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AN ELEMENTARY APPROACH TO WIENER'S THIRD TAUBERIAN THEOREM FOR THE EUCLIDEAN n-SPACE

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§1. INTRODUCTION

It is the purpose of this paper to present an extension of N. Wiener's third Tauberian Theorem (Theorem 29 in his book «The Fourier Transform and certain of its Applications»). It applies to functions $f \in M^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, i.e. to f with bounded p-means on the Euclidean n-space. The approach will be elementary, relying only on the basic properties of the ordinary Fourier transform and standard techniques from functional analysis. The structure of the proof will be quite similar to that of Wiener's first general Tauberian theorem involving bounded functions (Theorem 4 in his book), as described in the first chapter of H. Reiter's book («Classical Harmonic Analysis and Locally Compact Groups»). As will be shown, the point of relevance lies in both situations in the fact that the functions under consideration belong to the dual of a Banach convolution algebra of integrable functions which admits local inversion of the Fourier transform (i.e. for which the Theorem of Wiener-Lévy is available). In the case of $M^p(\mathbb{R}^n)$ this predual may be characterized as a dyadic decomposition space (this gives already some information on its Banach space structure as well as of that of $M^p(\mathbb{R}^n)$), but also as a minimal element in a certain family of Banach spaces (this gives an atomic characterization for its elements).

Although the results of this note fit into the general (abstract) frame concerning Tauberian theorems as presented in [7] we shall not make use of terminology and results of that earlier note before the final part of this note.

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As far as possible we shall make use of the notations and methods used in H. Reiter's book ([23]). However, in order to make this note self-contained let us start by recalling the most important notations.

As main results of this paper the reader will find Tauberian theorems related to (and extending) those given by N. Wiener (cf. [26] - [28]) as Theorems 1, 12, 13, 19, 20, 30 and Corollaries 10.11.

§2. NOTATIONS

Throughout the paper p denotes any real number satisfying $1 \leqslant p < \infty$, and q denotes the conjugate of p, given by $1/p + 1/q = 1$. $(L^p(\mathbb{R}^n), \| \ \|_p)$ stands for the Lebesgue space of equivalence classes of) measurable functions f such that

$$\|f\|_p := \left(\int_{\mathbb{R}^m} |f(x)|^p \, dx \right)^{1/p} \leqslant \infty. \tag{1}$$

Each of these spaces contains $K(\mathbb{R}^n)$, the space of continuous complex-valued functions with compact support on $\mathbb{R}^n$ as a dense subspace. Since $(L^p)' = L^q$, these Banach spaces are reflexive for $1 < p < \infty$. We write $L^p_{\ell oc}$ for the space of locally p-integrable functions and recall that $L^p_{\ell oc} \subseteq L^r_{\ell oc}$ for $p \geqslant r$. For y, $t \in \mathbb{R}^n$ the translation and (character) multiplication operators, given by

$$L_y f(x) := f(x - y), \tag{2}$$

and

$$M_t f(x) := \langle x, t \rangle f(x), \tag{3}$$

with

$$\chi(t) := \langle x, t \rangle := \exp\left(2\pi i \, \Sigma_{j=1}^n x_j t_j\right) \tag{4}$$

act isometrically on each L^p-space. The same is true or the inversion operator $\check{}$, given by $\check{f}(x) = f(-x)$. Furthermore, for $\rho > 0$ the following dilation operators will be useful:

$$M_\rho f(x) := \rho^n f(\rho x), \tag{5}$$

$$D_\rho f(x) := \quad f(x/\rho). \tag{6}$$

One has

$$\|M_\rho f\|_p = \rho^{n/q} \|f\|_p \quad \text{for} \quad f \in L^p(\mathbb{R}^n). \tag{7}$$

In particular, M_ρ acts isometrically on $(L^1(\mathbb{R}^n), \| \ \|_1)$. The operators M_t, $t \in \mathbb{R}^n$ and M_ρ, $\rho > 0$ also act as automorphisms with respect to the convolution product, given by

$$(8) \qquad g * f(x) := \int_{\mathbb{R}^n} g(y) f(x-y)\, \mathrm{d}y \quad \text{for} \quad f, g \in K(\mathbb{R}^n),$$

for which one has

$$(9) \qquad \|g * f\|_p \leqslant \|g\|_1 \|f\|_p \quad \text{for} \quad 1 \leqslant p \leqslant \infty.$$

In particular, $(L^1(\mathbb{R}^m), \| \ \|_1)$ will be considered as a commutative Banach algebra with respect to convolution. The Fourier transform F, given by

$$(10) \qquad Ff(t) := \hat{f}(t) := \int_{\mathbb{R}^n} f(x) \overline{\langle x, t\rangle}\, \mathrm{d}x \quad \text{for} \quad f \in K(\mathbb{R}^n), t \in \mathbb{R}^n,$$

extends to a contractive, injective homomorphism of Banach algebras

$$(11) \qquad F : (L^1(\mathbb{R}^n), \| \ \|_1) \to (C^0(\mathbb{R}^n), \| \ \|_\infty)$$

into the space of continuous functions vanishing at infinity (as multiplication algebra with the sup-norm). The basic operators mentioned above change their role under the Fourier transform, i.e.

$$(12) \qquad (L_y f)^\wedge = M_{-y}\hat{f}; \quad (M_y f)^\wedge = L_y \hat{f} \quad \text{for} \quad y \in \mathbb{R}^n$$

$$(13) \qquad (M_\rho f)^\wedge = D_\rho \hat{f}; \quad (D_\rho f)^\wedge = M_\rho \hat{f} \quad \text{for} \quad \rho > 0.$$

We write as usual $\mathcal{D}$ $(= C_0^\infty)$ and S for the space of test functions or rapidly decreasing functions on $\mathbb{R}^n$ and observe that they are invariant under these basic operators. The formulas $\langle M_\rho k, h\rangle = \langle k, D_\rho h\rangle$, $\langle L_y k, h\rangle = \langle k, L_{-y} h\rangle$ and $\langle M_t k, h\rangle = \langle k, M_t k\rangle$ allow to extend them to the dual spaces $\mathcal{D}'$ and S'.

In the second part of our paper intensive use of the extended Fourier transform $F : S' \to S'$ and formulas involving it will be made (cf. [15]). We also recall that a continuous function w on $\mathbb{R}^n$ will be called a weight function, if

$$(14) \qquad w(x) \geqslant 1 \text{ and } w(x+y) \leqslant w(x)\, w(y) \quad \text{for} \quad x, y \in \mathbb{R}^n.$$

These are of importance, because the corresponding weighted L^1-spaces

$$(15) \qquad L^1_w(\mathbb{R}^n) := \{ f \mid f w \in L^1 \}$$

are Banach algebras (densely embedded into $L^1(\mathbb{R}^n)$ and containing $K(\mathbb{R}^n)$

as dense subalgebra) for the convolution product, with the norm

(16) $$\|f\|_{1,w} := \|fw\|_1$$

called *Beurling algebras* (cf. [23], Chap. 1, §6). For us only the weight functions

(17) $$w_\alpha(x) := (1 + |x|)^\alpha, \quad \alpha \geqslant 0$$

will be of relevance, and it will be convenient to write L^1_α instead of $L^1_{w_\alpha}$.

Finally we mention that we shall call a Banach space $(B, \| \ \|_B)$ a *BF-space* (on $\mathbb{R}^n$) if it is continuously embedded into $L^1_{loc}(\mathbb{R}^n)$, i.e. if one has:

(18) $$\begin{cases} \text{For } T > 0 \text{ there exists } C = C(T) > 0 \text{ such that} \\ \displaystyle\int_{|x| \leqslant T} |f(x)|\, dx \leqslant C \|f\|_B \quad \text{for all} \quad f \in B. \end{cases}$$

$(B, \| \ \|_B)$ will be called a *solid* BF-space, if further

(19) $$\begin{cases} f \in B, g \in L^1_{loc}, |g(x)| \leqslant |f(x)| \quad \text{a.e.} \\ \qquad \text{implies } g \in B \text{ and } \|g\|_B \leqslant \|f\|_B. \end{cases}$$

A BF-space is called *translation (dilation) invariant* if $L_y B \subseteq B$ ($M_\rho B \subseteq B$ for all $\rho > 0$). The operator norm of these (as for other operators) as a bounded linear mapping on $(B, \| \ \|_B)$ is denoted by $|||L_y|||_B$ ($|||M_\rho|||_B$). For convenience we agree to use the self-explaining symbol $|||\sigma *|||_p$ for the norm of the convolution operator $f \mapsto \sigma * f$ as operator on $(L^p, \| \ \|_p)$. A translation invariant BF-space B is said to have *continuous translation (dilation)* if the mapping $y \mapsto L_y f$ ($\rho \mapsto M_\rho f$) is continuous as a mapping into $(B, \| \ \|_B)$ for all $f \in B$. Since in that case the convolution (as given by (8)) may be interpreted as a vector-valued integral with values in $(B, \| \ \|_B)$, i.e.

(20) $$g * f = \int_{\mathbb{R}^n} L_y f g(y)\, dy \quad \text{for} \quad g \in K(\mathbb{R}^n), f \in B$$

it follows that one has for the weight function $w(x) := \max(1, |||L_y|||_B)$

(21) $$\|g * f\|_B \leqslant \|g\|_{1,w} \|f\|_B \quad \text{for} \quad g \in L^1_w, f \in B.$$

In fact, B becomes in this a Banach convolution module over the Banach algebra L^1_w. It is even an essential Banach module, because any bounded approximate unit $(e_\alpha)_{\alpha \in I}$ in $L^1(\mathbb{R}^n)$ with joint compact support (i.e. a suitable net of functions tending vaguely to the Dirac measure) is also bounded in L^1_w and one has

$$\text{(22)} \qquad \lim_{\alpha \to \infty} \| e_\alpha * f - f \|_B = 0.$$

If the weight function w is of polynomial growth, i.e. if there exist $\alpha \geqslant 0$ such that $\||L_y\||_B \leqslant C w_\alpha(y)$ it can easily be shown that B is continuously embedded into the $S'(\mathbb{R}^n)$. If B is furthermore a solid BF-space it is not difficult to verify that S is embedded into B. Furthermore, in this case it is sufficient to choose $k \in S(\mathbb{R}^n)$ with $\int k(y)\,dy = 1$ in order to have the more concrete relation

$$\text{(22')} \qquad \lim_{\rho \to \infty} \| M_\rho k * f - f \|_B = 0.$$

(For a short summary of relevant facts concerning Banach modules the interested reader is referred to [6]).

Of particular importance is the family of *homogeneous Banach spaces*, which are defined as isometrically translation invariant BF-spaces with continuous translation (cf. [17]). Thus they are (essential) Banach convolution modules over $L^1(\mathbb{R}^n)$: It follows that a homogeneous Banach space $(B, \| \ \|_B)$ is a *Segal algebra* in the sense of H. Reiter (cf. [23], Chap. VI) if and only if it is densely embedded (and thus even a Banach ideal) in $(L^1(\mathbb{R}^n), \| \ \|_1)$.

If $(B, \| \ \|_B)$ is an arbitrary translation invariant BF-space one writes $B_c := \{ f \mid y \to L_y f$ is a continuous function from $\mathbb{R}^n$ to $B \}$. If $y \to \||L_y\||_B$ is a locally bounded function, it is clear that B_c is a closed (the maximal) subspace of B having continuous translation.

The *suppot* of $f \in L^1_{loc}$ (considered as a measure) is denoted by $\operatorname{supp} f$. Given $B \subseteq L^1_{loc}$ and a set $Q \subseteq \mathbb{R}^n$ the symbol B_Q stands for $\{ f \mid f \in B, \operatorname{supp} f \subseteq Q \}$. The Lebesgue measure of Q is denoted by $|Q|$, and c_Q stands for the indicator function of Q. Positive constants are denoted by C, C', $C_1, \ldots$ and may vary from occurence to occurence in different proofs.

§3. THE MAIN RESULT

In this section variants of the following extension of Wiener's third Tauberian Theorem concerning functions with bounded quadratic means is to proved by elementary means:

THEOREM 1. Let $f \in L^p_{loc}(\mathbb{R}^n)$, $1 < p < \infty$ such that

$$\|f\|_{[p]} := \sup_{T \geqslant 1} \left(T^{-n} \int_{|x| \leqslant T} |f(x)|^p \, dx \right)^{1/p} < \infty \tag{23}$$

and $K_1, K_2 \in L^1_{n/p}$ be given. If K_1 satisfies the *Tauberian condition*

$$\hat{K}_1(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{R}^n, \tag{24}$$

then one may conclude from the relation

$$\lim_{T \to \infty} \left(T^{-n} \int_{|x| \leqslant T} |K_1 * f(x)|^p \, dx \right)^{1/p} = 0, \tag{25}$$

that the same relation holds true for K_2.

REMARK A). As will be seen, actually less stringent integrability conditions concerning K_1, K_2 will be sufficient. However, already the above assumptions are strictly weaker than those made by N. Wiener and subsequent authors, where for $n = 1$, $p = 2$ it is assumed that $(1 + |x|)\, K_i(x)$ belongs to $L^1 \cap L^2(\mathbb{R})$, for $i = 1, 2$ (cf. [27], p. 177). Besides this, our approach is a different one.

Following the notation of earlier authors we are writing for the *p-means*

$$A_p(f, T) := \left(\frac{1}{T^n} \int_{|x| \leqslant T} |f(x)|^p \, dx \right)^{1/p}$$

and denote the space satisfying (23), i.e. of elements of $L^p_{loc}(\mathbb{R}^n)$ with *bounded p-means* by $M^p(\mathbb{R}^n)$. It is a normed space with respect to the norm $\|f\|_{[p]} := \sup_{T \geqslant 1} A_p(f, T)$. The subspace of elements satisfying the stronger condition

$$\lim_{T \to \infty} A_p(f, T) = 0 \tag{26}$$

is denoted by $I^p(\mathbb{R}^n)$. Furthermore it will be appropriate to identify M^∞ with L^∞ and I^∞ with C^0.

THEOREM 2. For $f \in L^p_{loc}(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, define the sequence (of nonnegative real numbers) $(d_k)_{k \geqslant 1}$ by

$$d_k := \left(\int_{P_k} |f(x)|^p \, dx \right)^{1/p}, \quad k \geqslant 1, \tag{27}$$

where $P_k := \{x \mid 2^{k-1} \leqslant |x| \leqslant 2^k\}$ for $k \geqslant 2$ and $P_1 := \{x \mid |x| \leqslant 2\}$, and write $\| \ \|_{(p)}$ for the norm given by

$$\|f\|_{(p)} := \sup_{k \geqslant 1} 2^{-nk/p} \, \mathrm{d}_k . \tag{28}$$

Then one has

A) $f \in M^p(\mathbb{R}^n)$ if and only if $\|f\|_{(p)} < \infty$.

B) The norms $\| \ \|_{[p]}$ and $\| \ \|_{(p)}$ are equivalent.

C) $(M^p(\mathbb{R}^n), \| \ \|_{[p]})$ is a Banach space.

D) For $1 \leqslant p < \infty$ $f \in L^p_{\ell oc}(\mathbb{R}^n)$ belongs to $I^p(\mathbb{R}^n)$ if and only if

$$\lim_{k \to \infty} 2^{-kn/p} \, \mathrm{d}_k = 0. \tag{29}$$

E) $I^p(\mathbb{R}^n)$ coincides with the closure of $K(\mathbb{R}^n)$ (or of $C^0(\mathbb{R}^n)$ or of $L^s(\mathbb{R}^n)$, for $p \leqslant s < \infty$) in $M^p(\mathbb{R}^n)$. In particular, $I^p(\mathbb{R}^n)$ is a closed subspace of $M^p(\mathbb{R}^n)$.

F) $(M^p, \| \ \|_{[p]})_{1 \leqslant p \leqslant \infty}$ is a decreasing family of spaces, with continuous embeddings $M^r \hookrightarrow M^p$ for $r \geqslant p$.

G) $M^p(\mathbb{R}^n)$ is translation invariant and isometrically inversion invariant, with I^p as an invariant subspace. For the translation operators the following estimate is available:

$$|||L_y|||_{I^p} \leqslant |||L_y|||_{M^p} \leqslant C(1 + |y|)^{n/p} \quad \text{for} \quad y \in \mathbb{R}^n . \tag{30}$$

Furthermore translation is continuous in $(I^p, \| \ \|_{[p]})$.

PROOF. A), B) Assume $\|f\|_{[p]} < \infty$. Then one has for all $k \geqslant 1$

$$\mathrm{d}_k 2^{-kn/p} \leqslant \left(\frac{1}{2^{kn}} \int_{|x| \leqslant 2^k} |f(x)|^p \, \mathrm{d}x \right)^{1/p} \leqslant \|f\|_{[p]} < \infty,$$

hence $\|f\|_{(p)} \leqslant \|f\|_{[p]}$. Conversely one has for $T \geqslant 1$: $T \in [2^{k-1}, 2^k]$ for some $k \geqslant 1$ and consequently

$$A_p(f, T) \leqslant 2^{n/p} \left(\frac{1}{2^{kn}} \int_{|x| \leqslant 2^k} |f(x)|^p \, \mathrm{d}x \right)^{1/p} =$$

$$= 2^{n/p} 2^{-kn/p} (\Sigma_{j=1}^k \mathrm{d}_j^p)^{1/p} \leqslant 2^{n/p} \|f\|_{(p)} 2^{-kn/p} (\Sigma_{j=1}^k 2^{jn})^{1/p} \leqslant$$

$$\leqslant C_{n,p} \|f\|_{(p)} \quad \text{for all} \quad T \geqslant 1,$$

showing the required equivalence of norms.

C) As a matter of routine it is left to the reader that $(M^p, \| \ \|_{(p)})$ and thus $(M^p, \| \ \|_{[p]})$ is a Banach space (cf. [8]; it is easy to verify that absolutely series are norm convergent).

D) It is clear that the space $\{ f \,|\, f \in L^p_{\ell oc}, \ \lim_{k\to\infty} 2^{-kn/p} \mathrm{d}_k = 0 \}$ is a closed subspace of M^p containing $I^p(\mathbb{R}^n)$. The converse inclusion follows from the fact that the estimate $\mathrm{d}_l \, 2^{-ln/p} < \epsilon$ for $l \geqslant k_0$ yields for $T \geqslant 2^{k_0-1}$: $T \in [2^{k-1}, 2^k]$ for some $k \geqslant k_0$ and therefore

$$A_p(f, T) \leqslant 2^{-(k-1)n/p} \|f\|_{(p)} (\Sigma_{j=1}^{k_0} 2^{jn})^{1/p} + 2^{-(k-1)n/p} \cdot \epsilon (\Sigma_{j=k_0+1}^{k} 2^{jn})^{1/p}.$$

Considering these two terms separately one observes that the second one is uniformly (with respect to k) bounded by some multiple of ϵ. On the other hand it is clear that the first one tends to zero if $k \to \infty$ (k_0 fixed). This shows that (26) (i.e. $f \in I^p$) can be derived from (29).

E) It remains to show that $K(\mathbb{R}^n)$ is dense in I^p. For $f \in I^p$ and $\epsilon > 0$ choose $k_0 \in N$ such that $2^{-kn/p} \mathrm{d}_k < \epsilon$ for $k \geqslant k_0 \geqslant 2$, hence $\|f - \varphi_{k-1}\|_{(p)} < < \epsilon$, where φ_{k-1} is the indicator function of $D_{k-1} := \{x \,|\, |x| \leqslant 2^{k-1}\}$. Since $\operatorname{supp}(g * \varphi_k f) \subseteq D_k$ whenever $g \in K(\mathbb{R}^n)$ with $\operatorname{supp} g \subseteq D_1$, and because (for obvious reasons) the norms $\| \ \|_p$ and $\| \ \|_{(p)}$ are equivalent on $L^p_{D_k} = M^p_{D_k}$, it follows that there exists $g \in K_{P_1}$ such that

$$\|g * \varphi_k f - \varphi_k f\|_{(p)} < \epsilon.$$

Since $g * \varphi_k f \in K(\mathbb{R}^n) * L^p_{D_k} \subseteq K(\mathbb{R}^n)$ the density of K in I^p follows therefrom.

F) Follows from Hölder's inequality. The assertions in G) follows from E) and the following estimate:

$$A_p^p(L_y f, T) \leqslant \frac{1}{T^n} \int_{|z| \leqslant 2\max(|y|, T)} |f(z)|^p \, dz \leqslant$$

$$\leqslant 2^n \max(1, (|y|/T)^n) \|f\|_{[p]}^p$$

which directly implies (30). Since one has $A_p(f, T) = A_p(\check{f}, T)$ the inversion invariance is obvious. After checking continuity of translation for $f \in K(\mathbb{R}^n)$ *it is a consequence of E) that I^p has continuous translation.*

REMARK B. Using the notations used in [8] one might say that $M^p = = F(\mathfrak{P}_2, L^p, l^\infty_{-1/p,0})$ and $I^p = F(\mathfrak{P}_2, L^p, c^0_{-1/p,0})$, both spaces being built over the dyadic decomposition $\mathfrak{P}_2$ of $\mathbb{R}^n$. The estimate (30) might thus be derived from Theorem 3.1 of [8] as well.

REMARK C. It is clear from E) that I^p is a separable Banach space, which contains a subspace isomorphic to c_0 (cf. [19], Prop. 2.3): Fixing a sequence $(f_k)_{k \geq 1}$ in $K(\mathbb{R}^n)$ with $\operatorname{supp} f_k \subseteq P_k$ and $\|f_k\|_p = 2^{kn/p}$, $\{f \mid f = \Sigma_{k=1}^{\infty} a_k f_k, (a_k)_{k \geq 1} \in c_0\}$ is such a space. Of course, $M^p(\mathbb{R}^n)$ is not separable.

In the next step we study duality relations involving the spaces $M^p(\mathbb{R}^n)$ and $I^p(\mathbb{R}^n)$.

THEOREM 3. A) Denoting as usual the conjugate index to p by q (with $1/p + 1/q = 1$), one has for $1 < p \leq \infty$:

$M^p(\mathbb{R}^n)$ is the Banach dual of $E^q(\mathbb{R}^n)$, defined for $1 \leq q \leq \infty$

$$E^q(\mathbb{R}^n) := \{h \mid h \in L^q_{\ell oc}, \|h\|_{E^q} := \Sigma_{k=1}^{\infty} 2^{kn/p} \|h\, \psi_k\|_q < \infty\},$$

where ψ_k is the characteristic function of P_k, $k \geq 1$.

B) $K(\mathbb{R}^n)$ and $S(\mathbb{R}^n)$ are dense in $(E^q, \| \ \|_{E^q})$ for $1 \leq q < \infty$ and thus E^q has continuous translation for $1 \leq q < \infty$.

C) $E^q(\mathbb{R}^n) = (I^p(\mathbb{R}^n))'$ as a Banach space for $1 \leq p < \infty$.

D) $(E^q, \| \ \|_{E^q})$ is translation and isometrically inversion invariant, satisfying

$$\text{(30')} \qquad |\!|\!| L_y |\!|\!|_{E^q} \leq C(1 + |y|)^{n/p} \qquad \text{for} \qquad y \in \mathbb{R}^n.$$

E) The Banach spaces E^q, I^p and M^p are Banach modules over the Beurling algebra $L^1_{n/p}(\mathbb{R}^n)$ with respect to convolution.

As a consequence one has the following extension of Theorem 4.5 of [19] to higher dimensions:

COROLLARY 4. $(I^p(\mathbb{R}^n))'' = M^p(\mathbb{R}^n)$ for $1 < p < \infty$.

PROOF OF THEOREM 3. It is easy to check that $(E^q, \| \ \|_{E^q})$ is a Banach space containing $K(\mathbb{R}^n)$ and $S(\mathbb{R}^n)$ as dense subspaces (cf. e.g. [8], Theorem 3.1. C). The duality assertions stated in A) and C) being again quite elementary we may leave them to the reader (cf. Theorem 3.6 of [8], or Theorem 2.8 in [12] for such results in full generality. The estimate (30') follows from (30), applying the duality stated in C). Continuity of translation then follows from the density of $K(\mathbb{R}^n)$. The structure of a Banach convolution module over the isometrically inversion invariant algebra $L^1_{n/p}$ for I^p and E^q now follows via vector-valued integration, as mentioned in the introduction (which has of course the same effect as pointwise definitions). In particular, one has (after renorming $L^1_{n/p}$ suitably with a norm, written again as $\| \ \|_{1,n/p}$):

$$\|g * f\|_{E^q} \leqslant \|g\|_{1,n/p} \|f\|_{E^q}$$

and for $f, g \in K(\mathbb{R}^n)$.

$$\|g * f\|_{[p]} \leqslant \|g\|_{1,n/p} \|f\|_{[p]}$$

In order to extend this last extimate to $f \in M^p$ we observe that for g, $k \in K(\mathbb{R}^n)$ and $f \in L^1_{loc} \subseteq S'$ the following relation holds true:

$$\langle g * f, k\rangle = \langle f, \check{g} * k\rangle.$$

It implies by means of A) not only

$$\|g * f\|_{[p]} = \sup_{\|k\|_{E^q} \leqslant 1} |\langle g * f, k\rangle| \leqslant \|\check{g}\|_{1,n/p} \|f\|_{[p]},$$

but shows as well that the module structure given by ordinary convolution on $M^p = (I^p)''$ coincides with the double transposition of the convolutive structure on I^p.

With these facts in mind one can now give the

PROOF OF THEOREM 1. It is well known that the Theorem of Wiener-Lévy on the local inversion of the Fourier transform holds true for the Beurling algebras $L^1_\alpha(\mathbb{R}^n)$, $\alpha \geqslant 0$ (cf. [23], Chap. 1, §6.5). Consequently the linear span of the set of translates of K_1 is dense in $L^1_{n/p}$ if (24) is satisfied. Since $(L_y K_1) * f = L_y (K_1 * f) \in I^p$ is f satisfies (25), for all $y \in \mathbb{R}^n$, it follows therefrom that (using Theorem 3. E)) that $K * f \in I^p$ for all $K \in L^1_{n/p}$.

REMARK $\mathcal{D}$. For the classical situation ($n = 1, p = 2$) the above argument implies not only that the integrability of $(1 + |x|) K_i(x)$, $i = 1, 2$ would have been sufficient for the conclusion, but that as well each of these assumptions might have been replaced by the condition $(1 + |x|)^\alpha K_i(x) \in L^2(\mathbb{R})$ for some $\alpha > 1/2$ (e.g. $\alpha = 1$). In fact, one has by Hölder's inequality

$$\|K\|_{1,1/2} \leqslant \left(\int |K(x)|^2 (1 + |x|)^{2\alpha} \mathrm{d}x\right)^{1/2} \left(\int (1 + |x|)^{-2\alpha} \mathrm{d}x\right)^{1/2} \leqslant \infty.$$

In order to show that other conditions concerning the decay (at infinity) of the kernels K_i, $i = 1, 2$ are sufficient as well, a more detailed analysis of E^q will be important. We shall show a minimality property of E^q first and derive therefrom that E^q is Banach convolution algebra. Before doing this we look for the behaviour of our spaces under dilations.

LEMMA 5. The Banach spaces E^q, M^p and I^p are invariant with respect to the group of dilations $(M_\rho)_{\rho>0}$ for $1<p,\ q\leqslant\infty$. For the norms of these operators one has the estimates

$$|||D_\rho|||_{[p]}\leqslant \max(1,\rho^{n/p}) \tag{31}$$

and

$$|||M_\rho|||_{E^q}\leqslant \max(1,\rho^{n/p}) \tag{31'}$$

PROOF. By the transformation formula one has for $T\geqslant 1$ (discussing the cases $\rho\leqslant 1$ and $\rho\geqslant 1$ separately):

$$A_p(D_\rho f,T)=A_p(f,T/\rho)\leqslant \max(\|f\|_{[p]},(\rho/T)^{n/p}A_p(f,1))\leqslant$$
$$\leqslant \max(1,\rho)^{n/p}\|f\|_{[p]},$$

which implies (31). The estimate (31′) follows from the duality $E^q=(I^p)'$:

$$\|M_\rho f\|_{E^q}=\sup_{\|g\|_{[p]}\leqslant 1}|\langle M_\rho f,g\rangle|=$$
$$=\sup_{\|g\|_{[p]}\leqslant 1}|\langle f,D_\rho g\rangle|\leqslant \max(1,\rho^{n/p})\|f\|_{E^q}.$$

REMARK E. Actually it is true that $|||D_\rho|||_{E^q}=|||M_\rho|||_{[p]}$ has exactly the asymptotic behaviour indicated by (31) or (31′) (consider, for example $\|M_\rho k\|_{E^q}$ for $\rho\to\infty$ for any $k\in K(\mathbb{R}^n)$).

The announced minimality property can now be described.

THEOREM 6. A) For $1\leqslant q\leqslant\infty$ the space $(E^q,\|\ \|_{E^q})$ is the minimal Banach space in $L^1_{\ell oc}$ (actually in $L^1(\mathbb{R}^n)$) with respect to the following two properties:

a) $L^q_K(\mathbb{R}^n)$ is continuously embedded for any compact set $K\subseteq\mathbb{R}^n$;

b) $(M_\rho)_{\rho\in(0,1]}$ acts uniformly bounded on the space.

In particular, the following 'atomic characterization' of E^q is available: Given any compact neighborhood Q of the origin in $\mathbb{R}^n$ (preferably the unit ball Q_1) one has $E^q=B^q$ (as a Banach space), which is given as follows.

$$B^q:=\{h\mid h=\Sigma_{j=1}^\infty M_{\rho_j}f_j \quad\text{in}\quad L^q(\mathbb{R}^n),\ \rho_j\in(0,1] \tag{32}$$
$$f_j\in L^q_Q \quad\text{for}\quad j\geqslant 1,\quad\text{with}\quad \Sigma_{j=1}^\infty\|f_j\|_q<\infty\},$$

with the norm

$$\|h\|_{\{q\}}:=\inf\{\Sigma_{j=1}^\infty\|f_j\|_q,\ldots\}, \tag{33}$$

the infimum being taken over all 'admissible' representations as in (32). Furthermore one has

$$(34) \qquad \|f\|_{\{q\}} = \|f\|_q \quad \text{for} \quad f \in L^q_Q,$$

and $(E^1, \| \ \|_{\{1\}}) = (L^1, \| \ \|_1)$.

B) $(E^q, \| \ \|_{\{q\}})$ is continuously embedded into $L^r(\mathbb{R}^n)$ for $1 \leqslant r \leqslant q$.

REMARK F. It is evident that different neighborhoods Q and Q' define equivalent norms on B^q. It is also an immediate consequence of the definition (assuming now for convenience $Q = Q_1 = \{x \mid |x| \leqslant 1\}$) that one has

$$(35) \qquad \|h\|_q \leqslant \|h\|_{\{q\}} \leqslant r^{n/p} \|h\|_q \quad \text{for all} \quad h \in L^q_{Q_r}(\mathbb{R}^n)$$

where $Q_r := \{x \mid |x| \leqslant r\} \subseteq \mathbb{R}^n$ for $r \geqslant 1$ (cf. step iii) below; the second estimate in (35) following from the representation h

$$h = M_{r^{-1}}(M_r f) + 0 + 0 + \ldots, \quad \text{and using (7)).}$$

PROOF OF THEOREM 6. i) First of all one may check that $(B^q, \| \ \|_{\{q\}})$ is a normed subspace of $(L^q, \| \ \|_q)$. That it is actually continuously embedded in L^r for $1 \leqslant r \leqslant q$ (and that any 'admissible' in actually absolutely convrgent in each L^r) follows from the following estimate (writing here r' for the conjugate of r):

$$\|h\|_r \leqslant \Sigma_j \|M_{\rho_j} f_j\|_r \leqslant \Sigma_j \rho_j^{n/r'} \|f_j\|_r \leqslant$$
$$\leqslant |Q|^\beta \Sigma_j \|f_j\|_q < \infty \quad (\text{with } \beta := 1/r - 1/p)$$

for any 'atomic representation' of $h \in B^q$, hence

$$\|f\|_r \leqslant |Q|^\beta \|f\|_{\{q\}} \quad \text{for} \quad f \in B^q.$$

Using this it is not difficult to verify that absolutely convergent series are convergent in $(B^q, \| \ \|_{\{q\}})$, i.e. that it is a Banach space which has of course the required invariance property under $(M_\rho)_{\rho \in (0,1]}$. It is also clear from the construction that B^q has the required minimality property (and is thus independent of the choice of Q, cf. [9] for a related situation).

ii) In order to show that B^q coincides with E^q we verify first that E^q is continuously embedded in B^q. In fact, one has for $h \in E^q$ the representation

$$h = \Sigma_{k=1}^\infty h\psi_K = \Sigma_{k=1}^\infty M_{2^{-k}} f_k, \quad \text{with} \quad f_k := M_{2^k}(h\psi_k).$$

This is actually an admissible representation, and since

$$\|f_k\|_q = 2^{kn/q}\,\| h\,\psi_k\|_q$$

one obtains the estimate

$$\| h \|_{\{q\}} \leqslant \Sigma_{k=1}^{\infty} \|f_k\|_q \leqslant \Sigma_{k=1}^{\infty} 2^{kn/q}\,\| h\,\psi_k\|_q = \| h \|_{E^q} < \infty .$$

Although it would be possible to verify the converse inclusion by a direct device (using the atomic structure of B^q) it is more convenient to observe that E^q satisfies the requirements a) and b) (cf. Lemma 5).

iii) In order to verify (34) note that $\| h \|_q \leqslant \| h \|_{\{q\}}$ is always true (cf. i) above); conversely the trivial representation $h = h + 0 + 0 + \ldots$ is an admissible one for $h \in L^q_Q$, yielding the other inequality. The equality $E^1 = L^1$ is obvious.

In order to improve Theorem above we shall stress the analogy between Wiener's Theorem 4 and 29 (cf. [27], p. 73/74 and p. 117). Actually, one has in both cases (besides the Tauberian condition on K_1) the element f is choosen from the dual of some solid BF-space B on $\mathbb{R}$ (concrete: $B = L^1 = E^1$ and $B = E^2$ respectively). As will be shows below additional algebraic structure (i.e. the convolutive structure) of that predual will allow to persue the analogy further. However, before discussing it let us show that these two results may in fact be considered as particular cases of a real-parameter scale of result:

PROPOSITION 7. The decreasing family $(E^q, \| \; \|_{\{q\}})$, $1 \leqslant q \leqslant \infty$ of Banach spaces is closed with respect to complex interpolation, i.e. for $\theta \in (0, 1)$ one has

$$(36) \qquad (E^{q_1}, E^{q_2})_{[\theta]} = E^q, \quad \text{with} \quad 1/q = (1-\theta)/q_1 + \theta/q_2 .$$

In particular

$$(37) \qquad (L^1, E^2)_{[\theta]} = E^q, \quad \text{with} \quad q = 2/(2-\theta) \ (\text{or } \theta = 2/p).$$

As a consequence, the family $(M^p, \| \; \|_{[p]})_{1 < p \leqslant \infty}$ forms a scale of interpolation spaces with respect to the upper complex method. In particular, one has

$$(38) \qquad (M^p, L^\infty)^{[\theta]} = M^r, \quad \text{with} \quad 1/r = (1-\theta)/p .$$

PROOF. It is clear from Theorem 5A) that $E^q = B^q \subseteq B^s = E^s$ for $q \geqslant s$. $(E^q, \| \; \|_{\{q\}})$ being the retract of a vector-valued sequence space the formulas (36) and (37) follow from corresponding results concerning the interpolation of such spaces (see [2], Theorems 6.4.2 and 5.6.3; cf. [8], Theorem 3.6). In view of Theorem 3.A) the result concerning the M^p-spaces is an immediate consequence of Theorem 4.5.1 in [2].

The convolutive structure of E^q is described now. Recall that a Banach algebra $(A, \| \ \|_A)$ of continuous functions on $\mathbb{R}^n$ satisfies the condition of Wiener-Ditkin (cf. [23], Chap. 2) if for $t \in \mathbb{R}^n$ there exists a sequence $(h_n)_{n \geqslant 1}$ in A such that $h_n(s) = 0$ near t and, whenever $h \in A$ satisfies $h(t) = 0$, $\| h_n h - h \|_A \to 0$ for $n \to \infty$.

THEOREM 8. A) For $1 \leqslant q < \infty$ the normed space $(E^q, \| \ \|_{\{q\}})$ is a Banach algebra in $L^1(\mathbb{R}^n)$ (i.e. with respect to convolution).

B) $\mathsf{F}E^q := \{\hat{h} \mid h \in E^q\}$, with the norm $\|\hat{h}\|_{\}q\{} := \|h\|_{\{q\}}$ is a pointwise Banach algebra on $\mathbb{R}^n$ satisfying the condition of Wiener-Ditkin.

C) The theorem of Wiener-Lévy applies to E^q, i.e. given $h_0 \in E^q$ with $\hat{h}_0(t) \neq 0$ on a compact set $M \subseteq \mathbb{R}^n$ there exists $h_1 \in E^q$ with $\hat{h}_1 = 1/\hat{h}_0(t)$ for all $t \in M$.

D) The set of translates of $h_0 \in E^q$ is total in $(E^q, \| \ \|_q)$ if and only if $\hat{h}_0(t) \neq 0$ for all $t \in \mathbb{R}^n$.

E) $(E^q)^{\mathsf{K}} := \{h \mid h \in E^q, \operatorname{supp} \hat{h} \text{ compact}\}$ is a dense ideal in $(E^q, \| \ \|_{\{q\}})$.

F) $(\mathsf{F}E^q, \| \ \|_{\}q\{})$ is a Wiener algebra in the sense of Reiter ([23], Chap. II).

COROLLARY 9. For $1 < p < \infty$ $(M^p, \| \ \|_{(p)})$ is a Banach module over the Banach algebra $(E^q, \| \ \|_{\{q\}})$ with respect to convolution, containing I^p and M_c^p as closed (essential) submodules.

PROOF. The first assertion follows from Theorem 8.A) combined with Theorem 3.A):

$$\begin{aligned} \| g * \sigma \|_{[p]} &\leqslant C \,.\, \sup \{ |\langle g * \sigma, f \rangle|, f \in E^q, \|f\|_{\{q\}} \leqslant 1 \} = \\ &= C \,.\, \sup \{ |\langle \sigma, \check{g} * f \rangle|, \dots \} \leqslant \\ &\leqslant C_1 \|\sigma\|_{[p]} \|\check{g}\|_{\{q\}} = C \|\sigma\|_{[p]} \|g\|_{\{q\}}. \end{aligned}$$

Since translation is continuous in E^q it follows that $E^q * M^p \subseteq M_c^p$. On the other hand, since $f \in M_c^p$ can be approximated by elements of $\mathsf{K} * M^p \subseteq E^q * M^p$ it is clear that M_c^p is the essential part of the E^q-module M^p. Finally, $\mathsf{K}(\mathbb{R}^n)$ being a convolution algebra, the density of K in E^q (Theorem 3.C) and I^p (Theorem 2.E) implies the relation $E^q * I^p \subseteq I^p$.

PROOF OF THEOREM 8.A). By Theorem 6.A E^q is embedded into L^1. In view of the 'atomic' structure of $E^q = B^q$ the relevant work to be done is to give an estimate in B^q for elements of the form $f_3 := M_{\rho_1} f_1 * M_{\rho_2} f_2 =$

$= M_{\rho_2}(M_{\rho_3} f_1 * f_2)$ with $\rho_3 := \rho_1 \rho_2^{-1}$. Assuming without loss of generality that $\rho_2 \leqslant \rho_1 \leqslant 1$, i.e. $\rho_3 \geqslant 1$ and by means of Remark F) (note that in this case $\operatorname{supp}(M_{\rho_3} f_1 * f_2) \subseteq Q_1 + Q_1 \subseteq Q_2$!) one obtains:

$$\|f_3\|_{\{q\}} \leqslant \|M_{\rho_3} f_1 * f_2\|_{\{q\}} \leqslant 2^{n/p} \|f_1\|_1 \|f_2\|_q \leqslant 2^{n/p} |Q|^{1/p} \|f_1\|_q \|f_2\|_q .$$

Summation over atoms then yields (using suitable representations for h_1, $h_2 \in B^q$) for $C_0 > 0$:

$$\|h_1 * h_2\|_{\{q\}} \leqslant C_0 \|h_1\|_{\{q\}} \|h_2\|_{\{q\}}. \tag{38}$$

(If convenient one may of course replace the norm $\| \ \|_{\{q\}}$ by the equivalent one $C_0 \| \ \|_{\{q\}}$ which is submultiplicative).

B) Since E^q is a subalgebra of $L^1(\mathbb{R}^n)$ by A) above, we may use the ordinary Fourier transform. It is than clear that FE^q, endowed with its natural norm $\| \ \|_{\}q\{}$ is a pointwise Banach algebra of continuous functions on $\mathbb{R}^n$. In order to show that it satisfies the condition of Wiener-Ditkin it has to be shown that for $f \in E^q$, $\epsilon > 0$ there exists $\tau_1 \in E^q$, with

$$\hat{\tau}_1(x) \equiv 1 \text{ near zero, satisfying } \|f * \tau_1 - \hat{f}(0)\tau_1\|_{\{q\}} < \epsilon. \tag{39}$$

(Since E^q is solid FE^q is translation invariant and the origin does not play a particular role for this question).

It turns out that (39) can be obtained by a slight modification of the arguments for Lemma 1.2.4 in Reiter's book ([23], p. 6): Let $k \in \mathcal{D}$ be any test function satisfying $k(t) \equiv 1$ near the origin. Then $\tau := F^{-1} k$ belongs to $S(\mathbb{R}^n)$, hence E^q, and the set $\{M_\rho \tau \mid \rho \in (0, 1]\}$ is bounded in E^q. As τ_1 will be choosen from this set it will therefore be sufficient to check the validity of (39) for $f \in K(\mathbb{R}^n)$ (the dense subspace). For $f \in K(\mathbb{R}^n)$, however, we may argue as follows: $M_{\rho^{-1}} f \to \hat{f}(0)\delta_0 (= (\int f(y)\,dy)\delta_0)$ in the vague sense, for $\rho \to 0$. Since the net $(M_{\rho^{-1}} \tau)_{\rho \in (0,1]}$ is concentrated on a compact set in $\mathbb{R}^n$ it follow by the usual argument (i.e. by the interpretation of the convolution product as a vector-valued integral with continuous integrand, because E^q has continuous translation) that one has

$$\lim_{\rho \to 0} \|M_{\rho^{-1}} f * \tau - \hat{f}(0)\,\tau\|_{\{q\}} = 0. \tag{40}$$

Applying now the (bounded) stretching operator M_ρ it is clear that the estimate in (39) is valid for any $\tau_1 := M_\rho \tau$, with $\rho \geqslant \rho_0$.

C) - F) Since the proof of the Theorem of Wiener-Lévy as well as of its consequences follows now exactly the same pattern as in the case of $FL^1(\mathbb{R}^n)$ (cf. [23], p. 7/8) we need not give any detail here.

REMARK G. Of course it would have been possible to take τ to be the classical de la Vallée-Poussin kernel for the real line (and certain products of tensor-products of these kernels for higher dimensions) in the above proof.

COROLLARY 10. Theorem 1 holds true if one assumes $K_1, K_2 \in E^q$ (all other conditions remaining unchanged).

REMARK H. For $p = 2$, $n = 1$ this is an other improvement of the classical special case (Theorem 29 of Wiener), because the condition $(1 + |x|)K(x) \in$ $\in L^2(\mathbb{R})$ used there implies that

$$d'_n \sim 2^{n/2}\left(\int_{2^n \leqslant |x| \leqslant 2^{n+1}} |K(x)|^2 \, dx\right)^{1/2}$$

defines a uniformly bounded sequence (for $n \geqslant 1$), i.e. that $K \in E^2$, but not necessarily $K \in L^1_{1/2}$.

It is perhaps appropriate to comment here on related results in the literature, which have been obtained by completely different methods:

REMARK K. That $(E^p, \| \ \|_{E^p})$ is a Banach convolution algebra for $1 \leqslant p <$ $< \infty$ can be seen as an immediate consequence of the fact that the space $F(\mathbb{P}_2, L^p, l^1_{a,b})$ (introduced in [8] and staying with the conventions of using p and q as in [8] in this remark only) is a Banach convolution algebra for $a = n/p'$ and $b \geqslant 0$ (see Theorem 4.3 of [8] $1/p' = 1 - 1/p$ there). Besides the general structural properties it is important to observe that the space is contained in $L^1(\mathbb{R}^n)$. Unfortunately just this exceptional case ([8], p. 196: $q = 1$, $a = n/p' = n(p-1)/p$, $b = 0$) has been stated incorrectly. Unlike this particular case one has for $q > 1$ (as stated there) indeed only for $b > 1/p'$ the embedding into $L^1(\mathbb{R}^n)$ for the critical index $a = n/p'$. It was for this reason, and in order to demonstrate the advantage arising from the atomic approach that we have given the proof in the above, conceptually different form. However, the use of the other Banach convolution algebras as described in [8] would allow to proof related Tauberain theorems not discussed in this paper.

REMARK L. In this paper [5] A. Beurling (cf. Theorem II) considered already the algebras $A^q(\mathbb{R}^n)$ which can be shown to coincide with E^q (e.g. by means of results of C. Herz in [14] and R. Johnson, cf. [16]). Again, it must be said that following their concepts from the beginnings would be an arduous task. We also mention that R. Johnson already conjectured that it should be possible to use decomposition spaces for the proof of general Tauberian theorems (cf. [16], p. 132).

REMARK M. In the case $q = 2$ the fact that $E^2 = F^{-1}B^2_{1,1/2}$ (cf. e.g. [25], §2.8.2) is a Banach convolution algebra corresponds to the fact that $B^2_{1,1/2}$ is an Banach algebra of (continuous) functions with respect to pointwise multiplication (cf. [24], §2.6.2, Theorem 1, p. 133). A detailed discussion of Ditkin conditions of such algebras (which should be possible for the algebras FE^q as well) has been carried out by G. Bennett and J. Gilbert (cf. [1]).

§.4. FURTHER VARIANTS AND EXTENSIONS

A comparison of the conditions imposed on the kernels K_1, K_2 in Theorem 1 and Corollary 10 shows that there are no implications between them. Thus it is desirable to look for a joint generalization that should apply at least to kernels $K_i = K_i^1 + K_i^2$, with $K_i^1 \in L^1_{n/p}$ and $K_i^2 \in E^q$ respectively, for $i = 1, 2$. That this is possible will be shown below. Because the arguments yielding this more general version of a Tauberian theorem are quite similar to those used in the proof of Theorem 5 in [7] we start by stating explicitely a special case of that result which is closely related to Theorem 6.2 of [20]. It is obtained by the choice $A = E^q, A' = M^p, A'_0 = I^p$ there. The assumptions concerning A are satisfied in this case and therefore one has:

COROLLARY 11. Assume that $\mu_0 \in S'(\mathbb{R}^n)$ defines a multiplier for I^p with nonvanishing Fourier transform (i.e. $\| \mu_0 * f \|_{[p]} \leqslant C(\mu_0) \| f \|_{[p]}$ for all $f \in S(\mathbb{R}^n)$ and $\hat{\mu}_0(t) \neq 0$ for all $t \in (\mathbb{R}^n)$). If a given $f \in M^p_c$ satisfies $\mu_0 * f \in I^p$ then one has $\mu * f \in I^p$ for any $\mu \in S'$ which defines a multiplier on I^p.

The above Corollary also follows immediately from the following result, which does not impose the additional condition $f \in M^p_c$ (in contrast to Theorem 1 and Corollary 7), or at least allows to replace it by weak additional conditions on μ.

THEOREM 12. Assume that $\mu_0 \in S'(\mathbb{R}^n)$ defines a multiplier for I^p with nonvanishing Fourier transform. Let $f \in M^p$ be given such that $\mu_0 * f \in I^p$. Then for any multiplier μ of I^p the following is true: $\mu * f \in I^p$ if and only if $\mu * f \in M^p_c$.

PROOF. We mention once more that any multiplier μ of I^p defines also a multiplier for E^q. Since E^q is contained in L^1 it follows that $\hat{\mu}$ (in particular $\hat{\mu}_0$) belongs locally to FE^q. In particular, it is a continuous function and therefore the pointwise condition $\hat{\mu}_0(t) \neq 0$ makes sense.

Since $I^p \subseteq M^p_c$ it is clear that only one implication has to be discussed. Assume thus $\mu * f \in M^p_c$. Then, for $\epsilon > 0$ there exists $k \in L^1_{n/p}$, with supp $\hat{k}$ compact and $\| k * \mu * f - \mu * f \|_{[p]} < \epsilon$. Since FE^q has local inversion (cf. Theorem 7.C) there exists $g \in E^q$ such that $\hat{g}(t) = 1/\hat{\mu}_0(t)$ for all $t \in$ supp k. It follows that $k = k * \mu_0 * g$. Consequently $k * f = k * g * \mu_0 * f \in L^1_{h/p} * I^p \subseteq I^p$ and therefore $k * \mu * f = \mu * (k * f) \in I^p$. Since I^p is closed in M^p the desired conclusion follows.

REMARK N. Since the interesting implication above starts with the assumption $\mu * f \in M^p_c$ it is useful to describe sufficient conditions for this to be true. Of course, it is satisfied for any multiplier μ of M^p, if f happens to belong to M^p_c. On the other hand it is satisfied if one has $\mu \in L^1_{n/p} + E^q$, because in that case one has

$$(47) \qquad \mu * f = \lim_{\alpha \to \infty} k_\alpha * f \quad \text{for all} \quad f \in M^p(\mathbb{R}^n)$$

for a suitable net $(k_\alpha)_{\alpha \in I}$ in $K(\mathbb{R}^n)$. Actually, the approximation of the multiplier $f \mapsto \mu * f$ in the strong operator topology by elementary multipliers of the form $f \mapsto k * f$ is sufficient in order to have $\mu * f \in M^p_c$ for any $f \in M^p$.

The following result is mentioned here, because practically the same proof applies (stated in the terminology of Wiener-Type spaces).

THEOREM 13. Let G be a locally compact abelian group.

Let a multiplier $f \mapsto \mu_0 * f$ of $W(L^p, C^0)$, $1 \leqslant p < \infty$, with non-vanishing Fourier transform (i.e. $\hat{\mu}_0(t) \neq 0$ for all $t \in \mathbb{R}^n$). Assume that $\mu_0 * f \in W(L^p, C^0)$ for some $f \in W(L^p, L^\infty)$. Then one has for any multiplier ν of $W(L^p, C^0)$: $\nu * f \in W(L^p, L^\infty)_c$ if and only if $\nu * f \in W(L^p, C^0)$.

PROOF. It is sufficient to observe that $W(L^p, C^0)$ is the closure of $K(G)$ in $W(L^p, l^\infty) = W(L^q, l^1)' = A'$. Since A is a Segal algebra (with its natural norm), hence a dense ideal in $L^1(G)$, the pointwise algebra FA coincides with the ordinary Fourier algebra $A(\hat{G}) = FL^1(\hat{G})$ and has thus local inversion (cf. [23], Chap. VI).

In view of Theorem 12 it will be of interest to give sufficient conditions implying that $\sigma \in S'(\mathbb{R}^n)$ defines a multiplier for I^p (hence on E^q, M^p, by transposition). A result in this direction is given in the next Proposition, which involves once more an atomic approach.

PROPOSITION 14. Let $1 < p < \infty$ be given. Then $\sigma \in S'(\mathbb{R}^n)$ defines a multiplier of I^p, E^q and M^p if it has a representation (in S') as a

$$\sigma = \Sigma_{j=1}^{\infty} M_{\rho_j} \sigma_j \tag{41}$$

with $(\rho_j)_{j \geqslant 1}$ being any sequence in $(0, 1]$ and $(\sigma_j)_{j \geqslant 1}$ a sequence of L^p-multipliers (hence in $S'(\mathbb{R}^n)$) with common compact support, satisfying the condition

$$\Sigma_{j=1}^{\infty} \| \! | \sigma_j * \| \! |_p \, \rho_j^{-n/p} < \infty. \tag{42}$$

If $\sigma_j \in L^1(\mathbb{R}^n)$ for all $j \geqslant 1$ then σ maps M^p into M_c^p.

PROOF. We show first that $f \to \sigma * f$ defines a multiplier for I^p. After normalization we may assume that $\operatorname{supp} \sigma_j \subseteq Q$ for $j \geqslant 1$. In order to estimate the action of a given 'atom' σ_j let us choose an appropriate representation of $f \in E^q : f = \Sigma_{k=1}^{\infty} M_{\rho_k} f_k$. This gives

$$\sigma_j * f = \Sigma_{k=1}^{\infty} \sigma_j * M_{\rho_k} f_k = \Sigma_{k=1}^{\infty} M_{\rho_k} (M_{\rho_k^{-1}} \sigma_j * f_k).$$

Since it is known (and not difficult to verify) that

$$\| \! | M_\rho \sigma * \| \! |_q = \| \! | \sigma * \| \! |_q = \| \! | \sigma * \| \! |_p$$

and because

$$\operatorname{supp} M_{\rho_k^{-1}} \sigma_j * f_k \subseteq \rho_k Q + Q \subseteq Q + Q$$

it follows that

$$\begin{aligned} \| \sigma_j * f \|_{E^q} &\leqslant C \cdot \Sigma_{k=1}^{\infty} \| M_{\rho_k^{-1}} \sigma_j * f_k \|_q \leqslant \\ &\leqslant \| \! | \sigma_j * \| \! |_p \cdot C \, \Sigma_{k=1}^{\infty} \| f_k \|_q \leqslant C' \| \! | \sigma_j * \| \! |_p \| f \|_{E^q}, \end{aligned}$$

hence $\| \! | \sigma_j * \| \! |_{E^q} \leqslant C' \| \! | \sigma_j * \| \! |_p$.

For σ as described above one has thus for $f \in E^q$ as a by means of Lemma 5:

$$\begin{aligned} \| \sigma * f \|_{E^q} &\leqslant \Sigma_{j=1}^{\infty} \| M_{\rho_j} \sigma_j * f \|_{E^q} = \Sigma_{j=1}^{\infty} \| D_{\rho_j} (\sigma_j * D_{\rho_j^{-1}} f \|_{E^q} \leqslant \\ &\leqslant \Sigma_{j=1}^{\infty} \| \! | D_{\rho_j} \| \! |_{E^q} \| \! | \sigma_j * \| \! |_{E^q} \| \! | D_{\rho_j^{-1}} \| \! |_{E^q} \| f \|_{E^q} \leqslant \end{aligned}$$

$$\leqslant C' \|f\|_{E^q} \Sigma_{j=1}^{\infty} \| \sigma_j * \|_p \rho_j^{-n/p},$$

as was required. Of course $\check{\sigma}$ is a multiplier on E^q as well and the infimum over all expressions (42) defines a norm majorizing the operator norm on E^q (up to some constant), and therefore $\sigma = (\check{\sigma})\check{}$ defines a multiplier on $M^p = (E^q)'$ (by transposition). Since furthermore the partial sums approximate $\sigma *$ even with respect to the operator norm on E^q (and M^p) (cf. above) it follows that I^p is left invariant by the multiplier $f \mapsto \sigma * f$. If, furthermore $\sigma_j \in L^1(\mathrm{IR}^n)$, then the partial sums may be approximated by elementary multipliers of the form $g * \Sigma_{j=1}^k \sigma_j \in K(\mathrm{IR}^n)$. Since $K * M^p \subseteq M_c^p$ the proof is now complete.

REMARK O. Without going into details we mention that σ has a representation as described above if and only if it belongs to a certain decomposition space (as described in [12]), with local component being the space of all L^p-multipliers, and a global weighted l^1-condition, and with the dyadic decomposition of IR^n as underlying scheme. Such spaces can be described in various different ways which are to be explained elsewhere.

A slight change in the perspective occurs if one considers – instead of multipliers on I^p which extend to M^p – those multipliers of M^p which leave I^p invariant. One result in this direction is given in the following characterization:

THEOREM 15. Let $\sigma \in S'$ be a multiplier for M^p. Then σ leaves I^p invariant if and only if there is a net $(k_\gamma)_{\gamma \in \Gamma}$ in $K(\mathrm{IR}^n)$ such that

$$\sigma * g = \lim_{\gamma \to \infty} k_\gamma * g \quad \text{for} \quad g \in K(\mathrm{IR}^n) \tag{43}$$

and

$$\sup_{\gamma \in \Gamma} \| k_\gamma * g \|_{[p]} \leqslant C \| g \|_{[p]} \quad \text{for} \quad g \in K(\mathrm{IR}^n). \tag{44}$$

PROOF. (43) implies that $\sigma * g \in (K * I^p)^- \subseteq I^p$ for $g \in K(\mathrm{IR}^n)$. On the other hand, (44) shows that the family $(k_\gamma *)_{\gamma \in \Gamma}$ is uniformly bounded in the operator algebra over I^p, and therefore (by the density of K in I^p) (43) holds for any $g \in I^p$, i.e. $\sigma * I^p \subseteq I^p$.

Assume now the converse, i.e. $\sigma * I^p \subseteq I^p$. Then given $\epsilon > 0$, and $g \in K \subseteq I^p$ one has $\| \sigma * g - \mathrm{d} * \sigma * g \|_{[p]} < \epsilon$ (cf. (22)) for a suitable test function $\mathrm{d} \in \mathcal{D} \subseteq K$. Density of K in I^p also implies continuity of $t \to M_t h$ for any $h \in I^p$ and consequently continuity of $t \to M_t (\sigma * \mathrm{d}) * g = M_t[(\sigma * \mathrm{d}) * M_{-t} g]$, by the estimate

$$\|M_t[(\sigma * \mathrm{d}) * M_{-t}g] - M_s[(\sigma * \mathrm{d}) * M_{-s}g]\|_{[p]} \leqslant$$
$$\leqslant \|(M_t - M_s)(\sigma * \mathrm{d} * M_{-t}g)\|_{[p]} + \|M_s[\sigma * \mathrm{d} * (M_{-t} - M_{-s})g]\|_{[p]}$$

using the fact that M_s, $s \in \mathbb{R}_t$, acts isometrically on the solid BF-space I^p, and that $T_1 : f \to \sigma * \mathrm{d} * f$ is a bounded operator on I^p. Let us fix a neighborhood U of zero such that

(45) $$\|M_t \sigma * \mathrm{d} * g - \sigma * \mathrm{d} * g\|_{[p]} < \epsilon \quad \text{for} \quad t \in U.$$

As an integral of a continuous, vector-valued function with values in I^p one has for any $h \in L^1(\mathbb{R}^n)$

(46) $$\int h(t)[M_t(\sigma * \mathrm{d})] * g \, \mathrm{d}t = [\hat{h}(\sigma * \mathrm{d})] * g$$

(a detailed justification of this formula is left to the reader). If $h \in \mathcal{S}(\mathbb{R}^n)$is choosen in such a way that $\hat{h} \in \mathcal{D}$, with $\hat{h}(0) = \int h(t)\,\mathrm{d}t = 1$ and supp $\hat{h} \subseteq U$ one has in view of (45) and

$$\|[\hat{h}(\sigma * k)] * g - \sigma * g\|_{[p]} < \epsilon.$$

Since $\hat{h}(\sigma * k) \in \mathcal{D}(\mathcal{S}' * \mathcal{D}) \subseteq \mathcal{D} \subseteq K(\mathbb{R}^n)$ assertion (45) is verified. The construction also yields (44), and the proof is thus complete.

§5. SUBORDINATIVE OPERATORS AND ABSTRACT TAUBERIAN THEOREMS

A still more abstract approach, indicated by the results given by Lau ([20]) and J.P. Bertrandias ([4]) suggests to deal with subordinative operators: A bounded, linear operator on a translation-invariant BF-space $(B, \| \ \|_B)$ is called *subordinative* (on B) if Tf belongs to the closed, translation invariant subspace of B generated by $\{L_x f, x \in \mathbb{R}^n\}$, for all $f \in B$. Several simple observations concerning subordinative operators are collected in the following proposition:

PROPOSITION 16. Let $(B, \| \ \|_B)$ be a translation invariant BF-space on $\mathbb{R}^n$. Then one has

i) Any closed, translation invariant subspace of $(B, \| \ \|_B)$ is invariant under subordinative operators.

ii) The set of all subordinative operators is a subalgebra of the operator algebra $L(B)$, closed respect to the strong operator topology.

In case $(B, \| \ \|_B)$ has continuous translation one has further:

iii) T is subordinative if and only if for any $f \in B$ Tf belongs to the closure of $\{k * f \mid k \in K(\mathbb{R}^n)\}$ in $(B, \| \ \|_B)$

iv) For any $k \in K(\mathbb{R}^n)$ the (elementary) multiplier $T_k : f \to k * f$ is a subordinative operator.

As a matter of routine (i.e. by working with vector-valued integrals) the proof is left to the reader. Of course, $\mathbb{R}^n$ might be replaced by any lca . group G.

REMARK P. From ii) and iv) above, applied to M^p, it follows $\sigma = g_1 + g_2$, with $g_1 \in L^1_{n/p}$, $g_2 \in E^q$, or σ as described at the end of Proposition 14, define subordinative operators on M^p.

The following characterization of subordinative operators will allow us to derive from Theorem 12 a result which is in a certain sense the n-dimensional extension of K. Lau's Theorem 6.2 (in [20]) as a corollary.

THEOREM 17. Let T be a bounded linear operator on I^p. Then the following conditions are equivalent:

i) T is subordinative on I^p;

ii) T commutes with translations, i.e. $TL_x = L_x T$ for all $x \in \mathbb{R}^n$;

iii) There exists a uniquely determined $\sigma \in S'(\mathbb{R}^n)$, such that $Tf = \sigma * f$ for all $f \in S(\mathbb{R}^n)$ (i.e. Tf is represented by the continuous function $x \to \to \langle L_x \check{f}, \sigma \rangle$);

iv) There exists (a uniquely determined function) $h \in C^b(\mathbb{R}^n)$ such that $(Tf)^\wedge = h\hat{f}$ for all $f \in S(\mathbb{R}^n)$.

PROOF. The implications iv) $\Rightarrow$ iii) $\Rightarrow$ ii) are easily checked. That iii) implies i) may be considered as a consequence of Theorem 15. In fact, as a consequence of (44) the relation (43) holds true for any $g \in I^p(\mathbb{R}^n)$. The most interesting implication is i) $\Rightarrow$ iv). Its proof requires several steps. That ii) $\Rightarrow$ iv) holds true will be proved at the end by similar arguments.

Step I). Given a subordinative operator T on I^p let us first consider the associated operator $\hat{T} : k \to F(TF^{-1}k)$ (or: $\hat{T}\hat{f} = (Tf)^\wedge$ for short) as mapping from the dense subspace $D(\mathbb{R}^n) \subseteq FI^p(\mathbb{R}^n)$ to $D'(\mathbb{R}^n) \supseteq FI^p(\mathbb{R}^n)$ (cf. Theorem 3.B). We claim that T does not increase supports. This can be shown as follows: Since I^p is-continuously embedded into D' with the weak ($= \sigma(S', S)$ - topology) one has for $f \in I^p : Tf = \text{weak-lim}_{\alpha \to \infty} v_\alpha * f$, for a suitable not of (discrete) finite measures. It follows that $\hat{T}\hat{f} = \text{weak-lim}_{\alpha \to \infty} \hat{v}_\alpha \hat{f}$. Let now $t \in \mathbb{R}^n \setminus \operatorname{supp} \hat{f}$ be given. Then $k\hat{f} = 0$ for all $k \in D_U$ (for some small neighborhood U of t), and consequently $k(\hat{T}\hat{f}) =$ weak $\lim_{\alpha\to\infty} k\hat{v}_\alpha \hat{f} = 0$, i.e. $t \notin \operatorname{supp} \hat{T}\hat{f}$, as required.

Step II) In view of the duality relations it is clear that $\hat{T}'$, as an operator on $\mathcal{F}E^q$ with its canonical norm $\| h \|_{\mathcal{F}E^q} := \| \mathcal{F}^{-1} h \|_{E^q}$. It is not difficult to check that $\hat{T}'$ does not increase supports as well. It may therefore be considered as a mapping from $\mathcal{D} \subseteq \mathcal{F}E^q$ to $\mathcal{K}(\mathbb{R}^n)$ satisfying for some $C > 0$:

$$(48) \qquad \| \hat{T}'h \|_\infty \leqslant C \| h \|_{\mathcal{F}E^q} \quad \text{for all} \quad h \in \mathcal{D}(\mathbb{R}^n)$$

Step III) Let now $K \in \mathcal{D}'(\mathbb{R}^{2n})$ be the kernel associated with $\hat{T}'$ by the Schwartz kernel theorem ([15], Theorem 5.2.1). It is characterized by the formula

$$(49) \qquad \langle \hat{T}'h, g \rangle = \langle K, h \otimes g \rangle \quad \text{for} \quad h, g \in \mathcal{D}(\mathbb{R}^n).$$

We intend to show that K is supported by the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$. In fact, assuming that there exists $(x, y) \in \operatorname{supp} K$ for some $x \neq y$ one obtains a contradiction as follows: Let U, V be disjoint, compact neighborhoods of x and y respectively. It is then possible to find $h \in \mathcal{D}_U$ and $g \in \mathcal{D}_V$ such that $\langle K, h \otimes g \rangle \neq 0$. This leads to a contradiction, because $\operatorname{supp}(\hat{T}'h) \cap \operatorname{supp} g \subseteq U \cap V = \phi$ implies $\langle \hat{T}'h, g \rangle = 0$.

Step IV) Since K is supported by the diagonal, $\hat{T}'$ has to be of the following form (by Theorem 5.2.3 in [15]):

$$(50) \qquad \hat{T}'h = \Sigma a_\alpha \, \partial^\alpha h, \quad \text{with} \quad a_\alpha \in \mathcal{D}', \quad \text{for} \quad h \in \mathcal{D}$$

the sum being locally finite. Thus, for any given $h \in \mathcal{D}$ one may assume that only derivates of order $|\alpha| \leqslant k$ (for some $k \in N$) appear in this sum. We intend to show that $a_\alpha \in C^b(\mathbb{R}^n)$ for any multiindex α and then that $a_\alpha = 0$ for $\alpha \neq 0$. That only regular distributions a_α, given by continuous functions may appear in our case is checked as follows (cf. [21] p. 126):

In view of the local character of this question and the translation invariance of the situation it will be sufficient to consider the family (a_α) near the origin. Starting with a test function $h \in \mathcal{D}(\mathbb{R}^n)$ satisfying $h_0(t) \equiv 1$ in an open neighborhood U of zero one has $\hat{T}h_0 = a_0 h_0 = a_0$ on U. But $\hat{T}'$ maps $\mathcal{D} \subseteq \mathcal{F}E^q$ to $\mathcal{F}E^q \subseteq C^0(\mathbb{R}^n)$ and therefore a_0 has to be continuous function on U. For higher indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ the same assertion can be proved by recurrence. Let α be given and assume that a_β is continuous on U for $\beta \neq \alpha$ satisfying $\beta \prec \alpha$ (i.e. if $\beta_j \leqslant \alpha_j$ for $1 \leqslant j \leqslant n$).

Without loss of generality we assume $\alpha_1 = \beta_1 + 1$, and $\alpha_j = \beta_j$ for $j \geqslant 2$. Then one has for the test function $h_\alpha := x^\alpha h_0/\alpha!$

$$\hat{T}h_\alpha = \Sigma_{\gamma \prec \beta} a_\gamma \ \partial^\gamma(h_\alpha) + a_\alpha \ \partial^\alpha h_\alpha =: g_\alpha + a_\alpha h_0 \quad \text{on} \quad U(!)$$

for some $g_\alpha \in S'$, which is a continuous function on U (by the assumption). It follows that $a_\alpha = a_\alpha h_0 = \hat{T}' h_\alpha - g_\alpha$ is continuous on U.

Step V) It now remains to show that $a_\alpha(t) \equiv 0$ for $\alpha \neq 0$. Assume the contrary, i.e. that for a certain $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $|\alpha| = \alpha_1 + \ldots + \alpha_n > 0$ one has $a_\alpha \neq 0$. We want to bring this to a contradiction with the estimate (50). For simplicity we assume that $a_\alpha(0) \neq 0$ (this is allowed due to the isometric translation invariance of FE^q). There are two estimates. For $h = h_\alpha$ as in Step II) one has as a consequence of (50) and Lemma 5:

$$\sup_{\rho \in [0,1]} \| \hat{T}(D_\rho h) \|_\infty \leqslant C \sup_{\rho \in [0,1]} \| M_\rho(F^{-1}h) \|_{E^q} = C \| F^{-1} h \|_{E^q}.$$

An other hand one has

$$\| \hat{T}(D_\rho h) \|_\infty \geqslant | a_\alpha(0) | \, | \ \partial^\alpha (D_\rho h)(0) | \geqslant | a_\alpha(0) | \rho^{-|\alpha|} \ \partial^\alpha h(0) = \rho^{-|\alpha|} | a_\alpha(0) |.$$

The contradiction resulting from these two estimates ($\rho \to 0$) implies the required assertion.

ii) ⇒ iv) If T commutes with translations it also commutes with convolutions by elements in $L^1_{n/p}$ or E^q, because I^p has continuous translation. As in the above proof it follows that $\hat{T} : \hat{f} \to (Tf)^\wedge$ does not increase supports: In fact, for $t \notin \operatorname{supp} \hat{f}$ there exists an neighborhood U of t such that $h \hat{f} = 0$ for all $h \in \mathcal{D}_U$. But $h = \hat{g}$, with $g \in S \subseteq E^q$, and therefore $g * Tf = T(g * f)$, hence $h(\hat{T}\hat{f}) = \hat{g}(Tf)^\wedge = (g * Tf)^\wedge = [T(g * f)]^\wedge = \hat{T}(\hat{g}\hat{f}) = \hat{T}(0) = 0$ for all $h \in \mathcal{D}_U$, i.e. $t \notin \operatorname{supp} \hat{T}\hat{f}$. Thus $\operatorname{supp} \hat{T}\hat{f} \subseteq \operatorname{supp} \hat{f}$ and the above arguments can be applied.

REMARK Q. The assertions concerning the operator $\hat{T}$ as given in Steps I - III) could have been derived directly from Theorem 2 in [22], due to J. Peetre. Also for the implication ii) ⇒ iii) a standard result, would have been sufficient (instead of ii) ⇒ iv) above). However, for the convenience of the reader, and, above all in view of possible further extensions (cf. Theorem 20 below) we have prefered to present the complete proof based on results available in Hörmander's book ([15]).

Step VI) We finally observe that (48) implies, together with the solidity of E^q:

$$\|\hat{T}'(L_t h)\|_\infty \leqslant C \sup_{t\in\mathbb{R}^n} \|L_t h\|_{FE^q} = C\|h\|_{FE^q},$$

and further

$$\|a_0\|_\infty \leqslant C_0 |\!|\!| \hat{T}' |\!|\!|_{FE^q} = C_0 |\!|\!| \hat{T} |\!|\!|_{FI^q}.$$

It is now clear that $\hat{T}\hat{f} = h\hat{f}$ for all $f \in E^q$, with $h \in C^b(\mathbb{R}^n)$, and the proof of i) → iv) is complete.

The above theorem implies among others the following result which is of great use in the proof of the abstract Tauberian Theorem stated below.

PROPOSITION 18. i) Any w^* – continuous, i.e. $\sigma(M^p, E^q)$ – continuous and subordinative operator T on M^p is of the form $f \to \sigma * f$, and leaves I^p as well as M^p invariant.

ii) Any subordinative operator on I^p extends to a bounded linear operator on M^p. There is only one w^* – continuous extension.

PROOF. i) We begin with the discussion of the problem of representing subordinative operators on M^p which is necessary because difficulties arise if a pseudomeasure has to be convolved with an element of M^p. Consider therefore any subordinative, w^* – continuous operator T on $M^p(\mathbb{R}^n)$. Because I^p is a translation invariant, closed subspace of M^p T leaves I^p invariant. By the preceding theorem there exists a pseudomeasure $\sigma \in$ $\in F^{-1}L^\infty(\mathbb{R}^n)$ such that $Tf = \sigma * f$ for all $f \in S \subseteq I^p$. As will be shown in a certain sense the same is true for any $f \in M^p$. Technically one proceeds as follows: For k, $h \in \mathcal{D}$, with $\int k(y)\,dy = 1$ and $h(0) = 1$ one has (cf. [6], Corollary 2.3 for such regularizations in a more abstract setting) as a consequence of the w^* – continuity of T:

$$Tf = w^* - \lim_{\rho\to\infty} T[M_\rho k * (D_\rho h) f] = \lim_{\rho\to\infty} \sigma * M_\rho k * [(D_\rho h) f]$$

($M_\rho k * (D_\rho h) f$ actually belongs to $\mathcal{D} * \mathcal{D} \cdot S' \subseteq I^p$!).

On the other hand, convolution by σ defines a bounded operator on I^p, hence convolution by $\check{\sigma}$ is a bounded operator on E^q (cf. the proof of Theorem 3 and observe that $\check{\sigma} * g$ is well defined as a product of pseudomeasures, i.e. through pointwise multiplication of Fourier transforms). The second adjoint of the action of σ on I^p may be written formally as a convolution product, because it extends the convolution given on $S \subseteq I^p$ in a natural way, but one should keep in mind that $\sigma * f$, for general σ and $f \in$ $\in M^p$ is now characterized by the relation $\langle \sigma * f, g\rangle := \langle f, \check{\sigma} * g\rangle$ for $g \in E^q$. It follows

$$\langle \sigma * M_\rho k * (D_\rho h) f, g\rangle = \langle f, D_\rho h (M_\rho k^{\check{}} * \check{\sigma} * g)\rangle \to$$
$$\to \langle f, \check{\sigma} * g\rangle = \langle \sigma * f, g\rangle \quad \text{for} \quad \rho \to \infty$$

because $\check{\sigma} * g \in E^q$ for any $g \in E^q$ and $\mathcal{D}$ is dense in E^q. (cf. [6], Lemma 3.4). It is now clear from these two limit relations that, in this sense, the representation $Tf = \sigma * f$ is to be interpreted in the sequel. In order to avoid problems below we mention that this kind of 'extended convolution' is of course associative, commutative and fully compatible with ordinary convolution whenever two interpretations are possible.

ii) is verified using similar arguments.

Using the terminology of subordinative operators we arrive at the following very general (and abstract) Tauberian theorem, which may be considered as an extension of Theorem 6.2 in [20] to the case of higher dimensions and $p \neq 2$.

THEOREM 19. Let T_1 be a subordinative, w^* – continuous operator on $M^p(\mathbb{R}^n)$. Assume that a given $f \in M^p(\mathbb{R}^n)$ satisfies $T_1 f \in I^p$. Provided T_1 satisfies the Tauberian condition $T_1(\chi_t) \neq 0$ for all $t \in \mathbb{R}^n$ [where χ_t is the character $x \mapsto \exp(i \sum_{j=1}^n x_j t_j)$] one may conclude therefrom: For any other subordinative, w^* – continuous operator T_2 on $M^p(\mathbb{R}^n)$ one has $T_2 f \in M_c^p(\mathbb{R}^n)$ if and only if $T_2 f \in I^p(\mathbb{R}^n)$.

REMARK R. It is not clear whether the w^* – continuity is actually relevant for the validity of the result.

PROOF OF THEOREM 19. Having concentrated the technical discussions to the above results we may come directly to the heart of the proof, where quite familiar arguments can be used. Since $I^p \subseteq M_c^p$ we only have to verify one implication. Let σ_1, σ_2 be the pseudomeasures associated with T_1 and T_2, respectively. Assume $T_2 f \in M_c^p$ for a given $f \in M^p$. Then for any $a \in$ $\in S(\mathbb{R}^n) \subseteq L^1_{n/p}$, with $\int a(y)\, dy = 1$ one has (cf. [22] and i) above:

$$\lim_{\rho \to \infty} M_\rho a * \sigma_2 * f = T_2 f.$$

It will be convenient to assume here that $Q := \operatorname{supp} a$ is a compact set (hence $\operatorname{supp} (M_\rho a)^\wedge = \operatorname{supp} (D_\rho \hat{a}) = \rho Q$). Fixing now ρ for a moment, one observe that $\hat{\sigma}_1$ belongs locally to FE^q because σ_1 defines a convolution operator on the convolution algebra E^q and $\hat{\sigma}_1$ is locally invertible (due to the Tauberian condition and Theorem 7). Choose $g_\rho \in E^q$ such that $\hat{g}_\rho = 1/\hat{\sigma}_1(t) = 1/T_1(\chi_t)$ for all $t \in \rho Q$. Then one has

$$M_\sigma a * \sigma_2 * f = \sigma_2 * (M_\rho a * f) = \sigma_2 * (M_\rho a * g_\rho * \sigma_1 * f).$$

But $M_\rho a * g_\rho * (\sigma_1 * f) = M_\rho a * g_\rho * T_1 f \in E^q * E^q * I^p \subset I^p$.

Since $\sigma_2 * I^p \subseteq I^p$ it follows altogether that $T_2 f \in \overline{I^p} = I^p$, and the proof is complete.

Without going into details we mention that a quite similar statement for subordinative operators on Wiener type spaces over lca. groups can be given. In general, the same line of arguments goes through with suitable modifications. Of course, one has to replace the approximate units used above (and obtained by dilations) by suitable approximate units for $L^1(G)$ or $C^0(G)$, for example. A decisive change concerns the use of the kernel theorem. Fortunately there is another kernel theorem, applicable in this situation (and for the case $G = \mathbb{R}^n$ in a sense much more easier to obtain, cf. [10], Theorem B). Again the limiting case $p = \infty$ corresponds to Theorem 4 in Wiener's book.

THEOREM 20. Let G be a (non-compact) locally Abelian group, and let T_1 be a subordinative, w^* – continuous operator on $W(L^p, L^\infty) = W(L^q, L^1)'$, for some p, $1 < p < \infty$. Assume that T_1 does not vanish on characters, i.e. $T_1(\chi) \neq 0$ for any $\chi \in \hat{G}$, and that $T_1 f \in W(L^p, C^0) = \overline{K(G)}$ (closure in $W(L^p, L^\infty)$) for some given $f \in W(L^p, L^\infty)$. Then one has for any other subordinative and w^* – continuous operator on $W(L^p, L^\infty)$:

$$T_2 f \in W(L^p, C^0) \quad \text{if and only if} \quad T_2 f \in W(L^p, L^\infty)_c$$

(i.e. if and only if $\{k(L_y(T_2 f)) \mid y \in G\}$ is relatively compact in $L^p(G)$ for any $k \in K(G)$).

A comparison of our results with those given in [20] by K.S. Lau suggests to consider not only multipliers (i.e. subordinative operators in his paper) on M^p or M^p_c, bit also on a certain subspace of 'regular' elements $f \in M^p$, satisfying the additional assumption

$$(51) \qquad \lim_{z \to \infty} \left(z^{-n} \int_{z+Q} |f(z)|^p \, dz \right)^{1/p} = 0.$$

Making once more use of the terminology of Wiener-type spaces as treated in [11] one is thus lead to consider operators on the following space

$$(52) \qquad M^p_{rc}(\mathbb{R}^n) := M^p_c \cap W(L^p, C^0_{-n/p}).$$

This is a Banach space and a closed subspace of $M^p_r := M^p \cap W(L^p, L^\infty_{-n/p})$

which in turn is endowed with the natural norm

$$\|f\|_{[p]r} := \|f\|_{[p]} + \|f\|_{W(L^p, L^\infty_{-n/p})} \tag{53}$$

the last term being given as the supremum over $z \in \mathbb{R}^n$ of the expression used in (51). It is also convenient to set

$$I^p_r := I^p \cap W(L^p, C^0_{-n/p}) \tag{54}$$

and

$$E^q_s := E^q + W(L^p, L^1_{n/p}) \tag{55}$$

the latter being endowed with its natural norm, to be written as $\| \ \|_{\{q\}s}$ (as a sum of two compatible Banach spaces, cf. [2], Chap. 2 for this construction).

With these notations in mind it is now possible to give the relevant preliminary informations concerning M^p_r in the following result:

PROPOSITION 21. For $1 < p, q < \infty$ one has:

(i) The spaces $\mathcal{D}$ and $\mathcal{S}$ are dense in I^p_r and E^q_s in their respective norms;

(ii) Both Banach space have continuous translation and are Banach modules over the Beurling algebra $L^1_{n/p}$ with respect to convolution;

(iii) $M^p_r = (E^q_s)'$ as a Banach space;

(iv) $E^q_s = (I^p_r)'$ as a Banach space;

(v) $(E^q, \| \ \|_{\{q\}})$ a dense Banach ideal in $(E^q_s, \| \ \|_{\{q\}s})$

(vi) $(E^q_s, \| \ \|_{\{q\}s})$ is a Banach convolution algebra in $L^1(\mathbb{R}^n)$ to which all statements given in Theorem 8 apply.

PROOF. The verification of i) – iv) is left to the reader (see [2], Chap. 2, cf. also [6]). Only obvious modifications of earlier proofs are required.

In order to verify v) observe first that E^q_s is continuously embedded into $L^1(\mathbb{R}^n)$. That it is a Banach convolution algebra containing E^q as a dense Banach ideal follows from the fact that $W(L^q, L^1_{n/p})$ is a Banach ideal in $L^1_{n/p}$ (cf. [11]), and hence one has for $f^1, f^2 \in E^q_s$, written as $f^i = g^i + h^i$ with $g^i \in E^q$, $h^i \in W(L^q, L^1_{n/p})$ for $i = 1, 2$:

$$\begin{aligned} f^1 * f^2 = g^1 * g^2 + (g^1 * h^2 + g^2 * h^1) + h^1 * h^2 \in \\ \in E^q * E^q + E^q * L^1_{n/p} + L^1_{n/p} * W(L^q, L^1_{n/p}) \subseteq \\ \subseteq E^q + W(L^q, L^1_{n/p}) = E^q_s. \end{aligned}$$

It is clear that the corresponding norm estimates can be verified as well.

In order to verify properties C) - F) as stated in Theorem 8 one may argue as follows. Since $\mathcal{F}E^q$ is a dense ideal in $\mathcal{F}E^q_s$ containing S these two algebras coincide locally. The Wiener-Lévy theorem being local in nature and valid for E^q it must hold for E^q_s as well.

Let now $h_0 \in E^q_s$ be given with nonvanishing Fourier transform. Write E_{h_0} for its closed translation invariant linear span in E^q_s. For any $h_1 \in S$ with nonvanishing Fourier transform it follows that $h_1 * h_0 \in S * E^q_s \cap E_{h_0} \subseteq E^q \cap E_{h_0}$. By Theorem 8.D) the closed, translation invariant linear span in E^q coincides with E^q, hence $E_{h_1 h_0} = E^q_s$, and consequently $E_{h_0} = E^q_s$. Conversely, it is clear that $E_{h_0} = E^q_s$ implies that $\hat{h}_0(t) \neq 0$ for all $t \in \mathbb{R}^n$.

The verification of conditions E) and F) can now be left to the reader.

Following the arguments given in the proof of Theorem 17 and using again the fact that $\mathcal{F}E^q_s$ coincides locally (with corresponding local equivalence of norms) with $\mathcal{F}E^q$ it is not difficult to verify that an analogue of Theorem 17 holds true for operators on I^p_r. One thus has

PROPOSITION 22. Any subordinative operator T on the Banach space I^p_r is of the form $T: f \to \mathcal{F}^{-1}(h\,\mathcal{F}f)$ for all $f \in G$, for some $h \in C^b(\mathbb{R}^n)$.

After these preliminaries it is possible to state the following abstract Tauberian theorem for the space of regular elements in M^p (recalling that the w^*-topology is understood to be the $\sigma(M^p_{rc}, E^q_s)$-topology on $M^p_{r,c}$).

THEOREM 23. Let T_1, T_2 be subordinative, w^*-continuous operators on M^p_r. Assume that $T_1(\chi_t) \neq 0$ for all $t \in \mathbb{R}^n$, and that for a given $f \in M^p_{rc}$ one has $T_1 f \in I^p_r$. Then one has $T_2 f \in I^p_r$ as well.

As the proof does not require unexpected modifications compared with those given earlier it is omitted. Let us mention however that even subordinativity with respect to the norm $\| \ \|_{[p]}$ would have been sufficient as assumption (as in [20]).

Furthermore it would have been possible to derive related results by replacing M^p by any of the spaces $M^p \cap W(L^r, L^\infty_{-\alpha})$, for any r with $p < r < \infty$ and $\alpha \geqslant n/p$, endowed with its natural norm. The decisive step consists in checking that the predual $E^q + W(L^t, L^1_\alpha)$, with $1/t = 1 - 1/r$, is once more a Banach convolution algebra fulfilling all required conditions.

§ 6. CONCLUDING REMARKS

REMARK S. It is of interest to investigate the spaces $E^q(\mathbb{R}^n)$ and $M^p(\mathbb{R}^n)$

as Banach spaces with a double module structure as considered in [6] (as convolution module over $L^1_{n/p}$, cf. (21), and as a pointwise module over $A = C^0(\mathbb{R}^n)$). As pointed out in [6] practically all relevant informations concerning this double module structure can be exhibited by means of the socalled main-diagram and the corresponding set of equalities. (cf. [6], p. 199 ff.). Since $E^1 = L^1$ and $M^\infty = L^\infty$ we restrict our attention to the case $1 < p, q < \infty$. The case $p = 1, q = \infty$ has to be discussed separately.

For $E^q(\mathbb{R}^n)$ this question is easily answered. Since E^q contains $A_0 = K(\mathbb{R}^n)$ as a dense subspace (cf. Theorem 3.C), but is on the other hand a dual space of the double module I^p (Theorem 3.E) the minimal and maximal space in the diagram coincide with E^q, i.e. the diagram is as simple as it can be.

For $M^p(\mathbb{R}^n)$, $1 < p < \infty$, the situation is more interesting. As we shall explain below the diagram for this space has the following form:

(56)
$$\begin{tikzcd}
& M^p \arrow[dl, dash, "5"'] \arrow[dr, dash, "+4"] & \\
M^p_c \arrow[dr, dash, "-4"'] & & (I^p)^G \arrow[dl, dash, "6"] \\
& I^p &
\end{tikzcd}$$

Its set of equalities is $E(M^p) = \{1, 2, 3\}$. In fact, as a dual module (cf. Theorem 3.A) it is clear that M^p satisfies E_1 and E_2, i.e. $M^p = (M^p)^{\sim} = (M^p)^A = (M^p)^G$. That E_3 holds true follows from the fact that $I^p = (M^p)_0 = (M^p)_{AG}$ and that $K(\mathbb{R}^n) \cdot M^p \subseteq I^p$. Evidently $M^p{}_G = M^p_c$. Completing the discussion of the left part of the diagram we note that the inclusions $I^p \subseteq M^p_c \subseteq M^p$ are proper ones (cf. Theorem 2.E) and consider functions of the form:

$$(57)\qquad \begin{cases} h := \Sigma_{k \geq 1} 2^{nk/p} L_{2^k} h_k , \\ \\ \text{with } (h_k)_{k \geq 1} \text{ in } L^p_Q(\mathbb{R}^n), \text{ and with } \| h_k \|_p = 1 \end{cases}$$

where the sequence may be choosen to be equicontinuous in $L^p(\mathbb{R}^n)$ or not, respectively. It thus follows that equalities 4 and 5 do not hold, and it suffices to show, that E_6 does not hold, i.e. (cf. [8], Proposition 4.4) $(M^p)_A{}^G = \{f \mid f \in M^p, g * f \in (M^p)_A \text{ for all } g \in L^1_w\} = \{f \mid f \in M^p,\ k * f \in I^p$ for all $k \in K(\mathbb{R}^n)\} = (I^p)^G$ contains I^p as a proper subspace. Again h of the form (57) can be used, now with $(h_k)_{k \geq 1}$ tending to zero in $L^1(\mathbb{R}^n)$ (this is possible since $p > 1$, take e.g. $(k^{n/p} c_{[-1/k, 1/k]^n})_{k \geq 1}$). It follows that the sequence of seminorms associated to $f := k * h$ by (27) satisfies $d_k \sim \sim 2^{nk/p} \| k * h_n \|_p$, i.e. $k * h \in I^p$ by (29), as was required.

Without giving details it may be mentioned that the exceptional case $p = 1$ would require a separate discussion. Very likely Corollary 11 does not hold in that case. However, it is not difficult to carry out the necessary modifications in order to formulate the correct duality statements. Concerning the diagram it is possible to show that one has for M^1 as the set $E(M^1) = \{2, 5\}$, i.e. the diagram consists of 6 different spaces in that ease, and $E(E^\infty) = \{1, 2, 3, 4\}$.

Of course, it would be of great interest to obtain a complete description of the set of all multipliers of $I^p(\mathbb{R}^n)$, considered as a subspace of $S'(\mathbb{R}^n)$ with double module structure.

There are certain similarities between the family of L^p-multiplier spaces and the corresponding spaces for I^p and E^q, $1 < p, q < \infty$. One might say that, for example, Theorem 17 holds for the L^p-spaces, in a slightly modified form. Other similarities are described in the following remarks.

REMARK T. There does not exist a compact multiplier on I^p or E^q for $1 < p, q < \infty$.

PROOF. Assume that $T : f \to \tau * f$ defines a nontrivial compact multiplier on I^p. Then the adjoint operator $T' : k \to \check{\tau} * k$ is a nontrivial compact operator on E^q. It is therefore sufficient to show the nonexistence of nontrivial compact multipliers $S : f \to \sigma * f$ on E^q. If $S \neq 0$ then $\hat{\sigma}(t) \neq 0$ on some compact set U with nonvoid interior ($\hat{\sigma} \in C^b(\mathbb{R}^n)$ by Theorem 17!). If $g \in E^q$ satisfies $\hat{g}(t) = 1/\hat{\sigma}(t)$ on U then $f \to g * \sigma * f = f$, i.e. the identity operator over the appearently infinite dimensional Banach space $\{f \mid f \in E^q, \operatorname{supp} \hat{f} \subseteq U\}$ in E^q would have to be a compact operator which is not possible.

REMARK U. There does not exist any nontrivial multiplier $T : E^{q_1} \to E^{q_2}$ for $q_1 < q_2$ or $T : I^{p_2} \to M^{p_1}$ for $p_1 > p_2$.

PROOF. The proof of the first assertion being similar we discuss only the second one. Assume that $Tf \neq 0$ for some $f \in I^p$. Since T commutes with translations and the representation $f = \Sigma_{j \in \mathbb{Z}^n} f(L_j \psi)$ (with ψ being a suitable test function satisfying $\Sigma_{j \in \mathbb{Z}^n} \psi(x - j) \equiv 1$) converges in the norm $\| \ \|_{[p]}$ for $f \in I^p$ one may assume that $\operatorname{supp} f \subseteq Q_1$ (and $Tf \neq 0$). For any sequence $(y_k)_{k \geq 1}$ in $\mathbb{R}^n$, satisfying $|y_k| = 2^{k-1} + 1$, this implies $\operatorname{supp} L_{y_k} f \subseteq P_k$ for $k \geq 2$, and therefore by Theorem 2

$$\|L_{y_k}f\|_{[p_2]} \leqslant C\,2^{-kn/p_2} \quad \text{for} \quad k \geqslant 2.$$

An estimate in the other direction can be obtained as follows. Because $Tf \neq 0$ there exists $z \in \mathbb{R}^n$ such that $g := c_{Q_1}(L_z Tf) \neq 0$, hence $|g(x)| \leqslant |L_z Tf(x)|$ for all $x \in \mathbb{R}^n$. Thus

$$\|L_{y_k} g\|_{[p_1]} \leqslant \|L_{y_k} Tf\|_{[p_1]} \||L_z\||_{[p_1]} = C_1 \|T(L_{y_k}f)\|_{[p_1]} \leqslant C_2 \|L_{y_k}f\|_{[p_2]}.$$

Finally, using now once more that Theorem 2 allows to calculate the norm of $L_{y_k} g$, because $\operatorname{supp} L_{y_k} g \subseteq P_k$ for $k \geqslant 2$, one notes that

$$\|L_{y_k} g\|_{[p_1]} \geqslant C_2\, 2^{-kn/p_1} \|g\|\, p_1.$$

Combining these three estimates one arrives of course at a contradiction (for $k \to \infty$).

REMARK V. If $q_1 < q_2$ there exists a multiplier on E^{q_1} which is not a multiplier for E^{q_2}.

PROOF. As already observed earlier in this paper the local structure of the algebra of Fourier multipliers for E^q is the same as that of FE^q. Therefore it suffices to show that these algebras $(FE^q)_{1 < q < \infty}$ form even locally an increasing scale (for $q \to \infty$). This can be shown as follows: Assume that $(FE^{q_1})_Q = (FE^{q_2})_Q$ for some open set $Q \subseteq \mathbb{R}^n$. Then, by the closed graph theorem, the norms $\|\ \|_{\{q_1\}}$ and $\|\ \|_{\{q_2\}}$ would have to be equivalent on the space $\{f \mid f \in S, \operatorname{supp} \hat{f} \subseteq Q\}$. That this cannot be the case follows from the fact that for $f \in E^q$ the function $y \to \|L_y f\|_{\{q\}}(1 + |y|)^{n/p}$ is bounded from above and below for any $f \in S$ (cf. Theorem 3.D).

REMARK W. For the nontrivial situations, i.e. for $q_1 \leqslant q_2$ or $p_1 \geqslant p_2$ it is possible to obtain a representation theorem for the space of multipliers from E^{q_1} to E^{q_2} or (what even can be shown to be the same, using abstract, Banach module theoretic arguments) from I^{p_2} to M^{p_1}, in the sense of Figà-Talamanca/Gaudry [13]. Of course, such results are based on Theorem 15, which is closely related to Theorem 1 of [13].

REMARK X. At the moment it is not clear whether there are inclusions between the multiplier spaces of E^q for different values of q, or what the relations to the corresponding spaces of L^q-multipliers are (beyond Proposition 14). Of course, it would be interesting to have also sufficient conditions for $h \in C^b(\mathbb{R}^n)$ to define a Fourier multiplier on E^q or I^p, for example.

REMARK Y. There are of course many variants of the results presented in this paper (in particular those in §3 and §4) which can be obtained by suitable modifications of our arguments. A general frame for their treatment has been presented in [7]. For example, it would be possible to treat anisotropic versions of the above spaces (cf. [8] for a general treatment of Banach convolution algebras related to E^q). The most interesting among them could be handled by means of anisotropic dilation operators, replacing the operators M_ρ and D_ρ, $\rho \in [0, 1]$. Another generalization concerns Theorem 23: the assertions stated there hold true if M^p_r is replaced by $M^p \cap W(L^r, L^\infty)$, for $1 < p, r < \infty$.

On the other hand, it would be of interest to compare discrete versions of our results (where $\mathbb{R}^n$ is replaced by $\mathbb{Z}^n$) with results to be found in the literature on summability theory. Actually, decompositions into dyadic blocks are quite familiar in this setting (see e.g. [18]).

REMARK Z. As a final observation let us mention that the approach choosen in this note allows to stress the analogy between Theorem 4 and 29 in Wiener's book. In contrast to the work of K.S. Lau ([20]) or J.P. Bertrandias ([3], [4]) we have worked with functions (and distribution) instead of the classes constituting the Marenikiewicz-space $\mathcal{M}^2 = M^2/I^2$. This has allowed us to make use of the ordinary Fourier transform instead of the 'Wiener transform'. This was perhaps decisive in order to go to higher dimensions and arbitrary p with $1 < p < \infty$ (not only $p = 2$). To get rid of such results by means of (reasonable) characterization of the Wiener transform of $\mathcal{M}^p(\mathbb{R}^n)$ (which first has to be defined in an appropriate way) seems to be an very delicate and elaborate task.

However, one must admit, that despite the appearent similarities between the results obtained in these two different settings there are also remarkable differences. Although some of them might go back to the technique of proofs (e.g. the w^*-continuity assumed in Theorems 19 and 23) there are at least certain structural differences. Only to mention here one of the most striking ones (cf. [3]): Whereas $\mathcal{M}^p$ is a Banach module over $L^1(\mathbb{R}^n)$ under convolution one cannot say this with respect to M^p, which is only a Banach convolution module over the Beurling algebra $L^1_{n/p}(\mathbb{R}^n)$ (cf. Corollary 9).

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