DECOMPOSITION OF FUNCTIONS INTO WAVELETS OF CONSTANT SHAPE, AND RELATED TRANSFORMS*

A. GROSSMANN

CPT II, CNRS-Luminy, Marseille, France

J. MORLET

ORIC (Elf - Aquitaine), Rueil-Malmaison
France

Center for Interdisciplinary Research
and
Research Center Bielefeld-Bochum-Stochastics
University of Bielefeld No. 11

December 1984

Morlet: angeschrieben am 30.4.85

*To be published in: "Mathematics + Physics, Lectures on Recent Results", L. Streit, ed., World Scientific Publishing Co., Singapore.

DECOMPOSITION OF FUNCTIONS INTO WAVELETS OF CONSTANT SHAPE, AND RELATED TRANSFORMS

A. Grossmann
CPT II, CNRS-Luminy, Marseille, France

J. Morlet
ORIC (Elf - Aquitaine), Rueil-Malmaison
France

CONTENTS:

(Contents)

0. INTRODUCTION

These notes describe methods of decomposing a function of one real variable into contributions that are mutually permuted under the action of the two-parameter group of shifts and dilations. Such decompositions are classical for pure shifts (Fourier transform) and for pure dilations (Mellin transform). The simultaneous consideration of both parameters alters drastically the picture, for reasons that are easy to understand from the point of view of group representations: while the action of, say, pure shifts in $L^2(\mathbb{R}, dt)$ is highly reducible, the simultaneous action of shifts and dilations is essentially irreducible.

The group-theoretical background is shortly described in Chapter I, and the results specialized to our case in Chapter II. Chapter III is devoted to "voice decompositions" which are less general but possibly more useful in practice.

Our methods are particularly well adapted to the study of functions of zero average $(\int f(t)dt = 0)$ which occur naturally when one considers the propagation of waves in elastic media.

A few references to related material are: A.L. Carey: Bull. Austral. Math. Soc. 15 (1976) 1-12; J.R. Klauder and Aslaksen: Journ. Math. Phys. 10, 2267 (1967); A. Grossmann and J. Morlet: SIAM Journ. of Math. Analysis 15, 723 (1984); T. Paul: Journ. Math. Phys. 25, 3252 (1984).

1. General R_g - Transforms and L_g - Transforms

1.1 Invariant Measures on a Locally Compact Group

G: locally compact group:

$$x \in G .$$

Left invariant measure:

$$dx: \quad d(yx) = dx \qquad (y \in G).$$

Modular function: $x \to \Delta(x)$, positive-valued, transforming left-invariant into right-invariant measure:

$$\Delta^{-1}(x)dx = d_R x$$

$$d_R(xy) = d_R x \ .$$

$\Delta(x)$ is a positive-valued character: $\Delta(x) > 1$, $\Delta(e) = 1$, $\Delta(x^{-1}) = \frac{1}{\Delta(x)}$, $\Delta(xy) = \Delta(x)\Delta(y)$.

One has:

$$d(x^{-1}) = d_R x \ .$$

1.2 Left and Right Regular Representation

Left regular: If $\Phi \in L^2(G,dx)$ and $a \in G$, define $\lambda(a)\Phi \in L^2(G,dx)$ by

$$(\lambda(a)\Phi)(x) = \Phi(a^{-1}x) \quad (x \in G) \ .$$

Right regular: If $\Psi \in L^2(G,d_R x)$ and $a \in G$, define $\rho(a)\Psi \in L^2(G,d_R x)$ by

$$(\rho(a)\Psi)(x) = \Psi(xa).$$

The two representations act in different spaces, in general, namely $L^2(G,dx)$ and $L^2(G,d_R x)$.

1.3 Square Integrable Representation

Let $x \to U(x)$ be a strongly continuous, unitary, irreducible representation of the locally compact group G in a Hilbert space $H(U)$.

A vector $g \in H(U)$ is said to be admissible if

$$\int |(U(x)g,g)|^2 \, dx < \infty \quad . \tag{1.1}$$

In (1.1), the left-invariant measure dx can be replaced by the right-invariant measure $d_R x$, since $(U(x)g,g) = (g,U(x^{-1})g)$; so

$$\begin{aligned} \int |(U(x)g,g)|^2 dx &= \int |(g,U(x^{-1})g)|^2 \, dx \\ = \int |(U(x^{-1})g,g)|^2 dx &= \int |(U(y)g,g)|^2 d_R y \\ = \int |(g,U(x)g|^2 dx \; . \end{aligned}$$

If there exists in $H(U)$ a non zero admissible vector, the representation U is called <u>square integrable</u>. Any representation unitarily equivalent to U is then also square integrable.

If G is compact, any irreducible representation U of G is square integrable.

We shall see below examples of square integrable representations of groups that are not compact.

As an example of irreducible representation that is <u>not</u> square integrable, consider the one-dimensional representation $x \to e^{iax}$ of $\mathbb{R}$.

1.4 L_g-Transform and R_g-Transform

Let $x \to U(x)$ be a square integrable representation of G. If $g \in H(U)$ is any non zero vector satisfying (1.1), define

$$c_g = \|g\|^{-2} \int |(U(x)g,g)|^2 dx \; . \tag{1.2}$$

Now let f be an arbitrary vector in $H(U)$. Associate with f the two complex-valued functions $L_g f$ and $R_g f$ on G, defined by

$$(L_g f)(x) = \frac{1}{\sqrt{c_g}} (U(x)g,f), \quad (x \in G) \tag{1.3}$$

$$(R_g f)(x) = \frac{1}{c_g} (g,U(x)f) . \tag{1.3'}$$

Clearly

$$(L_g f)(x) = (R_g f)(x^{-1}) . \tag{1.4}$$

Even though $L_g f$ and $R_g f$ are so closely related, it will be useful to consider them both.

The function $L_g f$ will be called the L-transform of f, with respect to g and to the representation U.

Similarly, $R_g f$ will be called the R-transform of f.

For fixed g, the correspondence $f \to L_g f$ is linear.

The correspondence $f \to R_g f$ is also linear. (With our conventions, the inner product (f_1,f_2) is linear in f_2 and antilinear in f_1).

Notice that for $\lambda \in \mathbb{C}$, $\lambda \neq 0$. $\lambda = |\lambda|e^{i\varphi}$,

$$R_{\lambda g} f = e^{-i\varphi} R_g f$$

$$L_{\lambda g} f = e^{-i\varphi} L_g f .$$

1.4 Isometry

The transform L_g is isometric from $H(U)$ into $L^2(G,dx)$.

The transform R_g is isometric from $H(U)$ into $L^2(G,d_R x)$.

In other words: Let f_1 and f_2 be in $H(U)$. Then

$$\int \overline{(L_g f_1)(x)}(L_g f_2)(x)dx = (f_1,f_2) \tag{1.5}$$

and

$$\int \overline{(R_g f_1)(x)}\ (R_g f_2)(x) d_R x = (f_1, f_2)\ . \qquad (1.6)$$

These isometry properties will be basic in many applications. They imply in particular that, on their ranges, the transforms L_g and R_g are inverted by their adjoints.

For a proof of (1.5), (1.6), and of more general "orthogonality relations", see e.g. the paper by Carey quoted above or a Marseille preprint by A. Grossmann, J. Morlet and T. Paul, submitted to J.M.P.

1.6 Intertwining

We have

$$\lambda(a) L_g = L_g\ U(a)$$

$$\rho(a) R_g = R_g\ U(a)\ .$$

In words: L_g intertwines U and the left regular representation. R_g intertwines U and the right regular representation.

1.7 Characterization of the Range of L_g and of R_g

a) Reproducing equation for functions in the range of L_g:
Define:

$$p_g(a) = \frac{1}{c_g}\,(U(a)g, g) \qquad (a \in G). \qquad (1.7)$$

Then, the necessary and sufficient condition for a function $\Phi \in L^2(G, dx)$ to be in the range of L_g, is

$$\Phi(a) = \int p_g(a'^{-1}a)\Phi(a')da'\ . \qquad (1.8)$$

b) Reproducing equation for functions in the range of R_g:
The necessary and sufficient condition for a function $\Psi \in L^2(G, dx_R)$ to be in the range of R_g is

$$\Psi(a) = \int p_g(a' \; a^{-1})\Psi(a')d_R a' \; . \tag{1.9}$$

1.8 Left-Invariant Interpolation

Let $a_1,\ldots,a_n$ be n elements of G. Assume that they have been chosen so that the matrix $A = \{A_{ij}\}$ $(i,j = 1,\ldots,n)$, with

$$A_{ij} = p_g((a_i)^{-1}a_j) \tag{1.10}$$

is invertible. Denote by B the matrix inverse of A:

$$B = A^{-1} \; .$$

Let $\phi_1,\ldots,\phi_n$ be any n complex numbers. Consider the function $h(x)$ defined by

$$h(x) = \sum_{i,j=1}^{n} \phi_i \; B_{ij} \; p_g(a_j^{-1} \, x) \quad (x \in G) \; . \tag{1.11}$$

Then:

(i) For $i = 1,\ldots,n$ we have

$$h(a_i) = \phi_i \tag{1.12}$$

i.e. the function $h(x)$ takes the preassigned values ϕ_i on the preassigned points a_i .

(ii) The function h belongs to the closed subspace $L_g H(U) = PL^2(G;dx) \subseteq L^2(G,dx)$ discussed in Sec. 1.7.

(iii) Among the functions that belong to $LH(U)$ and solve the interpolation problem, h has minimal norm: if $h^{(\mathrm{other})}$ belongs to $LH(U)$ and satisfies (1.12), then one has

$$\|h\| \le \|h^{(\mathrm{other})}\| \; .$$

(iv) Finally, consider the left-shifted interpolation problem. If b is any element of G, consider

$$a_1' = b\,a_1, \ldots, a_n' = b a_n \ .$$

Then the interpolation problem $h'(a_i') = \phi_i'$ $(i = 1,\ldots,n)$ is solved by

$$h'(x) = \sum_{i,j=1}^{n} \phi_i' \, B_{ij} \, p_g((a_j')^{-1} x)$$

with the same matrix B, i.e. by

$$h'(bx) = \sum_{i=1}^{n} \phi_i' \, B_{ij} \, p_g(a_j^{-1} x) \ .$$

The r.h.s. of this equation does not contain b; the interpolation procedure is invariant under group multiplication from the left.

In other words, the formula (1.11) gives the projection of $L_g H(U)$ on the subspace spanned by the functions $p_g(a_j^{-1} x)$ (corresponding to the function $U(a_j)\phi$ in $H(U)$).

1.9 Right-Invariant Interpolation

The notations being as above, consider the matrix C

$$C_{ij} = p_g(a_i \, a_j^{-1}) \ .$$

Assume that C is invertible, and define

$$D = C^{-1}.$$

Then the function

$$r(x) = \sum_{i,j=1}^{n} \phi_i \, D_{ij} \, p_g(x \, a_j^{-1})$$

belongs to $R_g H \subseteq L^2(G, d_R x)$, satisfies the interpolation property (1.12), and is of minimal norm. This interpolation procedure is invariant under right shifts.

1.10 Bounds

By the unitarity of $U(x)$ we have

$$|(L_g f)(x)| \leq c_g^{-1/2} \|f\| \quad \|g\|$$

$$|(R_g f)(x)| \leq c_g^{-1/2} \|f\| \quad \|g\|$$

for every $x \in G$.

If

$$g = g_o + g_1$$

- where g_o and g_1 need not be admissible - and if

$$\|g_1\| \leq \varepsilon_1 \quad (\varepsilon_1 > 0)$$

then:

$$(L_g f)(x) = c_g^{-1/2}(U(x)g_o, f) + c_g^{-1/2}(U(x)g_1, f)$$

and consequently

$$|(L_g f)(x) - c_g^{-1/2}(U(x)g_o, f)| \leq c_g^{-1/2} \|f\|$$

for every $x \in G$. Similarly for $R_g f$.

If

$$f = f_o + f_1$$

with $\|f_1\| \leq \varepsilon$, then $(L_g f)(x) = (L_g f_o)(x) + (L_g f_1)(x)$, and so

$$|(L_g f)(x) - (L_g f_o)(x)| \le c_g^{-1/2} \|g\| \varepsilon$$

and

$$|(R_g f)(x) - (R_g f_o)(x)| \le c_g^{-1/2} \|g\| \varepsilon .$$

2. A SPECIAL CASE: SHIFTS AND DILATIONS

2.1 The Group of Shifts and Dilations

Consider the set G of all pairs $\{x,y\}$, with $x \in \mathbb{R}$, $y \in \mathbb{R}$, $y \neq 0$. The pair $\{x,y\}$ is the element $x \in G$ of the preceding sections. Define a product

$$\{x_1,y_1\}\ \{x_2,y_2\} = \{y_1x_2 + x_1, y_1y_2\} \tag{2.1}$$

and the inverse

$$\{x,y\}^{-1} = \{-y^{-1}x,\ y^{-1}\} . \tag{2.2}$$

With these operations, G becomes a group.

The identity (unit element in the group) is $\{0,1\}$.

We have

$$\{x_1,y_1\}^{-1}\{x_2,y_2\} = \{\frac{x_2-x_1}{y_1}\ ,\ \frac{y_2}{y_1}\} \tag{2.3}$$

$$\{x_1,y_1\}\{x_2,y_2\}^{-1} = \{\frac{x_1y_2-y_1x_2}{y_2}\ ,\ \frac{y_1}{y_2}\} \tag{2.4}$$

$$\{x_1,y_1\}\{x_2,y_2\}\{x_1,y_1\}^{-1}\{x_2,y_2\}^{-1} = \{x_1-x_2+y_1x_2-x_1y_2\ ,\ 1\} . \tag{2.4'}$$

In $\{x,y\}$ we call x the shift parameter, and y the dilation parameter. The law (2.1) corresponds to the multiplication of inhomogeneous linear transformations $t \to yt + x$ in one dimension.

Notice that y is allowed to be negative.

The set of pairs $\{x,y\}$ with $x \in \mathbb{R}$ and $y > 0$ is a normal subgroup $G_+ \subset G$ with $G_+/G = Z_2$ (the two-element group). The group G_+, contains the normal subgroup of translations, consisting of all the pairs $\{x,1\}$ $(x \in \mathbb{R})$.

G_+ is known as the "ax + b" - group.

The left-invariant measure in G is

$$d\{x,y\} = \frac{dx\ dy}{y^2} \quad . \tag{2.5}$$

The right-invariant measure in G is

$$d_R\{x,y\} = \frac{dx\ dy}{|y|} \quad . \tag{2.6}$$

Comparison between left and right action:

a) Left multiplication by $\{b,a\}$ =

$$\{b,a\}\{x,y\} = \{ax+b,\ ay\} \ .$$

b) Right multiplication by $\{b,a\}$

$$\{x,y\}\{b,a\} = \{by+x,\ ay\} \ .$$

In the usual metric of the $\{x,y\}$-plane, left multiplication stretches or compresses a set on a line y = const. and moves it to a line y = a const. The right multiplication does not stretch or compress lines y = const. The term by is an additive constant.

2.2 Shifts and Dilations as Operators; the Representation U

a) Notations: Fourier transform

F: $(Ff(\omega) = (2\pi)^{-1/2}\int f(t)e^{-i\omega t}\,dt$, also written as $\tilde{f} = Ff$
$(f \in L^2(\mathbb{R},dt))$ · (2.7)

$$F^{-1}: (F^{-1}\tilde{f})(t) = (2\pi)^{-1/2}\int f(\omega)e^{i\omega t}\, d\omega \; . \qquad (2.8)$$

Convolution:

$$(f*g)(t) = \int f(t-x)g(x)dx \; . \qquad (2.9)$$

If $g \in L^2(\mathbb{R},dt)$, and $\|g\|_1 = \int |g(t)|dt$, then, for every p $(1 \le p \le \infty)$, the assumption $f \in L^p(\mathbb{R},dt)$ implies $f*g \in L^p(\mathbb{R},dt)$, and

$$\|f*g\|_p \le \|g\|_1 \; \|f\|_p \; . \qquad (2.10)$$

$$F(f*g) = (2\pi)^{1/2}(Ff)(Fg) \; .$$

Shift operator:

$$T^x: \quad (T^x f)(t) = f(t-x) \quad (x \in \mathbb{R}) \; .$$

Multiplication by exponential:

$$E^x: \quad (E^x f)(t) = e^{ixt}f(t) \quad (x \in \mathbb{R}) \; .$$

Unitary dilation operator (with possible parity):

$$D^y: \quad (D^y f)(t) = |y|^{-1/2}f(y^{-1}t) \; (y \in \mathbb{R}, \; y \neq 0).$$

b) Formulas:

All the above operators are unitary. They satisfy

$$\begin{aligned} T^x T^{x'} &= T^{x+x'} , \; T^{-x} = (T^x)^* = (T^x)^{-1} \\ E^x E^{x'} &= E^{x+x'} , \; E^{-x} = (E^x)^* = (E^x)^{-1} \\ D^y D^{y'} &= D^{yy'} , \; D^{1/2} = (D^y)^* = (D^y)^{-1} \end{aligned} \qquad (2.11)$$

D^{-1} is the parity operator.

Products:

$$E^bT^x = e^{ibx}\, T^x\, E^b \,, \qquad T^x\, E^b = e^{-ibx}\, E^b\, T^x \tag{2.12}$$

$$FT^x = E^{-x}\, F \,, \qquad F\, E^b = T^b\, F \tag{2.13}$$

$$F^{-1}\, T^x = E^x\, F^{-1} \,, \qquad F^{-1}\, E^b = T^{-b}\, F^{-1} \tag{2.13'}$$

$$D^y\, T^x = T^{xy}\, D^y \,, \qquad T^x\, D^y = D^y\, T^{x/y} \tag{2.14}$$

$$D^y\, E^b = E^{b/y}\, D^y \,, \qquad E^b\, D^y = D^y\, E^{by} \tag{2.15}$$

$$F\, D^y = D^{1/y}\, F \,, \qquad F^{-1}\, D^y = D^{1/y}\, F^{-1} \tag{2.16}$$

$$F^2 = D^{(-1)} \,. \tag{2.17}$$

Writing out:

$$(D^y\, T^x\, f)(t) = |y|^{-1/2}\, f(\frac{t}{y} - x) \,, \; (T^x\, D^y\, f)(t) = |y|^{-1/2}\, f(\frac{t-x}{y}).$$

Relationship with complex conjugation: ($\bar{z}$ is complex conjugate of z)

$$\overline{T^x f} = T^x \qquad \overline{D^y f} = D^y \bar{f} \tag{2.18}$$

$$\overline{E^b f} = E^{-b} \bar{f} \tag{2.19}$$

$$\overline{Ff} = D^{(-1)} F \bar{f} = F D^{(-1)} \bar{f} = F \check{\bar{f}} \tag{2.20}$$

where $\check{f}(t) = f(-t)$.

It will be convenient to introduce the function $f^*(t)$ defined by

$$f^*(t) = \bar{f}(-t).$$

So

$$\overline{Ff} = Ff^* \,.$$

Remark on dilations: Imagine $f(t)$ as the air pressure near a loud-speaker playing a record. Then $(D^{(1/2)}f)(t)$ corresponds to the record being played at twice the speed (so it will be heard an octave higher; but this is not transposition in the sense of a musical instrument). $D^{(-1)}f$ corresponds to the tape being played in reverse.

b) The representation U:

Define

$$U(x,y) = T^x D^y \text{ , i.e. } (U(x,y)f)(t) = |y|^{-1/2} f(\tfrac{t-x}{y}) \text{ .} \tag{2.21}$$

Then

$$\tilde{U}(x,y) = F^{-1}U(x,y)F = E^x D^{1/2} \text{ i.e. } (\tilde{U}(x,y)\tilde{f})(\omega) = |y|^{1/2} e^{i\omega x}\tilde{f}(y\omega) \text{ .} \tag{2.22}$$

Notice that

$$(D^{(-1)})^{-1} = D^{(-1)} \text{ , } D^{(-1)}T^x D^{(-1)} = T^{-x} \text{ .} \tag{2.23}$$

The representation U is irreducible.

2.3 The Admissibility Condition

A) We have defined the representation $U(x,y) = T^x D^y$ of G in $L^2(\mathbb{R},dt)$ and the unitarily equivalent representation $\tilde{U}(x,y) = F^{-1}U(x,y)F = E^x D^{1/y}$ in $L^2(\mathbb{R},d\omega)$. In order to determine whether this representation is square integrable in the sense of Section 1.3 we have to write down the admissibility condition (1.1) with x replaced by the pair $\{x,y\}$ and dx by $dxdy/y^2$.

A straightforward calculation shows that

$$\iint |(T^x D^y g,g)|^2 \frac{dxdy}{y^2} = \iint |(E^x D^{1/y}\tilde{g},\tilde{g})|^2 \frac{dxdy}{y^2} = 2\pi \, \|g\|^2 \int |\tilde{g}(\omega)|^2 \frac{d\omega}{|\omega|}$$

and we obtain

Proposition: The representation (2.21) is square integrable. A function $g \in L^2(\mathbb{R}, dt)$ is admissible if and only if

$$\int \frac{|\tilde{g}(\omega)|^2}{|\omega|} d\omega < \infty \quad . \tag{2.25}$$

The number c_g defined, in general, by (1.2) is now

$$c_g = 2\pi \int \frac{|\tilde{g}(\omega)|^2}{|\omega|} d\omega \quad . \tag{2.26}$$

Remark: We see that admissibility is a restriction on the behavior of $\tilde{g}$ around zero frequency. In particular, if $g \in L^1(\mathbb{R}, dt)$ then admissibility implies

$$\tilde{g}(0) = \int g(t)dt = 0 \quad . \tag{2.27}$$

Conversely, if (2.27) holds and if $\tilde{g} \in L^2(\mathbb{R}, d\omega)$ has a bounded derivative, then g is admissible.

2.4 L_g and R_g - Transform

L_g-transform: Assume that the function $g \in L^2(\mathbb{R}, dt)$ satisfies (2.25). Define c_g by (2.26). To every function $f \in L^2(\mathbb{R}, dt)$, associate the function $(L_g f)(x,y)$ $(x \in \mathbb{R},\ y \in \mathbb{R},\ y \neq 0)$ defined by

$$(L_g f)(x,y) = (c_g)^{-1/2}(T^x D^y g, f) = (c_g)^{-1/2}(E^{-x} D^{1/y} \tilde{g}, \tilde{f}) \tag{2.28}$$

that is,

$$\begin{aligned} (L_g f)(x,y) &= (c_g)^{-1/2} |y|^{-1/2} \int \bar{g}((t' - x)/y) f(t') dt' \\ &= (c_g)^{-1/2} |y|^{-1/2} \int g^*((x - t')/y) f(t') dt' \\ &= (c_g)^{-1/2} |y|^{1/2} \int e^{i\omega x}\, \bar{\tilde{g}}(\omega y) f(\omega) d\omega \end{aligned} \tag{2.29}$$

with $g^*(t) = \bar{g}(-t)$.

For fixed y, $L_g f$ can be written as a convolution:

$$(L_g f)(\cdot,y) = (c_g)^{-1/2}(D^y g^*)*f \ . \tag{2.30}$$

It is important to notice that $(L_g f)(x,y)$ - and also $(R_g f)(x,y)$ - can be expressed either in terms of $f(t)$ or in terms of $\tilde{f}(\omega)$. These functions are sensitive to concentration or support properties both in the "time domain" and the "frequency domain". This will be seen in more detail in Section 2.6.

a') $\underline{R_g\text{-transform}}$: Define, in accordance with (1.31), $R_g f$ as

$$\begin{aligned}(R_g f)(x,y) &= c_g^{-1/2}(g,T^x D^y f) = (L_g f)(-\tfrac{x}{y},\tfrac{1}{y}) \\ &= c_g^{-1/2}\ (D^y E^x \tilde{g},\tilde{f}) \\ &= c_g^{-1/2}|y|^{1/2}\int \overline{g}(t'y+x)f(t')dt' \\ &= c_g^{-1/2}|y|^{1/2}\int g^*(-x-t'y)f(t')dt' \\ &= c_g^{-1/2}|y|^{-1/2}\int e^{-i\omega x/y}\,\overline{\tilde{g}}(\tfrac{\omega}{y})f(\omega)d\omega \ .\end{aligned} \tag{2.31}$$

For fixed y, $R_g f$ can be written as

$$(R_g f)(x,y) = c_g^{-1/2}((D^{1/y}g^*)*f)(-\tfrac{x}{y}) \ . \tag{2.32}$$

The function $L_g f$ is expressed in terms of $R_g f$ as

$$(L_g f)(x,y) = (R_g f)(-\tfrac{x}{y},\tfrac{1}{y}) \ .$$

b) Simplest properties: (i) $L_g f$ and $R_g f$ depend linearly on f. If λ is real and non zero, then $L_{\lambda g} f = L_g f$, and $R_{\lambda g} f = R_g f$. (ii) If $\tilde{g}$ is real and f is real, then, by (2.29),

$$\overline{(L_g f)(x,y)} = (L_g f)(x,-y)$$

and

$$(R_g f)(x,y) = (R_g f)(x,-y)$$

i.e. it is enough to know, say, $(L_g f)(x,y)$ on the half-plane $y > 0$.

(iii) If g and f are real, then $L_g f$ and $R_g f$ are real.

(iv) If g and f are Hardy, (i.e. $\tilde{g}(\omega) = 0$ for $\omega < 0$ and $\tilde{f}(\omega) = 0$ for $\omega < 0$) then $L_g f$ and $R_g f$ vanish in the half-plane $y < 0$.

c) <u>Isometry of L_g and R_g</u>: The general isometry results (1.5) read now

$$\|f\|^2 = \iint |(L_g f)(x,y)|^2 \frac{dxdy}{y^2} = \iint |(R_g f)(x,y)|^2 \frac{dxdy}{|y|} \tag{2.33}$$

and the corresponding equalities for scalar products:

$$\begin{aligned}\int \overline{f_1(t)} f_2(t)dt &= \iint \overline{(L_g f_1)(x,y)}(L_g f_2)(x,y) \frac{dxdy}{y^2} \\ &= \iint \overline{(R_g f_1)(x,y)}(R_g f_2)(x,y) \frac{dxdy}{|y|} \\ &= \int \overline{\tilde{f}_1(\omega)} \tilde{f}_2(\omega)d\omega .\end{aligned} \tag{2.33'}$$

d) <u>Inversion of L_g and of R_g</u>

L_g and R_g are isometric operators. Consequently each of them is inverted, <u>on its range</u>, by its adjoint. This gives, formally,

$$\begin{aligned} f(t) &= (c_g)^{-1/2} \iint (L_g f)(x,y) |y|^{-1/2} g((t-x)/y) \frac{dxdy}{y^2} \\ &= (c_g)^{-1/2} \iint (L_g f)(x,y) |y|^{-5/2} g((t-x)/y) dxdy \end{aligned} \tag{2.34}$$

$$\begin{aligned} \tilde{f}(\omega) &= (c_g)^{-1/2} \iint (L_g f)(x,y) |y|^{1/2} e^{-i\omega x} g(\omega y) \frac{dxdy}{y^2} \\ &= (c_g)^{-1/2} \iint (L_g f)(x,y) |y|^{-3/2} e^{-i\omega x} \tilde{g}(\omega y)\ dxdy \end{aligned} \tag{2.35}$$

$$f(t) = (c_g)^{-1/2} \iint (R_g f)(x,y)|y|^{1/2}\ g(ty+x)\ \frac{dxdy}{|y|}$$

$$= (c_g)^{-1/2} \iint (R_g f)(x,y) g(ty+x)\ \frac{dxdy}{|y|^{1/2}} \tag{2.36}$$

$$\tilde{f}(\omega) = (c_g)^{-1/2} \iint (R_g f)(x,y)|y|^{-1/2}\ e^{i\omega x/y}\ g(\tfrac{\omega}{g})\ \frac{dxdy}{|y|}$$

$$= (c_g)^{-1/2} \iint (R_g f)(x,y)|y|^{-3/2}\ e^{-i\omega x}\ g(\tfrac{\omega}{y})\ dxdy\ . \tag{2.37}$$

The integrals (2.34) to (2.37) may fail to converge at some points. By (2.33'), however, they give rise to convergent expression when "tested" with any square integrable function.

In particular, of for any $\delta > 0$ we consider the averaged values

$$f_\delta(t) = \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} f(t')dt' = \frac{1}{2\delta}\ (\chi_t^\delta, f)$$

with χ_t^δ, the characteristic function of the interval $[t-\delta, t+\delta]$.

Similarly, consider

$$g_\delta(t,x,y) = \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} g((t'-x)/y)dt' = \frac{1}{2\delta}\ (L_g \chi_t^\delta)(x,y);$$

then (2.34) is replaced by

$$f_\delta(t) = c_g^{-1/2} \iint (L_g f)(x,y)|y|^{-5/2}\ g_\delta(t,x,y) dxdy\ . \tag{2.38}$$

Similar results hold for (2.35) to (2.37).

e) $\underline{L_g}$-Transform of Shifted and Dilated Functions:

We have

$$(L_g T^{t_o} f)(x,y) = c_g^{-1/2}(T^x D^y g, T^{t_o} f) = c_g^{-1/2}(T^{x-t_o} D^y g, f) = (L_g f)(x-t_o, y), (t_o \in R). \tag{2.39}$$

If the graph of the function f is shifted, say, to the right, the function $L_g f$ is shifted by the same amount along the x-axis in the x,y-plane.

For dilations, we have

$$(L_g D^z f)(x,y) = c_g^{-1/2}(T^x D^y g, D^z f) = c_g^{-1/2}(T^{x/z} D^{y/z} g, f)$$

$$= (L_g f)(x/z, y/z) \quad (x \in \mathbb{R},\ y \in \mathbb{R},\ y \neq 0,\ z \in \mathbb{R}, z \neq 0). \tag{2.40}$$

If the function f is, say, raised by an octave ($z = 1/2$), then the values of the function $L_g f$ are pulled in by a similarity transform which halves all the vectors of the x,y-plane.

f) R_g-Transform of Shifted and Dilated Functions:

If f is shifted, we have

$$(R_g T^t f)(x,y) = c_g^{-1/2}(g, T^x D^y T^t f)$$

$$= c_g^{-1/2}(g, T^{x+yt} D^y f) = (R_g f)(x+yt, y) . \tag{2.41}$$

If f is dilated, we obtain

$$(R_g D^z f)(x,y) = c_g^{-1/2}(g, T^x D^y D^z f) = (R_g f)(x, yz) . \tag{2.42}$$

If the function f is raised by an octave, then $R_g f$ is spread out along the direction of the y-axis, by a factor of two.

All these transformations are unitary, i.e. do not change the total energy, which is proportional to $\iint |(L_g f)(x,y)|^2 \frac{dxdy}{y^2} = \iint |(R_g f)(x,y)|^2 \frac{dxdy}{|y|}$.

g) The Function p_g and Reproducing Equations

The function p_g, defined in Section 1.7 for an arbitrary group and any square integrable representation, is now

$$p_g(x,y) = \frac{1}{c_g}(T^x D^y g,g) = \frac{1}{c_g}|y|^{-1/2}\int \overline{g}(\frac{t-x}{y})g(t)dt =$$

$$= \frac{1}{c_g}(E^{-x} D^{1/y}\tilde{g},\tilde{g}) = \frac{1}{c_g}|y|^{1/2}\int e^{i\omega x}\,\overline{\tilde{g}}(\omega y)\tilde{g}(\omega)d\omega \;. \quad (2.43)$$

For $y = 1$, one has

$$p_g(x,1) = \frac{1}{c_g}\int \overline{g}(t-x)g(t)dt = \frac{1}{c_g}\int e^{i\omega x}\,|\tilde{g}(\omega)|^2\,d\omega \;. \quad (2.44)$$

Then: A function $\Phi(x,y) \in L^2(G, \frac{dxdy}{y^2})$ is in the range of L_g if and only if it satisfies, for every x,y, the equation

$$\Phi(x,y) = \iint p_g(\frac{x-x'}{y'}, \frac{y}{y'})\,\Phi(x',y')\,\frac{dx'dy'}{(y')^2} \quad . \quad (2.45)$$

This follows from (1.8) and (2.3).

A function $\Psi(x,y) \in L^2(G, \frac{dxdy}{|y|})$ is in the range of R_g if and only if it satisfies, for every x,y, the equation

$$\Psi(x,y) = \iint p_g(\frac{x'y-y'x}{y}, \frac{y'}{y})\,\Psi(x',y')\,\frac{dx'dy'}{|y'|} \quad . \quad (2.46)$$

This follows from (1.9) and (2.4).

2.5 Cycle-Octave Transform:

In view of the property (2.42) (behaviour of $R_g f$ under dilations of f), it is natural to work with the logarithm of the dilation parameter. We choose base 2 for convenience in numerical implementation, and by analogy with the terminology of music.

The notations and assumptions being as above, associate to $f \in L^2(\mathbb{R},dt)$ the pair of functions

$$(R_g^{(+)}f)(v,u) = (R_g f)(-v,2^u) \quad (2.47)$$

$$(R_g^{(-)}f)(v,u) = (R_g f)(v,-2^u) \qquad (2.48)$$

$$(v \in \mathbb{R},\ u \in \mathbb{R}) .$$

The pair $R_g^+ f, R_g^- f$ is called the cycle-octave transform of f with respect to g.

If f and g contain only positive frequencies, then $R_g^- f$ vanishes identically.

Suppose now that f is raised by an octave. Then $R_g^+ f$ is shifted upward by 1 :

$$(R_g^+ D^{1/2} f)(v,u) = (R_g^+ f)(v,u-1) . \qquad (2.49)$$

The - sign in the first argument of (2.47) has no intrinsic significance; it has been chosen to give customary behaviour under shifts of f.

An important property of the cycle-octave transform is related to the problem of storing only the values of the transform at a discrete set of points and reconstructing the original function from these values. It turns out that the appropriate discrete sets are best described in the cycle-octave parametrization. These matters will be discussed elsewhere.

Digital

2.6 Support and Concentration Properties:

a) Dilation Parameter

Under suitable assumptions on f and g, the functions $(L_g f)(x,y)$ and $(R_g f)(x,y)$ are non-negligible only in certain domains of the cut x,y-plane.

We consider first the limitations on the dilation parameter y. Fix $y_0 \neq 0$. From the last equation (2.29) and (2.31), one can obtain the integral of $|L_g f(x,y_0)|^2$ and of $|(R_g f)(x,y_0)|^2$ along the horizontal line $y = y_0$:

$$\int |(L_g f)(x,y_0)|^2 \, dx = \frac{2\pi}{c_g} |y_0| \int |\tilde{g}(y_0\omega)|^2 \; |\tilde{f}(\omega)|^2 \, d\omega \tag{2.52}$$

$$\int |(R_g f)(x,y_0)|^2 dx = \frac{2\pi}{c_g |y_0|} \int |g(\frac{\omega}{y_0})|^2 \; |\tilde{f}(\omega)|^2 d\omega \; . \tag{2.53}$$

In concrete applications, the function $\tilde{g}$ i. explicitly known (compare Section 2.5), and so "a priori" bounds on $\tilde{f}$ can be translated into estimates of the quantities (2.52) and (2.53).

We shall consider here only the extreme case where $\tilde{f}$ and $\tilde{g}$ have compact supports. The results below can be extended with the help of bounds in Section 1.10.

Proposition: Assume that $\tilde{g}(\omega)$ vanishes outside the compact set $K_{\tilde{g}}$ which does not contain the point $\omega = 0$. Assume similarly that $\tilde{f}$ vanishes outside the compact set $K_{\tilde{f}}$ which does not contain the point $\omega = 0$. Then $(L_g f)(x,y_0)$ vanishes for every x, if y_0 is outside the compact set $K_{\tilde{g}}/K_{\tilde{f}}$ (the set of all ratios ω/ω' with $\omega \in K_{\tilde{g}}$ and $\omega' \in K_{\tilde{f}}$). Similarly, $(R_g f)(x,y_0)$ vanishes for all x, if y_0 is outside the compact set $K_{\tilde{f}}/K_{\tilde{g}}$.

Expressed simpler:

If $(L_g f)(x,y)$ is to be different from zero, the dilation parameter y must be a ratio of a frequency occurring in g and of a frequency occurring in f.

If $(R_g f)(x,y)$ is to be different from zero, the dilation parameter y must be a ratio of a frequency occurring in f and of a frequency occurring in g.

b) Shift Paramter

Consider now the opposite case, and assume that f and g have compact support. The corresponding limitation on the shift parameter can be read from (2.30) and (2.32):

Assume that $g^*(t) = \bar{g}(-t)$ vanishes outside the compact set K_{g^*}, and that $f(t)$ vanishes outside the compact set K_f. Then

$(L_g f)(x,y)$ vanishes if x is outside the compact set $K_f + yK_{g*}$, and $(R_g f)(x,y)$ vanishes if x is outside the compact set $-yK_f - K_{g*} = -yK_f + K_g$.

The support properties just described become concentration estimates under more general assumptions on f and g.

3. VOICE DECOMPOSITIONS

$g \in L^1 \cap H^{1/2}$

3.1 Strengthened Admissibility Conditions

We consider now "analyzing wavelets" g such that

$$g \in L^1(\mathbb{R}, dt) \cap L^2(\mathbb{R}, dt) \tag{3.1}$$

and that

$$\tilde{g} \in L^1(\mathbb{R}\setminus\{0\}, \frac{d\omega}{|\omega|}) \cap L^2(\mathbb{R}\setminus\{0\}, \frac{d\omega}{|\omega|}) . \tag{3.2}$$

The requirements $g \in L^2(\mathbb{R}, dt)$ and $\tilde{g} \in L^2(\mathbb{R}\setminus\{0\}, \frac{d\omega}{|\omega|})$ ensure that g is admissible in the sense of Section 2.3. The condition $\tilde{g} \in L^1(\mathbb{R}\setminus\{0\}, \frac{d\omega}{|\omega|})$ enables us to define the number

$$k_g = \sqrt{2\pi} \int \frac{\overline{\tilde{g}(\omega)}}{|\omega|} \, d\omega \tag{3.3}$$

which will play an important role in this section. Notice that k_g is, in general, not equal to the multiple of the norm of $\tilde{g}$ in $L^1(\mathbb{R}\setminus\{0\}; \frac{d\omega}{\omega})$, namely $\sqrt{2\pi} \int \frac{|\tilde{g}(\omega)|}{|\omega|} \, d\omega$. This is the case if $\tilde{g}(\omega)$ is positive.

From now on, we shall assume that g satisfies (3.1), (3.2) and that $k_g \neq 0$.

3.2 V_g - Transform:

a) Definition: For any $f \in L^2(\mathbb{R}, dt)$ define $V_g f$ as the function of two variables

$$(V_g f)(t,y) = \frac{1}{k_g} |y|^{-1} \int \overline{g}\left(\frac{t'-t}{y}\right) f(t')dt' \qquad (3.4)$$

$$(t \in \mathbb{R},\ y \in \mathbb{R}\setminus\{0\}) .$$

The function $V_g f$ will be called the V_g-transform of f (with respect to g).

b) Comparison with the L_g - transform: For every $f \in L^2(\mathbb{R}, dt)$ one has

$$(V_g f)(t,y) = \frac{\sqrt{c_g}}{k_g} |y|^{-1/2} (L_g f)(t,y) . \qquad (3.5)$$

This is seen immediately, by comparing (3.5) and (2.29). The same comparison shows that $(V_g f)(t,y)$ is also

$$(V_g f)(t,y) = \frac{1}{k_g} \int e^{i\omega t}\, \overline{\tilde{g}}(\omega y) \tilde{f}(\omega) d\omega \qquad (3.6)$$

$$(t \in \mathbb{R},\ y \in \mathbb{R}\setminus\{0\}).$$

c) An isometry property: From (3.5) and (2.33) one sees that

$$\|f\|^2 = \frac{k_g^2}{c_g} \iint |(V_g f)(t,y)|^2 \frac{dtdy}{|y|} . \qquad (3.7)$$

3.3 Voice Decomposition Formula

a) Statement

The reason for introducing $V_g f$ is an extremely simple formula which allows to reconstitute f from $V_g f$, and to decompose f into contributions from disjoint ranges of the dilation parameter. Such contributions will be called voices; they merge to recreate f by addition. We shall see that, in the terminology of music, a voice has some concentration properties but no sharp cut-off in frequency.

The advertized formula is

$$f(t) = \int (V_g f)(t,y) \frac{dy}{|y|} \tag{3.8}$$

with suitable definition of convergence.

b) Derivation of the formula

i) Consider first $(V_g f)(t,y)$ as a function of t, for fixed $y \neq 0$. The definition (3.4) can be written as a convolution:

$$(V_g f)(\cdot,y) = \frac{1}{k_g} |y|^{-1/2} (D^y g^*) * f \tag{3.9}$$

with $g^*(t) = \bar{g}(-t)$. By (2.10) and by $g \in L^1(\mathbb{R}, dt)$, if $f \in L^p(\mathbb{R}, dt)$, it follows that $(V_g f)(\cdot,y) \in L^p(\mathbb{R}, dt)$ $(1 \leq p \leq \infty)$. In particular, $f \in L^2(\mathbb{R}, dt)$ gives $(V_g f)(\cdot,y) \in L^2(\mathbb{R}, dt)$. In (3.9), one can then take the Fourier transform with respect to t. Defining

$$(V_g f)^{\sim} (\omega,y) = (2\pi)^{-1/2} \int e^{-i\omega t} (V_g f)(t,y) dt \tag{3.10}$$

and using $FD^y = D^{1/y}F$, one obtains, by (2.11)

$$(V_g f)^{\sim}(\omega,y) = \frac{\sqrt{2\pi}}{k_g} \bar{\tilde{g}}(\omega y) \tilde{f}(\omega) \tag{3-11}$$

for almost every ω.

ii) One has to notice now that the integral

$$\int \bar{\tilde{g}}(\omega y) \frac{dy}{|y|} \tag{3.12}$$

is independent of ω. This can be verified directly or deduced from the fact that $\frac{dy}{|y|}$ is the invariant measure for the (one-parameter, two connected components) group of dilations with inversion. Consequently (3.12) is equal to its value at $\omega = 1$, which is $\frac{k_g}{\sqrt{2\pi}}$ by (3.3).

One obtains then, by multiplying (3.11) with $|y|^{-1}$ and integrating with respect to y,

$$\tilde{f}(\omega) = \int (V_g f)^{\sim}(\omega,y) \frac{dy}{|y|} \quad . \tag{3.13}$$

The assertion (3.8) is, formally, the inverse Fourier transform of (3.13). The convergence of (3.8) will be discussed elsewhere.

3.4 Relationship Between Band-Limited and Voice Limited Approximations

By (3.6) or (3.11), one has

$$\int |(V_g f)(t,y)|^2 \, dt = \frac{2\pi}{|k_g|^2} \int |\tilde{g}(\omega y)|^2 \, |\tilde{f}(\omega)|^2 \, d\omega \tag{3.14}$$

which shows (as in Section 2.6), that $(V_g f)(t,y)$ is non-negligible only if the dilation parameter connects the support of $\tilde{g}$ and the support of $\tilde{f}$.

We consider now a compact domain $Y \subset \mathbb{R}\backslash\{0\}$ of the dilation parameter and ask how well f is approximated by

$$f_Y(t) = \int_Y (V_g f)(t,y) \frac{dy}{|y|} \, . \tag{3.15}$$

One starts again from (3.11). Integration over the compact set Y gives

$$\int_Y (V_g f)^{\sim}(\omega,y) \frac{dy}{|y|} - \tilde{f}(\omega) = r_Y(\omega)\tilde{f}(\omega) \tag{3.16}$$

where

$$r_Y(\omega) = 1 - \frac{\sqrt{2\pi}}{k_g} \int_Y \bar{\tilde{g}}(\omega y) \frac{dy}{|y|} \tag{3.17}$$

depends only on g. For any fixed ω, $r_Y(\omega)$ tends to zero as Y grows toward $\mathbb{R}\backslash\{0\}$. This convergence is uniform on compact subsets of the ω-axis.

Let now Ω be a compact subset of the ω-axis. The band limited approximation f^Ω of $f \in L^2(\mathbb{R}, dt)$ is defined as

$$f^\Omega(t) = (2\pi)^{-1/2} \int_\Omega e^{i\omega t} \tilde{f}(\omega) d\omega \ . \tag{3.18}$$

If Y is a compact subset of $\mathbb{R}\setminus\{0\}$ (the range of the dilation parameter) then the voice limited approximation f_Y of $f \in L^2(\mathbb{R}, dt)$ is defined as

$$f_Y(t) = \int_Y (V_g f)(t,y) \frac{dy}{|y|} \ . \tag{3.19}$$

Equation (3.16) gives then

$$\| f_Y - f^\Omega \| \leq \| f^\Omega \| \sup_{\omega\in\Omega} |r_Y(\omega)| \leq \| f \| \sup_{\omega\in\Omega} |r_Y(\omega)| \tag{3.20}$$

where $r_Y(\omega)$ is defined by (3.17). For given g, the function $r_Y(\omega)$ is explicitly known and so (3.20) gives us control over the approximation of f by f_Y.

3.5 Examples

We shall now briefly describe some families of admissible wavelets.

a) Linear combinations of shifted and dilated gaussians.

The gaussian

$$h_0(t) = \exp(-\frac{t^2}{2}) \tag{3.21}$$

is not admissible, since $\tilde{h}_0(\omega) \neq 0$. It is easy, however, to construct admissible linear combinations of shifted and dilated gaussians.

Let φ, s, a, b be real, with $a \neq 0$. Define

$$h(\varphi, s, a, b) = e^{i\varphi} T^s D^a E^b h_0 \tag{3.22}$$

i.e.

$$h(\varphi,s,a,b;t) = e^{i(\varphi+b(t-s)/a)}|a|^{-1/2}\, h_0\left(\frac{t-s}{a}\right) . \tag{3.23}$$

Then

$$T^x h(\varphi,s,a,b) = h(\varphi,s+x,a,b) \tag{3.24}$$

$$D^y h(\varphi,s,a,b) = h(\varphi,sy,ay,b) \tag{3.25}$$

$$E^z h(\varphi,s,a,b) = h(\varphi+zs,s,a,az+b) \tag{3.26}$$

$$\overline{h}(\varphi,s,a,b) = h(-\varphi,s,a,-b) \tag{3.27}$$

$$\begin{aligned}(Fh)(\varphi,s,a,b) &= h\left(\varphi-\frac{sb}{a}, \frac{b}{a}, \frac{1}{a}, -\frac{s}{a}\right) \\ &= e^{i\varphi}\, E^{-s}\, D^{1/a}\, T^b\, h_0\end{aligned} \tag{3.28}$$

i.e.

$$\tilde{h}(\varphi,s,a,b;\omega) = e^{i\varphi}\, e^{-is\omega}\, |a|^{1/2}\, h_0(a\omega-b) . \tag{3.29}$$

The linear span of the functions $h(0,s,a,b)$ is an algebra with respect to pointwise and convolution product. One has

$$h(\varphi_1,s_1,a_1,b_1;t)\, h(\varphi_2,s_2,a_2,b_2;t) =$$

$$= \frac{1}{(a_1^2+a_2^2)^{1/4}}\, h_0\left(\frac{s_1-s_2}{\sqrt{a_1^2+a_2^2}}\right) h(\varphi,s,a,b;t) \tag{3.30}$$

with

$$a = \frac{a_1 a_2}{\sqrt{a_1^2+a_2^2}} \quad \text{i.e.} \quad \frac{1}{a^2} \quad \frac{1}{a_1^2} \quad \frac{1}{a_2^2} \tag{3.31}$$

$$s = \frac{s_1 a_2^2 + s_2 a_1^2}{a_1^2 + a_2^2} \quad \text{i.e.} \quad \frac{s^2}{a^2} = \frac{s_1^2}{a_1^2} + \frac{s_2^2}{a_2^2} \tag{3.32}$$

$$b = \frac{b_1 a_2 + b_2 a_1}{\sqrt{a_1^2+a_2^2}} \quad \text{i.e.} \quad \frac{b}{a} = \frac{b_1}{a_1} + \frac{b_2}{a_2} \tag{3.33}$$

and

$$\varphi = \varphi_1 + \varphi_2 + \frac{sb}{a} - \frac{s_1 b_1}{a_1} - \frac{s_2 b_2}{a_2} \quad . \tag{3.34}$$

The inner product of $h(1) = h(0,s_a,a_1,b_1)$ and $h(2) = h(0,s_2,a_2,b_2)$ is

$$(h(1),h(2) = \sqrt{2\pi} \left| \frac{a_1 a_2}{a_1^2+a_2^2} \right|^{1/2} \exp\left(- \frac{(s_1-s_2)^2 + (b_2 a_1 - b_1 a_2)^2}{2(a_1^2 + a_2^2)} \right) e^{i\psi} \tag{3.35}$$

with

$$\psi = \frac{(s_1-s_2)(a_1 b_1 + a_2 b_2)}{a_1^2 + a_2^2} \quad . \tag{3.36}$$

In particular,

$$(h(0,s,a,b),h_o) = \sqrt{2\pi} \;\; e^{iabs/(a^2+1)} \left| \frac{a}{a^2+1} \right|^{1/2} \exp(- \frac{s^2 + b^2}{2(a^2+a)}) . \tag{3.37}$$

A linear combination

$$g = \sum_n \; c_n \, h(0,s_n,a_n,b_n) \tag{3.38}$$

endlich?

is admissible if and only if

$$\sum_n \; c_n \left| a_n \right|^{1/2} h_o(b_n) = 0 \; . \tag{3.39}$$

The above formulas allow a discussion of the reproducing kernel and of other quantities of interest.

b) <u>The wavelets $h(b)$</u>

Define $h_o(t) = \exp(-t^2/2)$, and $h_1(t) = th_o(t)$. Then $Fh_o = h_o$, and $Fh_1 = -ih_1$.

For any $b \neq 0$, define $h(b)$ as

$$h(b) = bE^b h_o + i\, E^b h_1 \tag{3.40}$$

i.e.

$$h(b;t) = (b + it)\, e^{ibt}\, e^{-t^2/2} \; . \tag{3.41}$$

Then $Fh(b) = bT^b h_o + T^b h_1$, which gives

$$\tilde{h}(b;\omega) = \omega h_o(\omega - b) \; . \tag{3.42}$$

Consequently the wavelet $h(b)$ is admissible.

The maximum of $\tilde{h}(b;\omega)$, if $b > 0$, is attained at

$$\omega^b_{max} = (b/2) + \sqrt{(b/2)^2 + 1} \; . \tag{3.43}$$

The value of c_b and of k_b:

We have

$$c_b = 2\pi \int \frac{|\tilde{h}(b;\omega)|^2}{|\omega|}\, d\omega = 2b\,\sqrt{\pi^3}\, \mathrm{erf}(b) + 2\pi e^{-b^2} \tag{3.44}$$

$$k_b = \sqrt{2\pi} \int \frac{|\tilde{h}(b;\omega)|}{|\omega|}\, d\omega = 2\pi\, \mathrm{erf}\, (\frac{b}{\sqrt{2}}) \; . \tag{3.45}$$

where $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$.