mathfrak{q}_X = \{A \in \mathfrak{h};\ [A, X] = 0\}. \tag{8}$$

In our case $\dim \mathfrak{q}_X = 1$ for a *generic* vector X. If we denote by $Q_X = q_1\tilde{A} + q_2 A + q_3 B + q_4 C$ a generator of the algebra $\mathfrak{q}_X$, then the components q_i form a solution of the homogeneous system of equations whose matrix is equal to $\mathbf{A}$ (see [3] or [9] for the details about the construction of a canonical geodesic graph). By the Cramer's rule we obtain the following particular solution:

$$\begin{aligned} q_1 &= x_1^2 + x_2^2 + x_3^2 + x_4^2 \\ q_2 &= x_1^2 + x_2^2 - x_3^2 - x_4^2 \\ q_3 &= 2(x_1x_4 - x_2x_3) \\ q_4 &= -2(x_1x_3 + x_2x_4). \end{aligned} \tag{9}$$

Now, the condition $\xi(X) \perp \mathfrak{q}_X$ can be described by the equation

$$\sum_{j=1}^{4} q_j \cdot \xi_j = 0. \tag{10}$$

The system of equations described by the matrix $\mathbf{A}$ and the vector $\mathbf{b}$ in (7) and the equation (10) give the system of four equations for four variables. In the generic case, we obtain also the components of the unique solution $\xi(X)$ in the form

$$\begin{aligned} \xi_1 &= -\frac{(p - \alpha\varepsilon)z_1}{2} \\ \xi_2 &= \frac{(p - \alpha\varepsilon)(x_1^2 + x_2^2 - x_3^2 - x_4^2)z_1}{2\|x\|^2} \\ \xi_3 &= \frac{(p - \alpha\varepsilon)(x_1x_4 - x_2x_3)z_1}{\|x\|^2} \\ \xi_4 &= -\frac{(p - \alpha\varepsilon)(x_1x_3 + x_2x_4)z_1}{\|x\|^2}. \end{aligned} \tag{11}$$

Here we denote $\|x\|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$. The formulas (11) make sense on the subset $U \subset \mathfrak{m}$, where $U = \{X \in \mathfrak{m};\ \|x\|^2 \neq 0\}$. On $\mathfrak{m} \setminus \{U\}$, we can put $\xi(X) = 0$ and we obtain a canonical geodesic graph.

This geodesic graph cannot be continuous on the subset $V \subset \mathfrak{m} \setminus \{U\}$, where $V = \{(0,0,0,0,z_1) \in \mathfrak{m};\ z_1 \neq 0\}$. Indeed, considering the curves $\gamma_1 = (t,0,0,0,z_1)$ and $\gamma_2 = (0,0,t,0,z_1)$ in $\mathfrak{m}$, we easily see that the limit of ξ_2 (for t going to 0) along γ_1 is $(p-\alpha\varepsilon)z_1/2$ and along γ_2 it is $-(p-\alpha\varepsilon)z_1/2$. Hence, on V, the geodesic graph cannot be defined continuously (unless $p = \alpha\varepsilon$, where $\xi = 0$ and the space is naturally reductive). It is not hard to see that ξ is continuous at the origin (see also the general Lemma 2 in Section 3).

Further, it is easily seen that if we put

$$\eta(X) = \xi(X) - \frac{(p-\alpha\varepsilon)z_1}{2\|x\|^2} \cdot Q_X \tag{12}$$

we obtain

$$\eta_1 = -(p-\alpha\varepsilon)z_1, \qquad \eta_2 = \eta_3 = \eta_4 = 0 \tag{13}$$

which is the linear geodesic graph. Clearly, it is defined on all $\mathfrak{m}$ and it is continuous. It implies the natural reductivity of the space G/H (for any p, α, ε).

2.2. Six Dimensional Nilpotent Examples

Let us consider the sixdimensional vector space $\mathfrak{m}$ with the pseudo-orthonormal basis $\{E_1,\dots,E_4,Z_1,Z_2\}$ with the signature $(1,1,1,1,\varepsilon_1,\varepsilon_2)$, where $\varepsilon_1,\varepsilon_2 = \pm 1$. Further, let $\mathfrak{h} = \operatorname{span}(A,B,C)$. Put $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ and define the Lie bracket by the relations

$$\begin{aligned}
&[E_1,E_2] = 0, && [E_2,E_3] = \beta Z_1 + \gamma Z_2\\
&[E_1,E_3] = \alpha Z_1, && [E_2,E_4] = -\alpha Z_1\\
&[E_1,E_4] = \beta Z_1 + \gamma Z_2, && [E_3,E_4] = 0\\
&[Z_1,E_i] = [Z_2,E_i] = [Z_1,Z_2] = 0 \quad \text{for} \quad i = 1,\dots,4
\end{aligned} \tag{14}$$

for arbitrary parameters α, β, γ ($\alpha \neq 0 \neq \gamma$). The scalar product on $\mathfrak{m}$ is $\operatorname{ad}(H)$-invariant for all possibilities of $\varepsilon_1,\varepsilon_2 = \pm 1$. If we consider N as the unique connected and simply connected group whose Lie algebra is $\mathfrak{m}$, $H = \mathrm{SU}(2)$ and $G = N \rtimes \mathrm{SU}(2)$, we obtain a three-parameter family of invariant pseudo-Riemannian metrics on the manifold G/H.

From the equation (6) we obtain a system of equations whose matrix $\mathbf{A}$ and the right-hand side vector $\mathbf{b}$ are

$$\mathbf{A} = \begin{pmatrix} -x_2 & x_3 & x_4\\ x_1 & x_4 & -x_3\\ x_4 & -x_1 & x_2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} (\alpha x_3 + \beta x_4)\varepsilon_1 z_1 + \gamma x_4 \varepsilon_2 z_2\\ (\beta x_3 - \alpha x_4)\varepsilon_1 z_1 + \gamma x_3 \varepsilon_2 z_2\\ -(\alpha x_1 + \beta x_2)\varepsilon_1 z_1 - \gamma x_2 \varepsilon_2 z_2 \end{pmatrix}. \tag{15}$$

By the Cramer's rule we obtain the components of the vector $\xi(X)$ in the form

$$\begin{aligned}
\xi_1 &= \frac{-2\alpha\varepsilon_1 z_1(x_1x_4 + x_2x_3) + 2(\beta\varepsilon_1 z_1 + \gamma\varepsilon_2 z_2)(x_1x_3 - x_2x_4)}{\|x\|^2} \\
\xi_2 &= \frac{\alpha\varepsilon_1 z_1(x_1^2 - x_2^2 + x_3^2 - x_4^2) + 2(\beta\varepsilon_1 z_1 + \gamma\varepsilon_2 z_2)(x_1x_2 + x_3x_4)}{\|x\|^2} \qquad (16) \\
\xi_3 &= \frac{2\alpha\varepsilon_1 z_1(x_3x_4 - x_1x_2) + (\beta\varepsilon_1 z_1 + \gamma\varepsilon_2 z_2)(x_1^2 - x_2^2 - x_3^2 + x_4^2)}{\|x\|^2}.
\end{aligned}$$

The formulas (16) make sense again on the subset $U = \{X \in \mathfrak{m};\ \|x\|^2 \neq 0\}$. For $\|x\|^2 = 0$ we put $\xi(X) = 0$. At the origin, the geodesic graph ξ is continuous (see Lemma 2). On the subset $V = \{(0,0,0,0,z_1,z_2) \in \mathfrak{m};\ z_1^2 + z_2^2 \neq 0\}$ it cannot be continuous: For $z_1 \neq 0$ we consider the component ξ_2 and the curves $\gamma_1 = (t,0,0,0,z_1,z_2)$, $\gamma_2 = (0,t,0,0,z_1,z_2)$. Then we see that the limits of ξ_2 along these curves for $t \to 0$ are different, as in the previous example. For $z_1 = 0$ and $z_2 \neq 0$ we consider the component ξ_3 and the same curves.

Finally, we observe that G/H is a pseudo-Riemannian g.o. space of degree two (for any α, β, γ, ε_1, ε_2, ε_3).

2.3. Six Dimensional Examples of Type $\mathrm{SO}(5)/\mathrm{U}(2)$

Let $\{E_1,\dots,E_4,Z_1,Z_2\}$ with the signature $(1,1,1,1,\varepsilon,\varepsilon)$ is the pseudo-orthonormal basis of the vector space $\mathfrak{m}$. We denote by B_{12} the operator from $\mathfrak{so}(\mathfrak{m})$, with the action given by the formula

$$B_{12}(Z_1) = Z_2, \quad B_{12}(Z_2) = -Z_1, \quad B_{12}(E_i) = 0 \quad \text{for} \quad i = 1,\dots,4 \qquad (17)$$

and we put $D = A_{12} + A_{34} + 2B_{12}$. The isotropy algebra is $\mathfrak{h} = \mathrm{span}(A,B,C,D) \simeq \mathfrak{u}(2)$. We define the Lie bracket on $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ by the relations

$$\begin{aligned}
&[E_1,E_2] = \lambda(D - A), \quad [E_3,E_4] = \lambda(D + A), \quad [Z_1,Z_2] = 2(\lambda^2/\rho^2)D \\
&[E_1,E_3] = \rho Z_1 + \lambda B, \qquad [E_2,E_4] = -\rho Z_1 + \lambda B \\
&[E_1,E_4] = \rho Z_2 + \lambda C, \qquad [E_2,E_3] = \rho Z_2 - \lambda C \qquad (18) \\
&[E_1,Z_1] = [E_2,Z_2] = -(\lambda/\rho)E_3, \quad [E_1,Z_2] = -[E_2,Z_1] = -(\lambda/\rho)E_4 \\
&[E_3,Z_1] = [E_4,Z_2] = (\lambda/\rho)E_1, \quad [E_3,Z_2] = -[E_4,Z_1] = (\lambda/\rho)E_2
\end{aligned}$$

for the parameters $\rho > 0$ and $\lambda \neq 0$ (for $\lambda = 0$ we obtain the example from the previous section). For $\lambda > 0$ the algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(5)$ and for $\lambda < 0$ it is isomorphic to $\mathfrak{so}(1,4)$. Hence we have two-parameter family of metrics on $\mathrm{SO}(5)/\mathrm{U}(2)$ and on $\mathrm{SO}(4,1)/\mathrm{U}(2)$.

From the equation (6) we obtain a system of equations whose matrix $\mathbf{A}$ and the right-hand side vector $\mathbf{b}$ are

$$\mathbf{A} = \begin{pmatrix} -x_2 & x_3 & x_4 & x_2 \\ x_1 & x_4 & -x_3 & -x_1 \\ x_4 & -x_1 & x_2 & x_4 \\ 0 & 0 & 0 & 2z_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} (\rho\varepsilon - \frac{\lambda}{\rho})(x_3 z_1 + x_4 z_2) \\ (\rho\varepsilon - \frac{\lambda}{\rho})(-x_4 z_1 + x_3 z_2) \\ -(\rho\varepsilon - \frac{\lambda}{\rho})(x_1 z_1 + x_2 z_2) \\ 0 \end{pmatrix}. \tag{19}$$

By the Cramer's rule we obtain the components $\xi(X)$ of the unique geodesic graph on $U = \{X \in \mathfrak{m};\ \|x\|^2 \neq 0\}$ in the form

$$\begin{aligned} \xi_1 &= \left(\varepsilon\rho - \frac{\lambda}{\rho}\right) \frac{-2z_1(x_1x_4 + x_2x_3) + 2z_2(x_1x_3 - x_2x_4)}{\|x\|^2} \\ \xi_2 &= \left(\varepsilon\rho - \frac{\lambda}{\rho}\right) \frac{z_1(x_1^2 - x_2^2 + x_3^2 - x_4^2) + 2z_2(x_1x_2 + x_3x_4)}{\|x\|^2} \\ \xi_3 &= \left(\varepsilon\rho - \frac{\lambda}{\rho}\right) \frac{2z_1(-x_1x_2 + x_3x_4) + z_2(x_1^2 - x_2^2 - x_3^2 + x_4^2)}{\|x\|^2} \\ \xi_4 &= 0. \end{aligned} \tag{20}$$

For $\|x\|^2 = 0$ we put again $\xi(X) = 0$. Geodesic graph is continuous on U and at the origin. As in the previous example, we see that it cannot be continuous on $V = \{(0,0,0,0,z_1,z_2) \in \mathfrak{m};\ z_1^2 + z_2^2 \neq 0\}$. Geodesic graph is linear (and it implies the natural reductivity of G/H) if $\rho^2 = \varepsilon\lambda$. Otherwise, G/H is a pseudo-Riemannian g.o. space of degree two.

2.4. Seven Dimensional Nilpotent Examples

This manifold was constructed by Gordon in [7]. Let us consider the pseudo-orthogonal basis $\{E_1, \dots, E_4, Z_1, \dots, Z_3\}$, where $E_1, \dots, E_4$ are unit and space-like and

$$\langle Z_1, Z_1\rangle = \varepsilon_1 k^2, \qquad \langle Z_2, Z_2\rangle = \varepsilon_2 l^2, \qquad \langle Z_3, Z_3\rangle = \varepsilon_3 m^2. \tag{21}$$

The Lie bracket satisfy

$$\begin{gathered} [E_1,E_2] = Z_1/k^2, \qquad [E_2,E_3] = Z_3/m^2, \qquad [E_1,E_3] = Z_2/l^2 \\ [E_2,E_4] = Z_2/l^2, \qquad [E_1,E_4] = Z_3/m^2, \qquad [E_3,E_4] = Z_1/k^2 \\ [Z_i, E_k] = [Z_i, Z_j] = 0 \end{gathered} \tag{22}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1$. We put again $\mathfrak{h} = \operatorname{span}(A, B, C)$. The matrix $\mathbf{A}$ of the coefficients of the corresponding homogeneous system and the vector $\mathbf{b}$ of the right-hand sides are given by

$$\mathbf{A} = \begin{pmatrix} -x_2 & x_3 & x_4 \\ x_1 & x_4 & -x_3 \\ x_4 & -x_1 & x_2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} \varepsilon_1 x_2 z_1 + \varepsilon_2 x_3 z_2 + \varepsilon_3 x_4 z_3 \\ -\varepsilon_1 x_1 z_1 + \varepsilon_2 x_4 z_2 + \varepsilon_3 x_3 z_3 \\ \varepsilon_1 x_4 z_1 - \varepsilon_2 x_1 z_2 - \varepsilon_3 x_2 z_3 \end{pmatrix}. \tag{23}$$

The components ξ_i of the unique geodesic graph are

$$\begin{aligned} \xi_1 &= \frac{\varepsilon_1(-x_1^2 - x_2^2 + x_3^2 + x_4^2)z_1 + 2\varepsilon_3(x_1x_3 - x_2x_4)z_3}{\|x\|^2} \\ \xi_2 &= \frac{2\varepsilon_1(x_2x_3 - x_1x_4)z_1 + 2\varepsilon_3(x_1x_2 + x_3x_4)z_3}{\|x\|^2} + \varepsilon_2 z_2 \\ \xi_3 &= \frac{2\varepsilon_1(x_1x_3 + x_2x_4)z_1 + \varepsilon_3(x_1^2 - x_2^2 - x_3^2 + x_4^2)z_3}{\|x\|^2}. \end{aligned} \tag{24}$$

For $\|x\|^2 = 0$ we put $\xi_1 = \xi_3 = 0$ and $\xi_2 = \varepsilon_2 z_2$. We easily check that geodesic graph is continuous on U and on the set $W = \{(0,0,0,0,0,z_2,0) \in \mathfrak{m};\ z_2 \in \mathbb{R}\}$. It cannot be continuous on the set $V = \{(0,0,0,0,z_1,z_2,z_3) \in \mathfrak{m};\ z_1 \neq 0 \text{ or } z_3 \neq 0\}$. We see that G/H is a pseudo-Riemannian g.o. space of degree two (for any k, l, m, ε_1, ε_2, ε_3).

2.5. Seven Dimensional Examples of Type $(\mathbf{SO}(5) \times \mathbf{SO}(2))/\mathbf{U}(2)$

Let $\{E_1, \dots, E_4, Z_1, Z_2, Z_3\}$ with the signature $(1,1,1,1,\varepsilon_1,\varepsilon_1,\varepsilon_2)$ be the pseudo-orthonormal basis of the seven-dimensional vector space $\mathfrak{m}$. We denote $\mathfrak{v} = \operatorname{span}(E_1, \dots, E_4)$, $\mathfrak{z} = \operatorname{span}(Z_1, Z_2, Z_3)$ and thus $\mathfrak{m} = \mathfrak{v} + \mathfrak{z}$. Now we put $D = A_{14} + A_{23} + 2B_{12}$ and we consider the algebra $\mathfrak{h} = \operatorname{span}(A, B, C, D) \simeq \mathfrak{u}(2)$. We define the Lie algebra structure on $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ by the additional relations

$$\begin{aligned} &[E_1, E_2] = p(Z_1 - A), \quad [E_2, E_3] = qZ_3 - pC, \quad [E_1, E_3] = p(Z_2 + B) \\ &[E_2, E_4] = -p(Z_2 - B), \quad [E_1, E_4] = qZ_3 + pC, \quad [E_3, E_4] = p(Z_1 + A) \\ &[Z_1, Z_2] = \frac{2q}{p} Z_3, \qquad [Z_2, Z_3] = \frac{2p}{q} Z_1, \qquad [Z_3, Z_1] = \frac{2p}{q} Z_2 \end{aligned} \tag{25}$$

where p and q are the parameters satisfying $p > 0$, $q \neq 0$ and $p \neq |q|$, and by the adjoint action of the elements from $\mathfrak{z}$ on $\mathfrak{v}$ given by

$$\operatorname{ad}(Z_1)|_{\mathfrak{v}} = (A_{12} + A_{34}), \quad \operatorname{ad}(Z_2)|_{\mathfrak{v}} = (A_{13} - A_{24}), \quad \operatorname{ad}(Z_3)|_{\mathfrak{v}} = \frac{p}{q}(A_{14} + A_{23}). \tag{26}$$

It can be shown (see [5]) that the algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(5) + \mathfrak{so}(2)$ for $q > 0$ or it is isomorphic to $\mathfrak{so}(4,1) + \mathfrak{so}(2)$ for $q < 0$.

From the equation (6) we obtain a system of equations whose matrix $\mathbf{A}$ and the right-hand side vector $\mathbf{b}$ are

$$\mathbf{A} = \begin{pmatrix} -x_2 & x_3 & x_4 & x_4 \\ x_1 & x_4 & -x_3 & x_3 \\ x_4 & -x_1 & x_2 & -x_2 \\ 0 & 0 & 0 & 2\varepsilon_1 z_2 \end{pmatrix} \tag{27}$$

$$\mathbf{b} = \begin{pmatrix} (\varepsilon_1 p - 1)\, x_2 z_1 + (\varepsilon_1 p - 1)\, x_3 z_2 + \left(\varepsilon_2 q - \frac{p}{q}\right) x_4 z_3 \\ -(\varepsilon_1 p - 1)\, x_1 z_1 - (\varepsilon_1 p - 1)\, x_4 z_2 + \left(\varepsilon_2 q - \frac{p}{q}\right) x_3 z_3 \\ (\varepsilon_1 p - 1)\, x_4 z_1 - (\varepsilon_1 p - 1)\, x_1 z_2 - \left(\varepsilon_2 q - \frac{p}{q}\right) x_2 z_3 \\ 2\left(\varepsilon_2 \frac{q}{p} - \varepsilon_1 \frac{p}{q}\right) z_2 z_3 \end{pmatrix}. \tag{28}$$

Then the corresponding vector $\xi(X)$ can be calculated by the Cramer's rule and its components are

$$\begin{aligned} \xi_1 &= \big[(\varepsilon_1 p - 1)(-x_1^2 - x_2^2 + x_3^2 + x_4^2) z_1 - 2(\varepsilon_1 p - 1)(x_1 x_4 + x_2 x_3) z_2 \\ &\quad + 2\left(\varepsilon_2 q - \frac{\varepsilon_2 q}{\varepsilon_1 p}\right)(x_1 x_3 - x_4 x_2) z_3\big]/\|x\|^2 \\ \xi_2 &= \big[2(\varepsilon_1 p - 1)(x_2 x_3 - x_1 x_4) z_1 + (\varepsilon_1 p - 1)(x_1^2 - x_2^2 + x_3^2 - x_4^2) z_2 \\ &\quad + 2\left(\varepsilon_2 q - \frac{\varepsilon_2 q}{\varepsilon_1 p}\right)(x_1 x_2 + x_3 x_4) z_3\big]/\|x\|^2 \\ \xi_3 &= \big[2(\varepsilon_1 p - 1)(x_1 x_3 + x_2 x_4) z_1 + 2(\varepsilon_1 p - 1)(x_3 x_4 - x_1 x_2) z_2 \\ &\quad + \left(\varepsilon_2 q - \frac{\varepsilon_2 q}{\varepsilon_1 p}\right)(x_1^2 - x_2^2 - x_3^2 + x_4^2) z_3\big]/\|x\|^2 \\ \xi_4 &= \frac{\varepsilon_2 q^2 - \varepsilon_1 p^2}{\varepsilon_1 p q} z_3. \end{aligned} \tag{29}$$

For $\|x\|^2 = 0$ we put $\xi_1 = \xi_2 = \xi_3 = 0$ and ξ_4 as in the formula (29). Geodesic graph is continuous on U and at the origin. It cannot be continuous on the set $V = \{(0,0,0,0,z_1,z_2,z_3) \in \mathfrak{m};\ z_1^2 + z_2^2 + z_3^2 \neq 0\}$. Geodesic graph is linear (and G/H is naturally reductive) only for $p = \varepsilon_1 = 1$ and arbitrary ε_2. Otherwise, G/H is a pseudo-Riemannian g.o. space of degree two.

3. Observations

We now want to show that all geodesic graphs presented in the previous section are continuous maps at the origin. For this purpose we need only the following

Lemma 2. *Let $x_1, \dots, x_p, z_1, \dots, z_s$ be coordinates in the space $\mathbb{R}^{p+s}$ and $\|x\| = \sqrt{\sum_{1 \le i \le p} x_i^2}$. Suppose that the geodesic graph of a given pseudo-Riemannian g.o.*

space G/H is expressed, for $\|x\| \neq 0$, by the formulas in which some of the graph components ξ_k (let us say $\xi_1, \dots, \xi_t$) are zeros, or linear functions of the variables $z_1, \dots, z_s$, and the others (let us say $\xi_{t+1}, \dots, \xi_u$), are rational functions of the form $P_{t+1}/\|x\|^2, \dots, P_u/\|x\|^2$, where the numerators are homogeneous polynomials of degree three in x_i, z_j which are linear in the variables $z_1, \dots, z_s$. Suppose that, for $\|x\| = 0$, the rest of the graph is defined by letting $\xi_1, \dots, \xi_t$ unchanged and putting $\xi_{t+1}, \dots, \xi_u$ equal to zero. Then this geodesic graph is continuous at the origin.

Proof: Put in general $(x_1, \dots, x_p) = \sigma(a_1, \dots, a_p)$, where the numerical vector $(a_1, \dots, a_p)$ on the right-hand side is a unit vector and $\sigma = \|x\|$ is the norm. Then we can write

$$\xi_{t+l} = \sum_{j=1}^{s} Q^j_{t+l}(a_1, \dots, a_p) z_j, \quad \text{for} \quad l = 1, \dots, u-t \tag{30}$$

where each $Q^j_{t+l}(a_1, \dots, a_p)$ is a quadratic homogeneous polynomial. Now, each $|Q^j_{t+l}(a_1, \dots, a_p)|$ is bounded from above by some positive constant k^j_{t+l}. Hence

$$|\xi_{t+l}| \leq \sum_{j=1}^{s} k^j_{t+l} |z_j|. \tag{31}$$

We see that this is a subcontinuous and hence continuous function. We see finally that the whole geodesic graph is continuous at the origin. □

Further, we add a conjecture whose proof might be not simple.

Conjecture 1. For every pseudo-Riemannian g.o. space G/H with compact isotropy group H, every canonical geodesic graph (in the sense of J. Szenthe) is continuous at the origin.

Remark. Recently, we started to investigate pseudo-Riemannian homogeneous spaces which are "g.o. spaces up to measure zero." Here the isotropy group H is not compact, the (incomplete) geodesic graph is not continuous at the origin and the discontinuities are of stronger character. The results will appear in a subsequent paper.

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