

## A characterization of Wiener's algebra on locally compact groups

By

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It is the purpose of this paper to show that the algebra  $W(\mathbb{R})$ , which has first been defined by N. Wiener, is the smallest among all Segal algebras on  $\mathbb{R}$  which are at the same time  $C^0(\mathbb{R})$ -modules with respect to pointwise multiplication. By means of a new characterization of  $W(\mathbb{R})$  this result can be extended to general locally compact groups.

Our notation follows that of [2] and [3].  $G$  denotes a locally compact group. In order to avoid trivialities we assume that  $G$  is nondiscrete.  $K(G)$  shall denote the space of all continuous functions  $k$  with compact support ( $\text{supp } k$ ). The Banach space  $C^0(G)$  consists of all continuous functions on  $G$  vanishing at infinity. It is the closure of  $K(G)$  with respect to the norm  $\|f\|_\infty = \sup_{x \in G} |f(x)|$ . The Haar measure of a measurable set  $M \subseteq G$  shall be denoted by  $|M|$ . For  $y \in G$  the left and right translation operators are defined by  $L_y f(x) = f(y^{-1}x)$  and  $R_y f(x) = f(xy^{-1}) \Delta^{-1}(y)$ ,  $\Delta$  being the Haar module on  $G$ . For facts concerning Segal algebras the reader is referred to [2] and [3]. A Segal algebra  $S(G)$  is a Banach module over  $C^0(G)$  (with respect to multiplication), if  $fg$  is in  $S(G)$  for every  $f \in S(G)$  and  $g \in C^0(G)$  and satisfies  $\|fg\|_S \leq \|f\|_S \|g\|_\infty$ . Finally let us recall that Wiener's algebra  $W(\mathbb{R})$  consists of all continuous functions  $f$  on  $\mathbb{R}$  satisfying

$$\|f\|'_W = \sum_{n \in \mathbb{Z}} \max_{0 \leq x \leq 1} |f(n+x)| < \infty$$

(cf. [2], Chap. I, § 5, ex. iii). With the norm

$$\|f\|_W = \sup_{y \in \mathbb{R}} \|L_y f\|'_W$$

$W(\mathbb{R})$  is a Segal algebra which is of course a  $C^0(\mathbb{R})$ -module.

**Theorem 1.** *Let  $S(\mathbb{R})$  be a Segal algebra on  $\mathbb{R}$  which is a  $C^0(\mathbb{R})$ -module. Then  $S(\mathbb{R})$  contains Wiener's algebra  $W(\mathbb{R})$  and  $\|f\|_S \leq C \|f\|_W$  holds for all  $f \in W(\mathbb{R})$ . Hence  $W(\mathbb{R})$  is the smallest among all Segal algebras which are  $C^0(\mathbb{R})$ -modules.*

*Proof.* Let  $f \neq 0, f \in S(\mathbb{R})$  be given. Then there is some  $k_1 \in K(\mathbb{R})$  with  $f_1 := k_1 * f \neq 0$ , in particular  $f_1$  is a continuous function in  $S(\mathbb{R})$ . Let  $U$  be an open set such that  $|f_1(x)| \neq 0$  for all  $x \in U$ . Then there is some  $h \in K(\mathbb{R})$ ,  $h \neq 0$  with  $0 \leq h(x) \leq 1$  and  $\text{supp } h \subseteq U$ . Consequently  $f_2 := |f_1| \cdot h = f_1 \cdot h |f_1|/|f_1|$  is in  $S(\mathbb{R})$ ,  $h|f_1|/|f_1|$  being in

$K(\mathbb{R})$ . Moreover  $f_2$  is positive. Thus we may choose an open set  $V \subseteq U$  with  $f_2(x) \geq \delta > 0$  for all  $x \in V$ . Let now  $k \in K(\mathbb{R})$  be given. Without loss of generality we may suppose that  $k$  is positive. Then there is a finite sequence  $(y_i)_{i=1}^n \subseteq \mathbb{R}$  with  $\text{supp } k = \bigcup_{i=1}^n y_i + V$ . Hence

$$k(x) \leq \|k\|_\infty \delta^{-1} \sum_{i=1}^n L_{y_i} f_2(x) := f_3(x) \quad \text{for all } x \in \mathbb{R}.$$

This implies that  $k/f_3$  is in  $K(\mathbb{R})$  and that  $k = f_3 \cdot k/f_3$  is in  $S(\mathbb{R})$ . In particular  $S(\mathbb{R})$  contains a function  $g \in K(\mathbb{R})$  with  $0 \leq g(x) \leq 1$  and  $g(x) \equiv 1$  on  $[0, 1]$ . Let now  $f \in W(\mathbb{R})$  be given. Then with  $a_n(f) = \max_{0 \leq x \leq 1} |f(n+x)|$  we have for every  $x \in \mathbb{R}$

$$|f(x)| \leq \sum_{n \in \mathbb{Z}} a_n(f) L_n g(x).$$

Since for  $f \in W(\mathbb{R})$  the sum on the right hand converges to an element  $g_1 \in S(\mathbb{R})$  with

$$\|g_1\|_S \leq \sum_{n \in \mathbb{Z}} \max_{0 \leq x \leq 1} |f(n+x)| \|L_n g\|_S \leq \|f\|'_W \|g\|_S,$$

$f = g_1 \cdot f/g_1$  is in  $S(\mathbb{R})$  and satisfies

$$\|f\|_S \leq \|g_1\|_S \|f/g_1\|_\infty \leq \|f\|'_W \cdot 1 \cdot \|g\|_S \leq C \|f\|_W \quad \text{with } C = \|g\|_S.$$

With some obvious modifications of the proof the above theorem is applicable to the generalization of Wiener's algebra to locally compact Abelian groups  $G$  which have a discrete subgroup  $\Gamma$  such that  $G/\Gamma$  is compact ([3], § 5, example v), in particular to  $G = \mathbb{R}^n$  (with  $\Gamma = \mathbb{Z}^n$ , cf. [2], chap. I, § 5, example iii). If we denote these spaces by  $W(G)$  theorem 1 remains true.

In order to give a generalization of  $W(G)$  to arbitrary locally compact groups  $G$  (such that the characterization of theorem 1 extends to this generalization) we introduce a new space  $W^1(G)$ . Let  $g$  be any positive function in  $K(G)$ . Then  $W^1(G)$  consists of all continuous functions  $f$  on  $G$  which satisfy

$$(*) \quad |f(x)| \leq \sum_{n \in \mathbb{N}} a_n L_{y_n} g(x) \quad \text{for all } x \in G$$

for a suitable sequence  $(y_n)_{n \in \mathbb{N}} \subseteq G$  and a sequence  $a = (a_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$  with  $\|a\|_1 = \sum |a_n| < \infty$ . It is clear that

$$\|f\| = \inf \{ \|a\|_1, a = (a_n)_{n \in \mathbb{N}} \text{ satisfies } (*) \}$$

defines a norm on  $W^1(G)$ . Moreover it is not difficult to see that  $W^1(G)$  does not depend on  $g$ ;  $g$  can even be replaced by the characteristic function of any open, relatively compact subset of  $G$ .

**Theorem 2.**  $W^1(G)$  is a pseudosymmetric Segal algebra on  $G$  which contains  $K(G)$  as a dense subspace and is continuously embedded into  $L^1 \cap C^0(G)$ .

**Proof.** Routine computations show that  $(W^1(G), \|\cdot\|)$  is a Banach space which is continuously embedded into  $L^1 \cap C^0(G)$  and contains  $K(G)$  as a dense subspace. In particular  $y \rightarrow L_y k$  and  $y \rightarrow R_y k$  is a continuous function from  $G$  to  $W^1(G)$  for all

$k \in K(G)$ . Since  $W^1(G)$  is left invariant and  $L_y$  defines an isometry on  $W^1(G)$  for all  $y \in G$  it follows therefrom that  $W^1(G)$  is a Segal algebra. On the other hand it follows from the relation  $L_y R_x = R_x L_y$  for all  $x, y \in G$  and the fact that for any compact set  $K \subseteq G$   $\sup_{x \in K} \|R_{xy}\| < \infty$  that  $y \rightarrow \|R_y\|$  (this denotes the norm of the operator  $R_y$ ) is locally bounded on  $G$ . This in turn implies that  $y \rightarrow R_y f$  is continuous for all  $f \in W^1(G)$ , i.e.  $W^1(G)$  is pseudosymmetric.

Of course  $W^1(G)$  is a  $C^0(G)$ -module.

**Theorem 3.**  *$W^1(G)$  is the smallest among all Segal algebras  $S(G)$  on  $G$  which are  $C^0(G)$ -modules, i.e.  $W^1(G) \subseteq S(G)$  and  $\|f\|_S \leq C \|f\|$  for all  $f \in W^1(G)$  ( $C$  only depending on  $S(G)$ ), if  $S(G)$  is a  $C^0(G)$ -module as well as a Segal algebra.*

**Proof.** As in the proof of theorem 1 one shows that any Segal algebra which is a  $C^0(G)$ -module contains  $K(G)$ . Consequently  $S(G)$  must contain the functions  $f_1 = \sum a_n L_{y_n} g$ . Since  $S(G)$  is a  $C^0(G)$ -module this implies  $f \in S(G)$  for all  $f \in W^1(G)$  and the corresponding norm inequality.

**Remark.** Note that it follows from theorem 1 (and the remark following it) and theorem 3 that  $W^1(G)$  coincides with  $W(G)$  if  $W(G)$  is defined. Thus it seems to be reasonable to call  $W^1(G)$  Wiener's algebra on  $G$ . Of course  $W^1(G)$  coincides with  $L^1(G)$  if  $G$  is discrete and with  $K(G) = C^0(G)$  if  $G$  is compact.

At the end of this paper we discuss another generalization of Wiener's algebra to general locally compact groups which is due to Liu-van Rooij-Wang ([1], cf. in particular theorem 5.1 which remains true for general locally compact groups with the definition of [3]). In [1] the space  $\mathcal{M}_1(G)$  is defined in the following way:

Let  $K$  be a compact subset of  $G$  with nonvoid interior. Then for a continuous function  $f$  the function  $f^*$  is defined by means of  $f^*(x) = \|f \xi_{xK}\|_\infty$ ,  $\xi_{xK}$  being the characteristic function of  $xK$ .  $\mathcal{M}_1(G)$  consists of all continuous functions  $f$  with  $f^* \in L^1(G)$ . As pointed out in [1]  $\mathcal{M}_1(G)$  does not depend on  $K$  and  $K(G)$  is a dense subspace of  $\mathcal{M}_1(G)$ . Moreover  $\mathcal{M}_1(G)$  is a Segal algebra with the norm  $\|f\|_1^* = \|f^*\|_1$  and evidently  $\mathcal{M}_1(G)$  is a  $C^0(G)$ -module. It follows from section 5 of [1] that  $\mathcal{M}_1(\mathbb{R})$  coincides with  $W(\mathbb{R})$ . This is a special case of the following result:

**Theorem 4.**  *$\mathcal{M}_1(G)$  coincides with  $W^1(G)$  and the norms of the two spaces are equivalent.*

**Proof.** In view of theorem 3 and the fact that both  $\mathcal{M}_1(G)$  and  $W^1(G)$  contain  $K(G)$  as a dense subspace it will be sufficient to show that there is a constant  $C$  such that  $\|k\| \leq C \|k\|_1^*$  for all  $k \in K(G)$ . The proof is based on the following lemma:

**Lemma.** *Let  $V$  and  $U$  be relatively compact open sets in  $G$  with  $VV^{-1}V \subseteq U$ . Then for any compact set  $M \subseteq G$  there exists a sequence  $(y_i)_{i=1}^n \subseteq G$  and a sequence  $(B_j)_{j=1}^n$  of measurable sets satisfying the following conditions:*

- 1)  $M \subseteq \bigcup_{j=1}^n B_j$ ;  $B_i \cap B_j = \emptyset$  for  $i \neq j$ ,
- 2)  $y_j V \subseteq B_j \subseteq y_j U$  for  $1 \leq j \leq n$ .

PROOF. Let  $V, U$ , and  $M$  be given with  $VV^{-1}V \subseteq U$ . Since  $M$  is compact there is a finite sequence  $(x_i)_{i=1}^m \subseteq G$  with  $M \subseteq \bigcup_{i=1}^m x_i V$ . It is now easy to find a subsequence  $(y_j)_{j=1}^n$  of  $(x_i)_{i=1}^m$  such that the family  $\{y_j V\}_{j=1}^n$  consists of pairwise disjoint sets and cannot be enlarged in the sense that for any  $i, 1 \leq i \leq m$  there is some  $j, 1 \leq j \leq n$  with  $x_i V \cap y_j V \neq \emptyset$ . If we denote the set  $\{i \mid 1 \leq i \leq m, x_i V \cap y_j V \neq \emptyset\}$  by  $A_j$  we have therefore  $\bigcup_{j=1}^n A_j = \{1, 2 \dots m\}$ . If we define further  $C_j = G \setminus \bigcup_{i \neq j} y_i V$  and  $D_j = \bigcup_{i \in A_j} x_i V \cap C_j$  it is easy to verify that we have

$$y_j V \subseteq D_j, \quad D_j \subseteq \bigcup_{i \in A_j} x_i V \subseteq y_j V V^{-1} V \subseteq y_j U,$$

$$\bigcup_{j=1}^n D_j \supseteq M \quad \text{and} \quad D_i \cap y_j V = \emptyset$$

for  $i \neq j$ . It follows therefrom that the sets  $B_j = D_j \setminus \bigcup_{i < j} D_i$  satisfy the required conditions.

PROOF OF THEOREM 4 (CONTINUED). Let  $k \in K(G)$  with  $\text{supp } k = M$  be given. For the compact set  $K$  used in the definition of  $f^*$  we can find two relatively compact, open sets  $U$  and  $V$  satisfying  $UU^{-1} \subseteq K$  and  $VV^{-1}V \subseteq U$ . The lemma now gives a sequence  $(B_j)_{j=1}^n$  satisfying 1) and 2). Let us denote the characteristic function of  $K$  by  $g$ , that of  $B_j$  by  $g_j$  for  $1 \leq j \leq n$ . Then we have  $\|g_j\|_1 \geq |y_j V| = |V| > 0$ . If we define  $a_j = \sup \{|k(z)|, z \in y_j U\}$  it is clear that  $|k(x)| \leq \sum a_j g_j(x) \leq \sum a_j L_{y_j} g(x)$  for all  $x \in G$ . This implies  $\|k\| \leq \sum_{j=1}^n a_j$ . On the other hand the condition  $UU^{-1} \subseteq K$  implies

$$a_j \leq \sup \{|k(z)|, z \in x K\} = k^*(x)$$

for every  $x \in B_j \subseteq y_j U$ . Therefore we have

$$\sum a_j g_j(x) \leq k^*(x) \quad \text{for all } x \in G.$$

This implies

$$\|k\|_1^* = \|k^*\|_1 \geq \|\sum a_j g_j\|_1 = \sum a_j \|g_j\|_1 \geq |V| \sum a_j \geq |V| \|k\|,$$

i.e.  $\|k\| \leq C \|k\|_1^*$  for all  $k \in K(G)$  with  $C = |V|^{-1}$ . The proof is now complete.

By theorem 3 and theorem 4 we have shown that the intersection of the family of all Segal algebras on  $G$  which are  $C^0(G)$ -modules is the Segal algebra  $W^1(G) = \mathcal{M}_1(G)$ . We note that a similar result for the family of all Segal algebras on  $G$  cannot be obtained. In fact, it is well known (cf. [3], § 5, vii) that the intersection of all Segal algebras on an Abelian group  $G$  consists exactly of all (continuous) functions in  $L^1(G)$  having a Fourier transform with compact support. Since any Segal algebra contains functions that do not have this property it is not difficult to derive therefrom that any Segal algebra on  $G$  contains another Segal algebra as a proper subspace.

The author gratefully acknowledges several discussions with H. Rindler on the subject of this paper that led to the proof of theorem 4 for general locally compact groups.

## References

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Eingegangen am 23. 3. 1976 \*)

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\*) Eine Neufassung ging am 22. 7. 1976 ein.