

ON THE GEOMETRIC STRUCTURE OF HYPERSURFACES OF CONULLITY TWO IN EUCLIDEAN SPACE

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Abstract. In this paper we introduce the notion of a semi-developable surface of codimension two as a generalization of the notion of a developable surface of codimension two. We give a characterization of the developable and semi-developable surfaces in terms of their second fundamental forms. We prove that any hypersurface of conullity two in Euclidean space is locally a foliation of developable or semi-developable surfaces of codimension two.

1. Introduction

The class of semi-symmetric spaces was first studied by Cartan [3] in connection with his research on locally symmetric spaces. All locally symmetric spaces and all two-dimensional Riemannian manifolds belong to this class. In 1968 Nomizu [7] conjectured that in all dimensions greater or equal to three every irreducible complete Riemannian semi-symmetric space is locally symmetric. His conjecture was refuted in 1972 by Takagi [11], who constructed a complete irreducible hypersurface in \mathbb{E}^4 , which is semi-symmetric but is not locally symmetric, and by Sekigawa [8], who gave counterexamples of arbitrary dimensions. In 1982 Szabó [9] gave a local classification of Riemannian semi-symmetric spaces, dividing them into three basic classes: trivial, exceptional and typical. Semi-symmetric spaces of the typical class were studied also by Boeckx *et al* in [2] under the name Riemannian manifolds of conullity two.

In the present paper we study the class of the typical semi-symmetric hypersurfaces (hypersurfaces of conullity two) in Euclidean space \mathbb{E}^{n+1} , considering them with respect to their second fundamental form.

In Section 3 we introduce the notion of a semi-developable surface of codimension two as a generalization of the notion “developable surface” of codimension two

and give a characterization of the developable and semi-developable surfaces of codimension two in terms of their second fundamental form.

In Section 4 we prove the following structure theorem:

Each hypersurface of conullity two in \mathbb{E}^{n+1} is locally a foliation (one-parameter system) of developable or semi-developable surfaces of codimension two.

2. Preliminaries

For an n -dimensional Riemannian manifold (M^n, g) we denote by $T_p M^n$ the tangent space to M^n at a point $p \in M^n$ and by $\mathfrak{X}M^n$ – the algebra of all vector fields on M^n . The associated Levi-Civita connection of the metric g is denoted by ∇ , the Riemannian curvature tensor R of type $(1, 3)$ is defined by

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z, \quad X, Y, Z \in \mathfrak{X}M^n$$

and the corresponding curvature tensor of type $(0, 4)$ is given by

$$R(X, Y, Z, U) = g(R(X, Y)Z, U), \quad X, Y, Z, U \in \mathfrak{X}M^n.$$

We remark that all manifolds, vector fields, differential forms, functions and surfaces are assumed to be smooth (i.e., of differentiability class C^∞).

A **semi-symmetric space** is a Riemannian manifold (M^n, g) , whose curvature tensor R satisfies the identity

$$R(X, Y) \cdot R = 0$$

for all $X, Y \in \mathfrak{X}M^n$. According to Szabó's classification (using the terminology of [2]) every locally irreducible semi-symmetric space belongs to one of the following three classes:

- 1) "trivial" class, consisting of all locally symmetric spaces and all two-dimensional Riemannian manifolds
- 2) "exceptional" class of all elliptic, hyperbolic, Euclidean and Kählerian cones
- 3) "typical" class of all Riemannian manifolds foliated by Euclidean leaves of codimension two.

The trivial semi-symmetric manifolds are well-known and the exceptional ones are described and constructed explicitly in [9] and [10]. For the class of foliated semi-symmetric spaces Szabó [10] had derived a system of non-linear partial differential equations, describing their metrics.

Foliated semi-symmetric spaces were studied by Boeckx *et al* [2] with respect to their metrics as Riemannian manifolds of conullity two.

A Riemannian manifold (M^n, g) is of **conullity two**, if at every point $p \in M^n$ the tangent space $T_p M^n$ can be decomposed in the form

$$T_p M^n = \Delta_0(p) \oplus \Delta_0^\perp(p)$$

where $\dim \Delta_0(p) = n - 2$, $\dim \Delta_0^\perp(p) = 2$ and $\Delta_0(p)$ is the nullity vector space of the curvature tensor R_p , i.e.,

$$\Delta_0(p) = \{X \in T_p M^n; R_p(X, Y)Z = 0, Y, Z \in T_p M^n\}.$$

The $(n - 2)$ -dimensional distribution $\Delta_0 : p \rightarrow \Delta_0(p)$ is integrable and its integral manifolds are totally geodesic and locally Euclidean. So, (M^n, g) is foliated by Euclidean leaves of codimension two.

In [2] the metrics of the Riemannian manifolds of conullity two are described by systems of non-linear partial differential equations. For some classes of manifolds of conullity two the metrics are determined in explicit form.

We study hypersurfaces of conullity two (or foliated semi-symmetric hypersurfaces) in Euclidean space \mathbb{E}^{n+1} with respect to their second fundamental form, considering them as one-parameter systems of geometrically determined surfaces of codimension two.

We denote the standard metric in \mathbb{E}^{n+1} by g and its Levi-Civita connection by ∇' . Let ∇ be the induced connection on a hypersurface M^n in \mathbb{E}^{n+1} and $h(X, Y) = g(AX, Y)$, $X, Y \in \mathfrak{X}M^n$ be the second fundamental tensor of M^n with corresponding shape operator A .

Foliated semi-symmetric hypersurfaces are characterized in terms of the second fundamental form as follows [5].

Proposition 1. *A hypersurface M^n in \mathbb{E}^{n+1} is of conullity two if and only if its second fundamental form h is*

$$h = \lambda\omega \otimes \omega + \mu(\omega \otimes \eta + \eta \otimes \omega) + \nu\eta \otimes \eta, \quad \lambda\nu - \mu^2 \neq 0$$

where ω and η are unit one-forms; λ, μ and ν are functions on M^n .

Here the Euclidean leaves of the foliation are the integral submanifolds of the distribution Δ_0 , determined by the one-forms ω and η

$$\Delta_0(p) = \{X \in T_p M^n; \omega(X) = 0, \eta(X) = 0\}, \quad p \in M^n.$$

We denote by Δ_0^\perp the distribution of M^n , orthogonal to Δ_0 . Since the second fundamental form h of M^n is symmetric, then locally there exist two mutually orthogonal unit vector fields $\xi_1, \xi_2 \in \Delta_0^\perp$ with corresponding unit one-forms η_1 and η_2 , respectively, such that

$$h = \nu_1\eta_1 \otimes \eta_1 + \nu_2\eta_2 \otimes \eta_2, \quad \nu_1\nu_2 \neq 0 \tag{1}$$

where ν_1 and ν_2 are functions on M^n . The vector fields ξ_1 and ξ_2 determine the principal directions of the shape operator A of M^n .

Using the Codazzi equations for a hypersurface with second fundamental form h , satisfying (1), in Section 4 we obtain all involutive $(n - 1)$ -dimensional distributions, containing Δ_0 , and prove that the integral surfaces of these distributions are developable or semi-developable surfaces of codimension two.

3. Developable and Semi-Developable Surfaces of Codimension Two

A $(k + 1)$ -dimensional surface M^{k+1} in Euclidean space \mathbb{E}^{n+1} , which is a one-parameter system $\{\mathbb{E}^k(s)\}$, $s \in J$ of k -dimensional linear subspaces of \mathbb{E}^{n+1} , defined in an interval $J \subset \mathbb{R}$, is said to be a *ruled $(k + 1)$ -surface* [4, 1]. The planes $\mathbb{E}^k(s)$ are called *generators* of M^{k+1} . A ruled surface M^{k+1} is said to be **developable** [1], if the tangent space $T_p M^{k+1}$ at all regular points p of an arbitrary fixed generator $\mathbb{E}^k(s)$ is the same. A developable ruled hypersurface $M^n = \{\mathbb{E}^{n-1}(s)\}$, $s \in J$ in \mathbb{E}^{n+1} is called a **torse**.

Now we shall consider a ruled $(n - 1)$ -surface $M^{n-1} = \{\mathbb{E}^{n-2}(s)\}$, $s \in J$ (ruled surface of codimension two). Let $\{N_1, N_2\}$ be a normal frame field of M^{n-1} , consisting of two mutually orthogonal unit vector fields. We denote by h_1 and h_2 the second fundamental forms of M^{n-1} corresponding to the vector fields N_1 and N_2 , respectively and by A_1 and A_2 their corresponding shape operators, i.e.,

$$h_1(x, y) = g(A_1 x, y), \quad h_2(x, y) = g(A_2 x, y), \quad x, y \in \mathfrak{X} M^{n-1}.$$

If D is the normal connection of M^{n-1} , then the Gauss and Weingarten formulas imply

$$\begin{aligned} \nabla'_x y &= \nabla_x y + h_1(x, y) N_1 + h_2(x, y) N_2, & x, y \in \mathfrak{X} M^{n-1} \\ \nabla'_x N_1 &= -A_1 x + D_x N_1, & \nabla'_x N_2 = -A_2 x + D_x N_2. \end{aligned} \quad (2)$$

Let p be an arbitrary point of M^{n-1} and $\mathbb{E}^{n-2}(s)$ be the generator of M^{n-1} containing p . We denote by $\Delta_0(p)$ the subspace of $T_p M^{n-1}$, tangent to $\mathbb{E}^{n-2}(s)$ and by Δ_0 – the distribution $\Delta_0 : p \rightarrow \Delta_0(p)$. The unit vector field on M^{n-1} , orthogonal to Δ_0 and its corresponding one-form are denoted by W and ω , respectively (W is determined up to a sign). Since the integral submanifolds $\mathbb{E}^{n-2}(s)$ of the distribution Δ_0 are auto-parallel, then $\nabla'_{x_0} y_0 \in \Delta_0$ for all $x_0, y_0 \in \Delta_0$. Hence, the first equality in (2) implies

$$h_1(x_0, y_0) = h_2(x_0, y_0) = 0, \quad x_0, y_0 \in \Delta_0. \quad (3)$$

Using the unique decompositions

$$x = x_0 + \omega(x) W, \quad y = y_0 + \omega(y) W, \quad x_0, y_0 \in \Delta_0$$

of arbitrary vector fields $x, y \in \mathfrak{X}M^{n-1}$, from (3) we get

$$\begin{aligned} h_1(x, y) &= p \omega(x) \omega(y) + \omega(x) h_1(y_0, W) + \omega(y) h_1(x_0, W) \\ h_2(x, y) &= q \omega(x) \omega(y) + \omega(x) h_2(y_0, W) + \omega(y) h_2(x_0, W) \end{aligned} \tag{4}$$

where $p = h_1(W, W)$, $q = h_2(W, W)$.

We denote by $\bar{\beta}_1$ and $\bar{\beta}_2$ the one-forms on M^{n-1} , defined by

$$\begin{aligned} \bar{\beta}_1(x_0) &= h_1(x_0, W), & \bar{\beta}_1(W) &= 0, & x_0 &\in \Delta_0 \\ \bar{\beta}_2(x_0) &= h_2(x_0, W), & \bar{\beta}_2(W) &= 0. \end{aligned}$$

Let β_1 and β_2 be their corresponding unit one-forms, i.e.,

$$\begin{aligned} \bar{\beta}_1 &= b_1 \beta_1, & b_1 &= \|\bar{\beta}_1\| \\ \bar{\beta}_2 &= b_2 \beta_2, & b_2 &= \|\bar{\beta}_2\|. \end{aligned}$$

Hence, the equalities (4) imply

$$h_1 = p \omega \otimes \omega + b_1 (\omega \otimes \beta_1 + \beta_1 \otimes \omega), \quad h_2 = q \omega \otimes \omega + b_2 (\omega \otimes \beta_2 + \beta_2 \otimes \omega). \tag{5}$$

Let B_1 and B_2 be the unit vector fields on M^{n-1} , corresponding to the one-forms β_1 and β_2 , respectively, i.e.,

$$\beta_1(x) = g(B_1, x), \quad \beta_2(x) = g(B_2, x), \quad x \in \mathfrak{X}M^{n-1}.$$

It is obvious that $B_1, B_2 \in \Delta_0$. We denote by θ the one-form on M^{n-1} , defined as follows

$$\theta(x) = g(\nabla'_x N_2, N_1), \quad x \in \mathfrak{X}M^{n-1}.$$

Using that $g(\nabla'_x N_i, N_i) = 0$, $i = 1, 2$, the Weingarten formulas and equalities (5), we obtain

$$\begin{aligned} \nabla'_x N_1 &= -b_1 \omega(x) B_1 - b_1 \beta_1(x) W - p \omega(x) W - \theta(x) N_2 \\ \nabla'_x N_2 &= -b_2 \omega(x) B_2 - b_2 \beta_2(x) W - q \omega(x) W + \theta(x) N_1. \end{aligned} \tag{6}$$

The developable surfaces of codimension two in \mathbb{E}^{n+1} are characterized [6] by

Lemma 1. *Let M^{n-1} be a surface in \mathbb{E}^{n+1} with normal frame field $\{N_1, N_2\}$. Then, M^{n-1} is locally a developable surface of codimension two if and only if*

$$\begin{aligned} \nabla'_x N_1 &= -p \omega(x) W - \mu \omega(x) N_2, & x &\in \mathfrak{X}M^{n-1} \\ \nabla'_x N_2 &= -q \omega(x) W + \mu \omega(x) N_1 \end{aligned} \tag{7}$$

where μ, p and q are functions on M^{n-1} , such that $p^2 + q^2 > 0$.

Remark. The planes \mathbb{E}^{n-1} of codimension two can be considered as trivial developable surfaces of codimension two, for which $p = q = 0$ [5].

Now, let M^{n-1} be a developable surface of codimension two with normal frame field $\{N_1, N_2\}$, satisfying (7). If $\{\bar{N}_1, \bar{N}_2\}$ is another normal frame field of M^{n-1} , such that

$$\bar{N}_1 = \cos \varphi N_1 + \sin \varphi N_2, \quad \bar{N}_2 = -\sin \varphi N_1 + \cos \varphi N_2, \quad \varphi = \angle(N_1, \bar{N}_1)$$

then

$$\begin{aligned} \nabla'_x \bar{N}_1 &= -\bar{p} \omega(x)W - \bar{\mu} \omega(x) \bar{N}_2, & x \in \mathfrak{X}M^{n-1} \\ \nabla'_x \bar{N}_2 &= -\bar{q} \omega(x)W + \bar{\mu} \omega(x) \bar{N}_1 \end{aligned}$$

where

$$\bar{p} = p \cos \varphi + q \sin \varphi, \quad \bar{q} = -p \sin \varphi + q \cos \varphi, \quad \bar{\mu} = \mu - d\varphi(W).$$

The last equalities imply

$$\bar{\mu} - d \arctan \frac{\bar{q}}{\bar{p}}(W) = \mu - d \arctan \frac{q}{p}(W).$$

Consequently, the function $\kappa = \mu - d \arctan \frac{q}{p}(W)$ does not depend on the choice of the normal frame field $\{N_1, N_2\}$ of M^{n-1} .

We call a developable surface M^{n-1} in \mathbb{E}^{n+1} **planar**, if there exists a hyperplane \mathbb{E}^n in \mathbb{E}^{n+1} , such that M^{n-1} lies in \mathbb{E}^n . The planar developable surfaces of codimension two are studied in [5] under the name torses of codimension two and are characterized as follows:

Lemma 2. *A developable surface of codimension two is planar iff $\kappa = 0$.*

It is easily seen that for each developable surface M^{n-1} of codimension two there exists locally a normal frame field $\{l_1, l_2\}$, with respect to which the equalities (7) take the form

$$\nabla'_x l_1 = -\nu_1 \omega(x)W, \quad \nabla'_x l_2 = -\nu_2 \omega(x)W, \quad x \in \mathfrak{X}M^{n-1}$$

where ν_1 and ν_2 are functions on M^{n-1} . Such a normal frame field is called a **canonical normal frame field** of M^{n-1} . It is determined up to a constant orthogonal matrix.

Now we shall consider non-developable ruled surfaces of codimension two. Let $M^{n-1} = \{\mathbb{E}^{n-2}(s)\}$, $s \in J$ be such a surface. If $\{N_1, N_2\}$ is an arbitrary normal frame field of M^{n-1} , then the equalities (6) hold good. As a generalization of the notion developable surface of codimension two we give the following

Definition 1. A ruled surface $M^{n-1} = \{\mathbb{E}^{n-2}(s)\}$, $s \in J$ in \mathbb{E}^{n+1} is called **semi-developable**, if there exists a unit normal vector field N of M^{n-1} , which is constant along each fixed generator $\mathbb{E}^{n-2}(s)$, i.e., $\nabla'_{x_0} N = 0$, $x_0 \in \Delta_0$.

We shall prove the following

Proposition 2. *Let M^{n-1} be a non-developable ruled surface in \mathbb{E}^{n+1} with normal frame field $\{N_1, N_2\}$. Then, M^{n-1} is semi-developable if and only if*

$$\theta(x_0) = \varepsilon \operatorname{d} \arctan \frac{b_2}{b_1}(x_0), \quad x_0 \in \Delta_0, \quad b_1^2 + b_2^2 \neq 0 \quad (8)$$

where $\varepsilon = \pm 1$.

Proof: 1) Let M^{n-1} be a semi-developable surface of codimension two with normal vector field $N = \cos \varphi N_1 + \sin \varphi N_2$, $\varphi = \angle(N_1, N)$, which is constant along each generator and N^\perp be the normal vector field on M^{n-1} , defined by $N^\perp = -\sin \varphi N_1 + \cos \varphi N_2$. Then the equalities (6) imply

$$\begin{aligned} \nabla'_x N &= -b_1 \cos \varphi \omega(x) B_1 - b_2 \sin \varphi \omega(x) B_2 \\ &\quad - (b_1 \cos \varphi \beta_1(x) + b_2 \sin \varphi \beta_2(x)) W \\ &\quad - (p \cos \varphi + q \sin \varphi) \omega(x) W - (\theta(x) - \operatorname{d}\varphi(x)) N^\perp \\ \nabla'_x N^\perp &= b_1 \sin \varphi \omega(x) B_1 - b_2 \cos \varphi \omega(x) B_2 \\ &\quad + (b_1 \sin \varphi \beta_1(x) - b_2 \cos \varphi \beta_2(x)) W \\ &\quad + (p \sin \varphi - q \cos \varphi) \omega(x) W + (\theta(x) - \operatorname{d}\varphi(x)) N. \end{aligned}$$

Using that $\nabla'_{x_0} N = 0$, $x_0 \in \Delta_0$, from the last equalities we get

$$b_1 \cos \varphi \beta_1(x_0) + b_2 \sin \varphi \beta_2(x_0) = 0, \quad \theta(x_0) - \operatorname{d}\varphi(x_0) = 0, \quad x_0 \in \Delta_0. \quad (9)$$

If we assume that $b_1 = b_2 = 0$, then the second equality of (9) implies $\theta(x) - \operatorname{d}\varphi(x) = \mu \omega(x)$, where $\mu = \theta(W) - \operatorname{d}\varphi(W)$. So, we obtain

$$\begin{aligned} \nabla'_x N &= -\bar{p} \omega(x) W - \mu \omega(x) N^\perp, \quad x \in \mathfrak{X} M^{n-1} \\ \nabla'_x N^\perp &= -\bar{q} \omega(x) W + \mu \omega(x) N \end{aligned}$$

where $\bar{p} = p \cos \varphi + q \sin \varphi$, $\bar{q} = -p \sin \varphi + q \cos \varphi$. Then, according to Lemma 1, M^{n-1} is locally a developable surface of codimension two, which contradicts the condition that M^{n-1} is non-developable. Hence, $b_1^2 + b_2^2 \neq 0$.

In the case when $b_2 = 0$ (or $b_1 = 0$) the equalities (9) imply

$$\cos \varphi = 0 \quad (\text{or } \sin \varphi = 0), \quad \theta(x_0) = 0, \quad x_0 \in \Delta_0.$$

Hence, the conditions (8) are fulfilled.

In the case when $b_1 b_2 \neq 0$, the first equality of (9) implies

$$\beta_1(x_0) = -\frac{b_2}{b_1} \tan \varphi \beta_2(x_0)$$

which shows that the one-forms β_1 and β_2 are collinear. Since β_1 and β_2 are unit one-forms, then $\beta_1 = \varepsilon \beta_2$, where $\varepsilon = \pm 1$. Hence, $\tan \varphi = -\varepsilon b_1/b_2$. Using the second equality of (9) we get (8).

2) Let M^{n-1} be a non-developable ruled surface with normal frame field $\{N_1, N_2\}$, satisfying (6), such that the conditions (8) hold good. For the curvature tensor R' of ∇' from (6) we calculate

$$\begin{aligned} R'(x, y, N_1, N_2) = & -d\theta(x, y) + b_1 b_2 (\beta_1 \wedge \beta_2)(x, y) \\ & + b_1 q (\beta_1 \wedge \omega)(x, y) - b_2 p (\beta_2 \wedge \omega)(x, y) \end{aligned} \quad (10)$$

where $x, y \in \mathfrak{X}M^{n-1}$. Using that $R' = 0$ and $d\theta(x_0, y_0) = 0$, $x_0, y_0 \in \Delta_0$, from (10) we obtain

$$b_1 b_2 (\beta_1 \wedge \beta_2)(x_0, y_0) = 0, \quad x_0, y_0 \in \Delta_0. \quad (11)$$

In the case when $b_2 = 0$ (or $b_1 = 0$) we get from (8) that $\theta(x_0) = 0$, $x_0 \in \Delta_0$. Hence, $\theta = \theta(W)\omega$. Denoting $\mu = \theta(W)$, we obtain from (6)

$$\nabla'_x N_2 = -q\omega(x)W + \mu\omega(x)N_1 \quad (\text{or } \nabla'_x N_1 = -p\omega(x)W - \mu\omega(x)N_2)$$

which implies $\nabla'_{x_0} N_2 = 0$ (or $\nabla'_{x_0} N_1 = 0$), $x_0 \in \Delta_0$. Consequently, M^{n-1} is a semi-developable surface.

In the case when $b_1 b_2 \neq 0$, the equality (11) implies $(\beta_1 \wedge \beta_2)(x_0, y_0) = 0$, $x_0, y_0 \in \Delta_0$, which shows that $\beta_1 = \varepsilon\beta_2$ ($\varepsilon = \pm 1$). Setting $\varphi = -\varepsilon \arctan \frac{b_1}{b_2}$ and considering the normal frame field $\{N, N^\perp\}$ of M^{n-1} , defined by

$$N = \cos \varphi N_1 + \sin \varphi N_2, \quad N^\perp = -\sin \varphi N_1 + \cos \varphi N_2$$

we get the formulas

$$\begin{aligned} \nabla'_x N &= -\bar{p}\omega(x)W - \mu\omega(x)N^\perp \\ \nabla'_x N^\perp &= -b\omega(x)B - b\beta(x)W - \bar{q}\omega(x)W + \mu\omega(x)N \end{aligned}$$

where $\beta = \beta_1 = \varepsilon\beta_2$, $B = B_1 = \varepsilon B_2$, $\bar{p} = p \cos \varphi + q \sin \varphi$, $\bar{q} = -p \sin \varphi + q \cos \varphi$, $b^2 = b_1^2 + b_2^2$, $\mu = \theta(W) - d\varphi(W)$. Consequently, $\nabla'_{x_0} N = 0$, $x_0 \in \Delta_0$, i.e., M^{n-1} is a semi-developable surface. \square

In the process of proving Proposition 2 we have obtained that for each semi-developable surface M^{n-1} there exists a normal frame field $\{N, N^\perp\}$, such that

$$\begin{aligned} \nabla'_x N &= -p\omega(x)W - \mu\omega(x)N^\perp \\ \nabla'_x N^\perp &= -b\omega(x)B - b\beta(x)W - q\omega(x)W + \mu\omega(x)N. \end{aligned} \quad (12)$$

We call such a normal frame field a **canonical normal frame field** of the semi-developable surface M^{n-1} and the normal vector field N we call **main normal vector field** of M^{n-1} . The main normal vector field N is determined up to a sign.

A semi-developable surface M^{n-1} in \mathbb{E}^{n+1} is said to be **planar**, if there exists a hyperplane \mathbb{E}^n in \mathbb{E}^{n+1} , such that M^{n-1} lies in \mathbb{E}^n . It is obvious, that if M^{n-1} is a planar semi-developable surface, lying in a hyperplane $\mathbb{E}^n \subset \mathbb{E}^{n+1}$ with normal

N , then N is the main normal vector field of M^{n-1} . The planar semi-developable surfaces of codimension two are characterized by

Lemma 3. *Let M^{n-1} be a semi-developable ruled surface in \mathbb{E}^{n+1} with a canonical normal frame field $\{N, N^\perp\}$ satisfying (12). Then, M^{n-1} is planar iff $p = 0$.*

Proof: 1) Let M^{n-1} be a planar semi-developable surface, lying in a hyperplane \mathbb{E}^n with normal vector field N . Then, $\nabla'_x N = 0$, $x \in \mathfrak{X}M^{n-1}$ and the first equality of (12) implies that $p = 0$, $\mu = 0$.

2) Let M^{n-1} be a semi-developable surface with normal frame field $\{N, N^\perp\}$ and $p = 0$. Then, the equalities (12) imply

$$R'(x, y, N, W) = b \mu(\beta \wedge \omega)(x, y), \quad x, y \in \mathfrak{X}M^{n-1}.$$

Using that $R' = 0$, we get

$$b \mu(\beta \wedge \omega)(x, y) = 0, \quad x, y \in \mathfrak{X}M^{n-1}.$$

Since M^{n-1} is non-developable, then $b \neq 0$ and $\beta \wedge \omega \neq 0$. So, the last equality gives $\mu = 0$. Hence, $\nabla'_x N = 0$, $x \in \mathfrak{X}M^{n-1}$, i.e., M^{n-1} is planar. \square

The non-planar semi-developable surfaces of codimension two are characterized by

Lemma 4. *Let M^{n-1} be a surface in \mathbb{E}^{n+1} with normal frame field $\{N_1, N_2\}$. Then, M^{n-1} is locally a non-planar semi-developable surface with canonical normal frame field $\{N_1, N_2\}$ if and only if*

$$\begin{aligned} \nabla'_x N_1 &= -p \omega(x) W - \mu \omega(x) N_2 \\ \nabla'_x N_2 &= -b \omega(x) B - b \beta(x) W - q \omega(x) W + \mu \omega(x) N_1 \end{aligned} \quad (13)$$

where b, p, q and μ are functions on M^{n-1} , such that $b \neq 0$, $p \neq 0$.

Proof: 1) It is obvious from the considerations above that for a non-planar semi-developable surface M^{n-1} with canonical normal frame field $\{N_1, N_2\}$ the formulas (13) hold true and $b \neq 0$, $p \neq 0$.

2) Let M^{n-1} be a surface in \mathbb{E}^{n+1} with normal frame field $\{N_1, N_2\}$, satisfying (13) and $b \neq 0$, $p \neq 0$. Using that $R'(x, y, N_1) = 0$, from (13) we get

$$\begin{aligned} p \omega(x) \nabla'_y W - p \omega(y) \nabla'_x W \\ + (b \mu(\beta \wedge \omega)(x, y) - d(p \omega)(x, y)) W - d(\mu \omega)(x, y) N_2 = 0 \end{aligned} \quad (14)$$

which implies that $d\omega(x_0, y_0) = 0$, $x_0, y_0 \in \Delta_0$. Hence, the distribution Δ_0 is involutive. Consequently, for each point $p \in M^{n-1}$ there exists a unique maximal integral submanifold S_p^{n-2} of Δ_0 containing p . With the help of formulas (13) and (14) we obtain

$$g(\nabla'_{x_0} y_0, N_1) = 0, \quad g(\nabla'_{x_0} y_0, N_2) = 0, \quad g(\nabla'_{x_0} y_0, W) = 0, \quad x_0, y_0 \in \Delta_0$$

which imply that $\nabla'_{x_0} y_0 \in \Delta_0$ for all $x_0, y_0 \in \Delta_0$, i.e., the integral submanifold S_p^{n-2} of Δ_0 is totally geodesic. So, S_p^{n-2} lies on an $(n-2)$ -dimensional plane \mathbb{E}_p^{n-2} . Hence, M^{n-1} is locally a one-parameter system $\{\mathbb{E}^{n-2}(s)\}$, $s \in J$ of planes of codimension three, i.e., M^{n-1} is locally a ruled surface of codimension two. More over, the first equality of (13) implies $\nabla'_{x_0} N_1 = 0$, $x_0 \in \Delta_0$. Hence, locally M^{n-1} is a semi-developable surface with main normal vector field N_1 . \square

4. Structure Theorem

In what follows we consider a hypersurface M^n of conullity two with unit normal vector field N (N is determined up to a sign). The second fundamental form h of M^n satisfies (1). The $(n-2)$ -dimensional distribution, determined by the one-forms η_1 and η_2 , is denoted by Δ_0 . Applying the Codazzi equations for a hypersurface with second fundamental form defined by (1), we obtain the equalities

$$\begin{aligned}
 1) \quad & \nabla_{x_0} \xi_1 = -\gamma(x_0) \xi_2 \\
 2) \quad & \nabla_{x_0} \xi_2 = \gamma(x_0) \xi_1 \\
 3) \quad & g(\nabla_{\xi_1} \xi_1, x_0) = d \ln \nu_1(x_0) \\
 4) \quad & g(\nabla_{\xi_2} \xi_2, x_0) = d \ln \nu_2(x_0) \\
 5) \quad & g(\nabla_{\xi_1} \xi_2, x_0) = \frac{\nu_2 - \nu_1}{\nu_2} \gamma(x_0) \\
 6) \quad & g(\nabla_{\xi_2} \xi_1, x_0) = \frac{\nu_2 - \nu_1}{\nu_1} \gamma(x_0) \\
 7) \quad & (\nu_1 - \nu_2)^2 g(\nabla_{\xi_1} \xi_1, \xi_2) = (\nu_1 - \nu_2) d\nu_1(\xi_2) \\
 8) \quad & (\nu_1 - \nu_2)^2 g(\nabla_{\xi_2} \xi_2, \xi_1) = -(\nu_1 - \nu_2) d\nu_2(\xi_1)
 \end{aligned} \tag{15}$$

where γ is a one-form on Δ_0 , defined by $\gamma(x_0) = g(\nabla_{x_0} \xi_2, \xi_1)$, $x_0 \in \Delta_0$.

We denote by Δ_{ξ_1} and Δ_{ξ_2} the $(n-1)$ -dimensional distributions, orthogonal to the vector fields ξ_1 and ξ_2 , respectively, i.e.,

$$\begin{aligned}
 \Delta_{\xi_1}(p) &= \{x \in T_p M^n; \eta_1(x) = 0\} = \Delta_0 \oplus \text{span}\{\xi_2\}, \quad p \in M^n \\
 \Delta_{\xi_2}(p) &= \{x \in T_p M^n; \eta_2(x) = 0\} = \Delta_0 \oplus \text{span}\{\xi_1\}, \quad p \in M^n.
 \end{aligned}$$

In general, Δ_{ξ_1} and Δ_{ξ_2} are not involutive. We shall find all involutive $(n-1)$ -dimensional distributions of M^n containing Δ_0 . An arbitrary unit vector field $\xi \in \Delta_0^\perp$ is decomposed in the form $\xi = \cos \varphi \xi_1 + \sin \varphi \xi_2$, where $\varphi = \angle(\xi_1, \xi)$. Let ξ^\perp denotes the unit vector field in Δ_0^\perp , orthogonal to ξ , i.e., $\xi^\perp = -\sin \varphi \xi_1 + \cos \varphi \xi_2$. Then, the distribution Δ_ξ , orthogonal to ξ , is presented by

$$\Delta_\xi(p) = \{x \in T_p M^n; g(\xi, x) = 0\} = \Delta_0 \oplus \text{span}\{\xi^\perp\}, \quad p \in M^n.$$

Proposition 3. *Let M^n be a hypersurface of conullity two in \mathbb{E}^{n+1} with principal directions ξ_1, ξ_2 and $\xi = \cos \varphi \xi_1 + \sin \varphi \xi_2$ be a vector field in Δ_0^\perp . Then, the*

distribution Δ_ξ , which is orthogonal to ξ , is involutive if and only if the function φ satisfies

$$d\varphi(x_0) = \left(\frac{1}{k} \cos^2 \varphi + k \sin^2 \varphi \right) \gamma(x_0) - \frac{1}{k} \sin \varphi \cos \varphi dk(x_0) \quad (16)$$

where $x_0 \in \Delta_0$ and $k = \frac{\nu_1}{\nu_2}$.

Proof: Since Δ_0 is an involutive distribution ($[x_0, y_0] \in \Delta_0$ for all $x_0, y_0 \in \Delta_0$), then the distribution Δ_ξ is involutive if and only if $[x_0, \xi^\perp] \in \Delta_\xi$ for all $x_0 \in \Delta_0$, i.e.,

$$g(\nabla'_{x_0} \xi^\perp, \xi) - g(\nabla'_{\xi^\perp} x_0, \xi) = 0, \quad x_0 \in \Delta_0$$

or equivalently

$$g(\nabla_{x_0} \xi^\perp, \xi) + g(\nabla_{\xi^\perp} \xi, x_0) = 0, \quad x_0 \in \Delta_0.$$

The vector fields $\nabla_{x_0} \xi^\perp$ and $\nabla_{\xi^\perp} \xi$ can be expressed as follows

$$\begin{aligned} \nabla_{x_0} \xi^\perp &= (\gamma(x_0) - d\varphi(x_0)) \xi \\ \nabla_{\xi^\perp} \xi &= \sin \varphi \cos \varphi (\nabla_{\xi_2} \xi_2 - \nabla_{\xi_1} \xi_1) - \sin^2 \varphi \nabla_{\xi_1} \xi_2 + \cos^2 \varphi \nabla_{\xi_2} \xi_1 \\ &\quad + (\cos \varphi d\varphi(\xi_2) - \sin \varphi d\varphi(\xi_1)) \xi^\perp. \end{aligned}$$

Using the last equalities and (15) we obtain that Δ_ξ is an involutive distribution if and only if φ satisfies (16). \square

As a corollary we obtain

Corollary 1. Let M^n be a hypersurface of conullity two in \mathbb{E}^{n+1} with principal directions ξ_1 and ξ_2 . The distributions Δ_{ξ_1} and Δ_{ξ_2} are involutive if and only if $\gamma = 0$.

Since the distribution Δ_0 of M^n is involutive, then locally there exist parameters $u, v, w^1, \dots, w^{n-2}$ on M^n , such that $\Delta_0 = \text{span} \left\{ \frac{\partial}{\partial w^\alpha} \right\}_{\alpha=1, \dots, n-2}$. We denote

$$\varphi_\alpha = d\varphi \left(\frac{\partial}{\partial w^\alpha} \right), \quad k_\alpha = dk \left(\frac{\partial}{\partial w^\alpha} \right), \quad \gamma_\alpha = \gamma \left(\frac{\partial}{\partial w^\alpha} \right).$$

So, the equalities (16) can be written in the form

$$\varphi_\alpha = \left(\frac{1}{k} \cos^2 \varphi + k \sin^2 \varphi \right) \gamma_\alpha - \frac{1}{k} \sin \varphi \cos \varphi k_\alpha, \quad \alpha = 1, \dots, n-2$$

or equivalently

$$\varphi_\alpha = \frac{1-k^2}{2k} \gamma_\alpha \cos 2\varphi - \frac{k_\alpha}{2k} \sin 2\varphi + \frac{1+k^2}{2k} \gamma_\alpha, \quad \alpha = 1, \dots, n-2. \quad (17)$$

Setting $a_\alpha = \frac{1-k^2}{2k} \gamma_\alpha$, $b_\alpha = -\frac{k_\alpha}{2k}$, $c_\alpha = \frac{1+k^2}{2k} \gamma_\alpha$, we rewrite (17) in the form

$$\varphi_\alpha = a_\alpha \cos 2\varphi + b_\alpha \sin 2\varphi + c_\alpha, \quad \alpha = 1, \dots, n-2. \quad (18)$$

Now, if we fix (u, v) the equalities (18) can be considered as a system of partial differential equations for the unknown function $\varphi(w^1, \dots, w^{n-2}, u, v)$, where u and v are parameters. We shall prove that the integrability conditions

$$\varphi_{\alpha\beta} = \varphi_{\beta\alpha}, \quad \alpha, \beta = 1, \dots, n-2$$

for the system (18) are fulfilled.

It is easy to calculate that

$$\begin{aligned} \varphi_{\alpha\beta} - \varphi_{\beta\alpha} &= (a_{\alpha\beta} - a_{\beta\alpha} + 2b_\alpha c_\beta - 2b_\beta c_\alpha) \cos 2\varphi \\ &\quad + (b_{\alpha\beta} - b_{\beta\alpha} + 2c_\alpha a_\beta - 2c_\beta a_\alpha) \sin 2\varphi \\ &\quad + (c_{\alpha\beta} - c_{\beta\alpha} + 2b_\alpha a_\beta - 2b_\beta a_\alpha) \end{aligned}$$

where

$$\begin{aligned} a_{\alpha\beta} &= \frac{\partial}{\partial w^\beta} \left(\frac{1-k^2}{2k} \gamma_\alpha \right) = -\frac{k^2+1}{2k^2} k_\beta \gamma_\alpha + \frac{1-k^2}{2k} \gamma_{\alpha\beta} \\ b_{\alpha\beta} &= \frac{\partial}{\partial w^\beta} \left(-\frac{k_\alpha}{2k} \right) = -\frac{1}{2} \frac{\partial}{\partial w^\beta} (\ln k)_\alpha = -\frac{1}{2} \frac{\partial}{\partial w^\alpha} (\ln k)_\beta = b_{\beta\alpha} \\ c_{\alpha\beta} &= \frac{\partial}{\partial w^\beta} \left(\frac{1+k^2}{2k} \gamma_\alpha \right) = \frac{k^2-1}{2k^2} k_\beta \gamma_\alpha + \frac{1+k^2}{2k} \gamma_{\alpha\beta} \\ \gamma_{\alpha\beta} &= \frac{\partial}{\partial w^\beta} (\gamma_\alpha). \end{aligned} \quad (19)$$

Using the fact that the one-form γ is closed and taking into account (19) we calculate

$$\varphi_{\alpha\beta} - \varphi_{\beta\alpha} = 0, \quad \alpha, \beta = 1, \dots, n-2.$$

So, the integrability conditions for the system (18) are fulfilled. Hence, if $\varphi_0(u, v)$ is a given function, then there exists a unique solution $\varphi(w^1, \dots, w^{n-2}, u, v)$ of (18), defined for each $|w^\alpha| < \varepsilon$, $(u, v) \in D_0$, where $\varepsilon > 0$, $D_0 \subset \mathbb{R}^2$, satisfying $\varphi(0, \dots, 0, u, v) = \varphi_0(u, v)$.

Consequently, locally there exists a vector field $\xi = \cos \varphi \xi_1 + \sin \varphi \xi_2$, whose orthogonal distribution Δ_ξ is involutive. The integral submanifolds of Δ_ξ determine M^n locally as a one-parameter system of $(n-1)$ -dimensional surfaces, i.e., locally M^n is a foliation of surfaces of codimension two. Moreover, each function $\varphi_0(u, v)$ determines an involutive distribution Δ_ξ of M^n , which generates the corresponding foliation of M^n .

Now we shall prove that the integral submanifolds M_ξ^{n-1} of each involutive distribution Δ_ξ of M^n are developable or semi-developable surfaces of codimension two.

Theorem 1 (Structure theorem). *Each hypersurface of conullity two in \mathbb{E}^{n+1} is locally a foliation (one-parameter system) of developable or semi-developable surfaces of codimension two.*

Proof: First we shall consider the case $\nu_1 = \nu_2$. Setting $\nu := \nu_1 = \nu_2$ the equalities (15) imply

$$\begin{aligned} 1) \quad \nabla_{x_0} \xi_1 &= -\gamma(x_0) \xi_2 & 4) \quad g(\nabla_{\xi_2} \xi_2, x_0) &= d \ln \nu(x_0) \\ 2) \quad \nabla_{x_0} \xi_2 &= \gamma(x_0) \xi_1 & 5) \quad g(\nabla_{\xi_1} \xi_2, x_0) &= 0 \\ 3) \quad g(\nabla_{\xi_1} \xi_1, x_0) &= d \ln \nu(x_0) & 6) \quad g(\nabla_{\xi_2} \xi_1, x_0) &= 0. \end{aligned} \quad (20)$$

We denote $p = g(\nabla_{\xi_1} \xi_1, \xi_2)$, $q = g(\nabla_{\xi_2} \xi_2, \xi_1)$. Applying Proposition 3, we get that the vector field $\xi = \cos \varphi \xi_1 + \sin \varphi \xi_2$ determines an involutive distribution Δ_ξ if and only if $\varphi_\alpha = \gamma_\alpha$, $\alpha = 1, \dots, n-2$, i.e., $d\varphi(x_0) = \gamma(x_0)$, $x_0 \in \Delta_0$. Each integral submanifold M_ξ^{n-1} of Δ_ξ is an $(n-1)$ -dimensional surface with normal frame field $\{N, \xi\}$. We denote $\xi^\perp = -\sin \varphi \xi_1 + \cos \varphi \xi_2$ and $\omega = -\sin \varphi \eta_1 + \cos \varphi \eta_2$. Using (20) we get

$$\nabla'_x N = -\nu \omega(x) \xi^\perp, \quad \nabla'_x \xi = -\bar{q} \omega(x) \xi^\perp, \quad x \in \mathfrak{X} M_\xi^{n-1}$$

where $\bar{q} = p \sin \varphi + q \cos \varphi + \sin \varphi d\varphi(\xi_1) - \cos \varphi d\varphi(\xi_2)$. Hence, according to Lemma 1, locally M_ξ^{n-1} is a developable surface of codimension two with canonical normal frame field $\{N, \xi\}$. Moreover, since $\nu \neq 0$, the surfaces M_ξ^{n-1} are non-trivial ($M_\xi^{n-1} \neq \mathbb{E}^{n-1}$).

Now let us consider the case $\nu_1 \neq \nu_2$. Let Δ_ξ be an involutive distribution of M^n , where $\xi = \cos \varphi \xi_1 + \sin \varphi \xi_2$. Its integral submanifolds M_ξ^{n-1} are $(n-1)$ -dimensional surfaces in \mathbb{E}^{n+1} with normal frame field $\{N, \xi\}$. Once again we denote $\xi^\perp = -\sin \varphi \xi_1 + \cos \varphi \xi_2$ and $\omega = -\sin \varphi \eta_1 + \cos \varphi \eta_2$. Then, $\Delta_\xi = \Delta_0 + \text{span}\{\xi^\perp\}$. Let $\{e_1, \dots, e_{n-2}\}$ be a local orthonormal basis of Δ_0 . Using (15) we obtain

$$\begin{aligned} \nabla'_{\xi_1} \xi_1 &= d \ln \nu_1(e_\alpha) e_\alpha + \frac{d\nu_1(\xi_2)}{\nu_1 - \nu_2} \xi_2 + \nu_1 N \\ \nabla'_{\xi_2} \xi_2 &= d \ln \nu_2(e_\alpha) e_\alpha - \frac{d\nu_2(\xi_1)}{\nu_1 - \nu_2} \xi_1 + \nu_2 N \\ \nabla'_{\xi_1} \xi_2 &= \frac{\nu_2 - \nu_1}{\nu_2} \gamma(e_\alpha) e_\alpha - \frac{d\nu_1(\xi_2)}{\nu_1 - \nu_2} \xi_1 \\ \nabla'_{\xi_2} \xi_1 &= \frac{\nu_2 - \nu_1}{\nu_1} \gamma(e_\alpha) e_\alpha + \frac{d\nu_2(\xi_1)}{\nu_1 - \nu_2} \xi_2. \end{aligned}$$

Further, we calculate

$$\begin{aligned} \nabla'_{\xi^\perp} \xi &= (d\varphi(e_\alpha) - \gamma(e_\alpha)) e_\alpha - (\sin \varphi d\varphi(\xi_1) - \cos \varphi d\varphi(\xi_2)) \xi^\perp \\ &\quad - \left(\sin \varphi \frac{d\nu_1(\xi_2)}{\nu_1 - \nu_2} - \cos \varphi \frac{d\nu_2(\xi_1)}{\nu_1 - \nu_2} \right) \xi^\perp + (\nu_2 - \nu_1) \sin \varphi \cos \varphi N. \end{aligned} \quad (21)$$

On the other hand, the equation (15) imply

$$\nabla'_{e_\alpha} \xi = (d\varphi(e_\alpha) - \gamma(e_\alpha)) \xi^\perp, \quad \alpha = 1, \dots, n-2. \quad (22)$$

We denote

$$\begin{aligned} q &= \sin \varphi d\varphi(\xi_1) - \cos \varphi d\varphi(\xi_2) + \sin \varphi \frac{d\nu_1(\xi_2)}{\nu_1 - \nu_2} - \cos \varphi \frac{d\nu_2(\xi_1)}{\nu_1 - \nu_2} \\ \varphi_\alpha &= d\varphi(e_\alpha), \quad \gamma_\alpha = \gamma(e_\alpha), \quad \alpha = 1, \dots, n-2. \end{aligned}$$

Using the unique decomposition of an arbitrary vector field $x \in \Delta_\xi$ in the form

$$x = g(x, e_\alpha) e_\alpha + \omega(x) \xi^\perp$$

we obtain from (21) and (22)

$$\begin{aligned} \nabla'_x \xi &= (\varphi_\alpha - \gamma_\alpha) \omega(x) e_\alpha + g(x, e_\alpha) (\varphi_\alpha - \gamma_\alpha) \xi^\perp \\ &\quad - q \omega(x) \xi^\perp + (\nu_2 - \nu_1) \sin \varphi \cos \varphi \omega(x) N. \end{aligned} \quad (23)$$

From the equality (1) we get

$$\nabla'_x N = -(\nu_1 \sin^2 \varphi + \nu_2 \cos^2 \varphi) \omega(x) \xi^\perp - (\nu_2 - \nu_1) \sin \varphi \cos \varphi \omega(x) \xi. \quad (24)$$

Let us denote $\bar{B} = (\gamma_\alpha - \varphi_\alpha) e_\alpha$ ($\bar{B} \in \Delta_0$) and let B be a unit vector field, such that $\bar{B} = bB$, $b = |\bar{B}|$. If β is the unit one-form, corresponding to B , then

$$b\beta(x) = bg(x, B) = (\gamma_\alpha - \varphi_\alpha)g(x, e_\alpha).$$

Hence, setting $p = \nu_1 \sin^2 \varphi + \nu_2 \cos^2 \varphi$, $\mu = (\nu_2 - \nu_1) \sin \varphi \cos \varphi$ and using (23) and (24), we get

$$\begin{aligned} \nabla'_x N &= -p \omega(x) \xi^\perp - \mu \omega(x) \xi \\ \nabla'_x \xi &= -b \omega(x) B - b \beta(x) \xi^\perp - q \omega(x) \xi^\perp + \mu \omega(x) N \end{aligned}$$

where $p \neq 0$. In general, when $b \neq 0$, Lemma 4 implies that locally M_ξ^{n-1} is a semi-developable surface of codimension two with main normal vector field N . In particular, when $b = 0$, according to Lemma 1 locally M_ξ^{n-1} is a developable surface of codimension two.

Consequently, locally M^n is a one-parameter system of developable or semi-developable surfaces of codimension two. \square

Acknowledgements

The research is financially supported by L. Karavelov Civil Engineering Higher School, Sofia, Bulgaria under Contract # 133/2006.

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