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Amalgam Spaces and Generalized Harmonic Analysis

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Abstract

It is the purpose of this short note to indicate in which way certain amalgam spaces, defined in the spirit of N. Wiener, can be used to deal with questions arising in generalized harmonic analysis in several dimensions. In some sense these results are natural extensions of corresponding one-dimensional results due to N. Wiener (e.g. Theorem 29 in his book [33i]). Amalgams are taken with respect to "dyadic" and "uniform" decompositions respectively.

1 Introduction

In this short note we collect several interesting mathematical results, involving certain function spaces with mixed norms, which in certain special cases have been used in N. Wiener's work. Essentially, we cover the following three topics:

- Tauberian Theorems (Wiener's Third theorem) in higher dimensions
- A *new* generalization (even for $\mathbb{R}^d$) of Wiener's algebra $A(\mathbb{T})$ to locally compact Abelian groups
- Some comments on Gabor analysis

2 Tauberian Theorems in higher dimensions

Wiener's Theory of Tauberian Theorems, as described for example in his famous book [33i] "The Fourier Integral and Certain of its Applications", published in 1933, made the subject an important one, with a number of interesting applications, the most famous of course being the application to the prime number theorem.

According to his presentation a Tauberian theorem contains basically a statement of the following form. Assume that for some function or distribution satisfying a certain boundedness condition, the convolution product with a specific kernel K_0 shows "decay at infinity". Then the same is true for a wide class of kernels K. The crucial condition for this to be true is the non-vanishing of the Fourier transform of the kernel K_0, and the key argument is the approximation of those more general kernels K by linear combinations of translates of K_0. For the classical (first) Tauberian theorem of Wiener (Theorem 4 in [33i]) boundedness and decay at infinity are

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understood in the uniform sense, and then the appropriate class of kernels is just the Banach convolution algebra $L^1(\mathbb{R})$.

Following the description of the Tauberian theorem as given in H. Reiter's book [17], it is clear that there are essentially four facts upon which one may base the arguments, namely

- $L^1(\mathbb{R})$ is translation invariant and also a Banach algebra with respect to convolution;
- $L^\infty(\mathbb{R})$ is just the dual space of this Banach algebra;
- (suitably normalized) stretching of functions is an isometry in $L^1(\mathbb{R})$, a fact which can be used to derive Wiener's inversion theorem (cf. [17], I.3.1), stating that
- for any function h arising as the Fourier transform of some L^1-function g which is non-vanishing over some interval I, it is possible to find another function $g_1 \in L^1(\mathbb{R})$ such that its Fourier transform h_1 satisfies $h_1(s) = 1/h(s)$ for all $s \in I$.

More or less the same structure is used for his second version, where the Banach algebra is somewhat smaller (local integrability of kernels K is replaced by local boundedness). In Wiener's notation we are talking about Theorem 5 (at the beginning of Chapter II) compared to Theorem 4. Wiener denotes the new class of kernels by M_1. We would like to refer to those two theorems as Wiener's second and first Tauberian theorems, respectively. Due to the dense embedding of M_1 into L_1, the dual space of M_1 is somewhat bigger (it is also referred to as the space of translation-bounded measures in the literature, see [1]). from the present point of view this similarity can be understood as the consequence of the fact that the new class of kernels, which H. Reiter has called "Wiener's algebra" $W(\mathbb{R})$, is a so-called Segal algebra in Reiter's sense ([17], I.5.iii) for this example). As such it is a Banach space, continuously embedded into $L^1(\mathbb{R})$ and also an ideal in the bigger convolution algebra $L^1(\mathbb{R})$. As a consequence the so-called "ideal theorem for Segal algebras" holds true, i.e. there is a natural one-to-one correspondence between the closed ideals of L^1 and the closed ideals of a Segal algebra S (see [17], VI.2.5 for details). The two inverse mappings are just taking intersections (of the ideal in L^1 with S) and forming the closure (of the small ideal in S, with respect to the L^1 norm). In this sense only minor modifications are required to obtain the second Tauberian theorem from the first one, or to establish both along very similar lines of arguments (as done in Wiener's book). It is also not difficult to generalize both results to higher dimensions.

The story appears to be different for "Wiener's Third Tauberian Theorem", where boundedness is understood in the sense of quadratic means (Thm. 29 in [33i]). Completely different methods are used for its proof. It also appears as quite natural that one has to restrict the attention to the L^2-case because in his arguments Wiener moves between the kernels and their Fourier transforms, and Plancherel's theorem is about the only one which allows to go back and forth without much loss of information (nothing of this kind can be said about Fourier transforms and L^p-spaces, with p different from 2). Also, it is not clear from the original approach in which sense convolution plays any important role in the class of kernels considered in preparation of that third Tauberian theorem. On the other hands, the discussion of this problem has certainly been influential for the development of his ideas about generalized harmonic analysis, long before the notions of generalized functions, tempered distributions, and related concepts from modern analysis were introduced. One might suspect that these tools are needed to get the extension to higher dimensions done, but surprisingly simple dyadic decompositions can be used to solve the problem. It is possible to give a positive answer to the question concerning Tauberian theorems (of the third type) in any dimension and for general values of $p > 1$, by just observing that those p-bounded functions can be identified as the elements of some dual space, whose predual is not only an easily characterizable Banach space of functions belonging locally to L^p, but it also has the same convolution and stretching invariance properties as L^1.

We now come to a more precise description of a result that has been obtained around 1985 by the author, but unfortunately has not been published appropriately, because the corresponding conference proceedings [6] are not well accessible.

The following extension of Wiener's Third Tauberian Theorem concerning functions with bounded quadratic means can be shown by "elementary means" (without distribution theory):

Theorem 1 *Let* $f \in L^p_{loc}(\mathbb{R}^d)$, $1 < p < \infty$, *be given such that*

$$\|f\|_{[p]} := \sup_{T\geq 1} \left(T^{-d} \int_{|x|\leq T} |f(x)|^p dx \right)^{1/p} < \infty$$

and assume that $K_0(x)(1+|x|)^{d/p}$ *and* $K(x)(1+|x|)^{d/p}$ *are integrable over* $\mathbb{R}^d$. *If* K_0 *satisfies the Tauberian condition* $\hat{K}_0(t) \neq 0$ *for all* $t \in \mathbb{R}^d$, *then one may conclude from*

$$\lim_{T\to\infty} \left(\frac{1}{T^d} \int_{|x|\leq T} |K_0 * f(x)|^p dx \right)^{1/p} = 0$$

that the same relation holds true for K.

In order to continue our discussion we *define* M^p as the collection of all functions which are locally p-integrable and satisfy $\|f\|_{[p]} < \infty$. Furthermore we denote the functions satisfying a relation such as the one at the end of Theorem 1 for $K_0 * f$ above, by I^p.

Theorem 2 *For* $f \in L^p_{loc}(\mathbb{R}^d)$, $1 \leq p \leq \infty$, *define the sequence* $(d_k)_{k\geq 1}$ *by*

$$d_k := \left(\int_{P_k} |f(x)|^p dx \right)^{1/p}, k \geq 1,$$

where $P_k := \{x | 2^{k-1} \leq |x| \leq 2^k\}$ *for* $k \geq 2$ *and* $P_1 := \{x \,|\, |x| \leq 2\}$, *and write* $\|f\|_{(p)}$ *for the norm expression* $\|f\|_{(p)} := \sup_{k\geq 1} 2^{-dk/p} d_k$. *Then*

1. $f \in M^p(\mathbb{R}^d)$ *if and only if* $\|f\|_{[p]<\infty}$.
2. *The norms* $\|\cdot\|_{[p]}$ *and* $\|\cdot\|_{(p)}$ *are equivalent.*
3. $(M^p(\mathbb{R}^d), \|\cdot\|_{[p]})$ *is a Banach space.*

4. *For* $1 \leq p < \infty$, *a function* $f \in L^p_{loc}(\mathbb{R}^d)$ *belongs to* $I^p(\mathbb{R}^d)$ *if and only if* $\lim_{k\to\infty} 2^{-kd/p} d_k = 0$.

5. $I^p(\mathbb{R}^d)$ *coincides with the closure of* $K(\mathbb{R}^d)$ *(or of* $C_0(\mathbb{R}^d)$ *or of* $L^s(\mathbb{R}^d)$, *for* $p \leq s < \infty$*) in* $M^p(\mathbb{R}^d)$. *In particular,* $I^p(\mathbb{R}^d)$ *is a closed subspace of* $M^p(\mathbb{R}^d)$.

6. $(M^p, \|\cdot\|_{[p]})_{1\leq p\leq\infty}$ *is a decreasing family of spaces, with continuous embeddings* $M^r \hookrightarrow M^p$ *for* $r \leq p$.

7. $M^p(\mathbb{R}^d)$ *is translation invariant, containing* I^p *as a closed invariant subspace. For the translation operators* $T_x f(u) = f(u - x)$, *we have the estimate*

$$\|T_y\|_{I^p} \leq \|T_y\|_{M^p} \leq C(1 + |y|)^{dp},$$

and translation is continuous in $(I^p, \|\cdot\|_{[p]})$ *in the sense that* $T_x f - f \to 0$ *for* $x \to 0$.

In order to carry out the proof of the Third Tauberian Theorem in its general form an important step is to identify the space M^p as the Banach dual of some other Banach space E^q (is is taking the role of L^1). Without going into details we mention that the natural norm $\|\cdot\|_{\{q\}}$ is again defined in terms of dyadic decompositions. E^q can be shown to be the smallest Banach space containing all the functions in L^q (where $1/p + 1/q = 1$) and has the property of being invariant with respect to (L^1-isometric) dilations: $D_a f(x) = a^{-d} f(x/a)$. We will just list a number of properties of this predual Banach convolution algebra in the next theorem:

Theorem 3

1. *For* $1 \leq q \leq \infty$ *the normed space* $(E^q, \|\cdot\|_{\{q\}})$ *is a Banach algebra with respect to convolution, densely embedded into* $L^1(\mathbb{R}^d)$

2. $\mathcal{F}E^q := \{\hat{h} | h \in E^q\}$, *with the norm* $\|\hat{h}\|_{\}q\{} := \|h\|_{\{q\}}$ *is a pointwise Banach algebra on* $\mathbb{R}^d$ *satisfying the condition of Wiener-Ditkin (see [17], Chap.II).*

3. *The Theorem of Wiener-Lévy applies to* E^q, *i.e., given* $h_0 \in E^q$ *with* $\hat{h}_0(t) \neq 0$ *on a compact set* $M \subseteq \mathbb{R}^d$, *there exists* $h_1 \in E^q$ *with* $\hat{h}_1 = 1/\hat{h}_0(t)$ *for all* $t \in M$.

4. *The set of linear combinations of translates of* $h_0 \in E^q$ *is dense in* $(E^q, \|\cdot\|_{\{q\}})$ *if and only if* $\hat{h}_0(t) \neq 0$ *for all* $t \in \mathbb{R}^d$.

5. $(E^q)^{\mathcal{K}} := \{h | h \in E^q, supp(\hat{h}) is\ compact\}$ *is a dense ideal in* $(E^q, \|\cdot\|_{\{q\}})$.

6. $(\mathcal{F}E^q, \|\cdot\|_{\}q\{})$ *is a Wiener algebra in the sense of Reiter([17], Ch.II).*

3 Another Fourier Algebra for LCA Groups

One of the spaces playing an important role in the work of N. Wiener is his Banach algebra $A(\mathbb{T})$ of functions having an absolutely convergent Fourier series, or in other words, the subspace of all continuous and periodic functions which have absolutely summable (or ℓ^1) Fourier coefficients. The famous "inversion theorem" is at the basis for the proof of his Tauberian theorems (cf. [17], I.3.1.) Since then the so-called "Fourier algebra" $A(G)$ has been defined for general lca. groups, just as the image of the convolution algebra $L^1(\hat{G})$ under the group Fourier transform (making use of Pontrjagin's duality theorem.) As far as one can judge from the literature, $A(G)$ is always considered the "natural" generalization of $A(\mathbb{T})$ to the case of general groups. We do not have enough place to substantiate our thesis in detail, that another space (to be discussed below) might be a better candidate, because it has much better functorial properties, and reduces to the Fourier algebra $A(G)$ only for the case of compact groups (such as the torus,) but to $\ell^1(G)$ for discrete groups.

The Segal algebra $S_0(G)$ has been introduced by the first author in 1980 at a "winter-school in harmonic analysis," held in Vienna. A detailed description of the basic properties of this Banach space of continuous and integrable functions (well-defined even for general lca. groups) has been given in [4]. A short summary of the properties of its dual are given in [3]. For an elementary approach in the case that the underlying group is the d-dimensional Euclidean space is given in [5]. The link to atomic decompositions (similar to wavelet expansions) is described in [7] and in [8], where this space is described as the Banach space of all L^1-functions f for which the *Short Time Fourier Transform* (also called Sliding Window Fourier Transform) with respect to the Gauss-function (or any other non-zero Schwartz function g) is integrable over the TF (=time/frequency) plane. More precisely we have the following.

Definition 1 *The STFT of a function f with respect to the window g is given by*

$$S_g f(x,y) := \int_{\mathbb{R}^d} e^{-2\pi i y\cdot z}\,\overline{g(z-x)}f(z)dz = \langle M_y T_x g, f\rangle \quad \textit{for } (x,y) \in \mathbb{R}^{2d}. \tag{1}$$

Here the modulation operator M_y is given by $M_y f(z) = e^{-2\pi i y\cdot z} f(z)$. Many interesting properties of the STFT can be found in Folland's book on "Harmonic Analysis in Phase Space" ([9].) In contrast to the Schwartz-Bruhat space of rapidly decaying functions, the space $S_0(G)$ can be defined *without* the use of structure theory for arbitrary lca. groups, and in many ways is a very natural function space over any given lca. group.

One way of defining $S_0(G)$ is the following.

Definition 2 *Let Q be a fixed, open, relatively compact subset of G. Then we define*

$$S_0(G) := \{f \mid f = \sum T_{y_n} f_n, (f_n) \subseteq A_Q(G), \sum \|f_n\|_A < \infty\},$$

$$\|f\|_{S_0(G)} := \inf\{\sum \|f_n\|_A, \ldots\},$$

the infimum being taken over all admissible representations.

Here $(A(G), \|\cdot\|_A)$ denotes the Fourier algebra on G and $A_Q(G)$ denotes the closed subspace of all functions in $A(G)$ with support in some fixed compact set Q (with non-void interior.)

Theorem 4 *$S_0(G)$ is the smallest strongly character invariant Segal algebra on G (i.e., $S_0(G) \subseteq S$ for any Segal algebra satisfying $\|M_y f\|_S = \|f\|_S$). In particular, different sets Q give the same space with equivalent norms.*

It is important that this space has a number of invariance properties, which even makes it unique as has been shown by V. Losert in [14].

Theorem 5 *Let* α *be an isomorphism from* G_2 *onto* G_1. *Then* $\alpha^* : f \mapsto f \circ \alpha$ *gives a bipositive algebra isomorphism* T *(i.e.,* $Tf \geq 0 \iff f \geq 0$*) from* $S_0(G_1)$ *onto* $S_0(G_2)$. *Conversely, any bipositive isomorphism between* $S_0(G_1)$ *and* $S_0(G_2)$ *is given in this way.*

Among the basic properties of $S_0(G)$ the following properties are most useful for applications: invariance under the Fourier transformation (moving to the dual group); invariance under translation, frequency translation, as well as group automorphisms; restriction property (moving to subgroups); product stability (moving to product groups); integration along subgroups (moving to quotient groups).

Theorem 6

1. $\mathcal{F}S_0(G) = S_0(\hat{G})$;
2. $T_H S_0(G) = S_0(G/H)$;
3. $R_H S_0(G) = S_0(H)$;
4. $S_0(G_1)\hat{\otimes}S_0(G_2) = S_0(G_1 \times G_2)$.

The first two of the above properties make $S_0(G)$ an appropriate *Banach* space of test functions, whose dual $S_0'(G)$ is also invariant under (the extended) Fourier transform, defined by a simple duality argument. $S_0(G)$ is also small enough such that any of the L^p-spaces (for $1 \leq p \leq \infty$) is contained in $S_0'(G)$, hence this generalized Fourier transform gives sense to the notion of a (generalized) Fourier transform on such spaces, with no restriction to the case $p \leq 2$. Just as a sample application indicating but one of the useful properties of $S_0(G)$, let us state the following general version of Poisson's formula.

Theorem 7 *Let* H *be a discrete subgroup of some lca. group* G *with compact quotient. Then, up to some constant* $C_H > 0$, *one has for any* $f \in S_0(G)$ *that the following identity, where both sides converge absolutely:*

$$\sum_{h \in H} f(h) = C_H \sum_{l \in H^\perp} \hat{f}(l).$$

The argument of Poisson's theorem in this form can be based on the result described in [17], V.5.1, together with one of the "atomic decompositions" available for $S_0(G)$.

Finally it should be mentioned that $S_0(G)$ is also an appropriate Banach space of test functions in order to model generalized stochastic processes, actually just as linear operators from $S_0(G)$ into some Hilbert space $\mathcal{H}$. The resulting theory turns out to be much more symmetric (with respect to the Fourier transform) then the well-known theory (following the idea of vector-valued Radon-measures), which uses the space of compactly supported (only) continuous functions as a test space. Details on this are worked out in the PhD thesis of W.Hörmann [12] (Vienna, 1989).

4 Gabor Analysis and Local Spectral Analysis

If one thinks of a function corresponding to an acoustic signal, let us say a piece of music, it is natural to think of a melody as a sequence of harmonies, changing over time. Although mathematically speaking any actual piece of music is finite, and periodic Fourier analysis (Fourier series expansion) would be possible, such an analysis would not be very useful, due to its total lack of locality. Problems of this kind (with classical Fourier analysis) lead to the notion of "local spectral analysis." There are different ways of doing this (besides the brutal one of just cutting a signal into pieces,) but one of the most interesting approaches was suggested by D. Gabor, almost 50 years ago. Although this was during N. Wiener's time and in spirit closely related to some of his work, there is no indication that Wiener himself was aware of Gabor's suggestion.

D. Gabor claimed in his 1946 paper [10] that any $f \in L^2(\mathbb{R})$ can be written as a series of time-frequency shifted Gauss-functions. Indeed, the believe was that it should be possible to expand any f as a superposition of such Gauss-functions, shifted along the integer lattice, and modulated with integer frequencies (note that modulation and frequency-shift operator commute in this case,) with square summable coefficients. He was only giving some heuristic arguments and suggested even an algorithmic approach. Larger lattice constants would not allow to expand all of L^2, and smaller lattice constants would not allow to have uniqueness of coefficients. Meanwhile it is known that in this critical case one has to expect a lot of problems (cf. [13],) and that the algorithm as such fails to work (see [11].) However, in recent years, mainly stimulated through applications in signal analysis, serious progress has been made on the mathematics of Gabor expansion, based on the insight that (modest) *oversampling*, i.e., the use of Gabor families with $ab < 1$, makes a lot of sense and has *almost* all the features that D. Gabor had in mind when he made his suggestion. To make things precise let us recall the following definitions.

Definition 3 *Given a pair of lattice constants (a, b) we call a family of functions of the form $M_{n\beta}T_{k\alpha}g$, $(n, k) \in \mathbb{Z}^2$, a* Gabor family *generated by the triple (g, a, b).*

Gabor's original suggestion corresponds (in a normalized version) to the Gauss-function $g(x) = e^{-\pi x^2}$, and the time-frequency shift operators defined by $T_z f(x) = f(x - z)$ and $M_s f(x) = e^{2\pi i s x} f(x)$ for $z, s, x \in \mathbb{R}$ (with natural modifications for higher dimensions.)

It turns out that for a given triple (g, a, b), the desired expansion is possible if and only if the corresponding Gabor family forms a *frame* for the Hilbert space $L^2(\mathbb{R})$, which is equivalent to the existence of two positive constants $A, B > 0$ such that

$$A \|f\|_2 \leq \sum_{n,k} |\langle f, g_{n,k} \rangle|^2 \leq B \|f\|_2$$

for all $f \in L^2(R)$, or equivalently, one has invertibility of the so-called *frame operator*

$$S = S(g, a, b) : f \mapsto \sum_{n,k} \langle f, g_{n,k} \rangle \, g_{n,k}.$$

Only recently it has been shown by Seip/Wallsten and Y. Lyubarskii that the original problem can be answered in this sense if and only if $ab < 1$ (cf. [2, 18, 16, 15].)

If we examine the continuity properties of the frame operator, it is surprising to find that for general $g \in L^2$, it may *not* be bounded on L^2. For a set of sufficient conditions for continuity on L^p-spaces we refer to a paper of D. Walnut [19]. However, using $S_0(\mathbb{R}^d)$, one can make the following positive statements (based on Walnut's result or the general theory in [8]).

Theorem 8 *For any* $g \in S_0(\mathbb{R}^d)$, *the corresponding frame operator* $S(g,a,b)$ *is a bounded linear map* $L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$, *for any* $p \in [1,\infty]$, *but also on* $S_0(\mathbb{R}^d)$ *itself or on* $S_0'(\mathbb{R}^d)$.

For a general atom $g \in L^2(\mathbb{R}^d)$, one can state at least continuity between different Banach spaces of distributions.

Theorem 9 *For arbitrary* $g \in L^2(\mathbb{R}^d)$, *the operator* $S(g,a,b)$ *defines a bounded linear mapping from* $S_0(\mathbb{R}^d)$ *to* $S_0'(\mathbb{R}^d)$.

Most recently the following result has been proved by K.Gröchenig and the first author. It is very much, in spirit, similar, and its proof is based on, Wiener's inversion theorem for $A(\mathbb{T})$ (to appear J.Funct. Anal., 1996).

Theorem 10 *Whenever the operator* $S(g,a,b)$ *is invertible on* $L^2(\mathbb{R}^d)$, *its restriction to* $S_0(\mathbb{R}^d)$ *is also invertible on* $S_0(\mathbb{R}^d)$ *if only* $g \in S_0(\mathbb{R}^d)$ *and ab is a rational number. Equivalently,* $S(g,a,b)^{-1} = S(\gamma,a,b)$ *for some* $\gamma \in S_0(\mathbb{R}^d)$ *under these conditions.*

This *dual Gabor atom* γ can be used in many ways, e.g., to obtain a Gabor expansion of $f \in L^2(\mathbb{R}^d)$ by the formula

$$f = \sum_{n,k} \langle f, \gamma_{n,k} \rangle \, g_{n,k} \, .$$

Since $\gamma \in S_0(\mathbb{R}^d)$ the square summability of the set of coefficients for general $f \in L^2$ is guaranteed. An alternative use of γ can be made by observing that it allows to reconstruct the function f from the sampled STFT with window g via

$$f = \sum_{n,k} \langle f, g_{n,k} \rangle \, \gamma_{n,k} .$$

It turns out that any of these two series expansions is absolutely convergent (in $L^2(\mathbb{R}^d)$ and then in $S_0(\mathbb{R}^d)$ also) if and only if $f \in S_0(\mathbb{R}^d)$. We can only mention here that this expansion is very similar to the spirit of Shannon's sampling theorem for band-limited functions.

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Special thanks go to Prof. P. Masani for encouraging the writing of this article and giving me the hint how to access the MIT-archives, from where Dr. E. Andrews provided me with a copy of the correspondence between D.Gabor and N.Wiener (in the years 1951 - 1961). It gives no evidence that these two great scientist where having any exchange on the mathematics of what is nowadays called Gabor analysis, or theory of Gabor-expansions, although from the present point of view their mathematical work shows a number of very interesting connections, as outlined in this note. A forthcoming book (edited by H.G.Feichtinger and T.Strohmer, 1997) will provide an overview over the present state of the mathematics of Gabor analysis, indicating also the wide range of applications, e.g. to signal processing.

Readers interested in more details may contact the author via E-mail at `fei@tyche.mat.univie.ac.at` .
More informations on Gabor analysis and related work carried out by NUHAG (= Numerical Harmonic Analysis Groups at Vienna/Austria) can be obtained through WWW at `http://tyche.mat.univie.ac.at`.

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