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Banach Convolution Algebras of Functions II

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With 1 Figure

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Abstract. Let G be a noncompact, locally compact group. By means of "generalized dyadic decompositions" of G, translation invariant Banach spaces $F(\mathfrak{P}, B, X)$ of (classes of) measurable functions on G are constructed, e. g. certain weighted amalgams of L^p-spaces. Basic properties of these spaces are derived and connections with spaces considered in the literature are indicated. As a main result, sufficient conditions are given which imply that a space of this type is a Banach algebra with respect to convolution.

1. Introduction

The study of convolution algebras on locally compact groups is certainly one of the central themes of abstract harmonic analysis. During the last decade a great number of publications appeared which are concerned with Segal algebras introduced by H. Reiter (see [20], VI § 2, [21]; cf. also [19] for a survey of certain aspects, with an extensive list of references). Beurling algebras $L^1_w(G)$ ([20], VI § 7) represent another type of group algebras that has been studied in some detail. These two classes of convolution algebras are certainly of great interest inasmuch as they are — with respect to different properties — very closely related to the group algebra $L^1(G)$; one may mention the ideal theorem for Segal algebras, or the existence of bounded approximate units and hence the factorization property for Beurling algebras.

Compared with these two types, more general translation-invariant, dense Banach subalgebras of $L^1(G)$, such as suitably weighted L^p-spaces $L^p_w(G)$, or spaces of the form $L^1 \cap L^p_w$ for example, have not received such a systematic treatment. Domar's fundamental paper [7] seems to be the only significant exception to this "rule". The present paper is intended as a contribution to such a theory of general Banach convolution algebras of functions

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on locally compact groups. In this sense it represents a continuation of the investigations begun in [9].

We shall introduce here a decomposition method which allows the construction of an extensive family of dense subspaces of $L^1(G)$, for a locally compact group G. A variety of Banach convolution algebras related to these spaces will arise. Because the "dyadic decomposition" of $G=\mathbb{R}^m$ appears as the typical special case of the type of partitions of G to be considered here, they will be called "generalized dyadic decompositions". Another kind of decompositions, related to the decomposition used in the definition of Wiener's Segal algebra ([20], I § 5, iii), is to be considered in a subsequent paper. The most important special cases of the spaces defined below are weighted L^p-spaces, as well as spaces which are closely related to Beurling-Herz spaces (cf. [15], [3], [1], and [17]). We are thus able to prove convolution results for these spaces as well as for anisotropic versions of them. Furthermore, connections with the spaces $A(A,B,X)$, as defined in [9], are shown. We want to emphasize the fact that certain weighted L^p-spaces do not only appear as special cases of the spaces defined here. Beyond that, a somewhat refined treatment of weighted L^p-spaces requires more or less automatically the consideration of spaces defined by means of decomposition methods (cf. for example Theorem 4.5, or Theorem 5.3).

Together with the spaces defined in [9] the examples given in this paper constitute a point of departure for the discussion of general Banach convolution algebras. In particular, it is indicated that several properties of Segal algebras, or Beurling algebras, carry over to a more extensive class of group algebras, while several new phenomena arise that do not exist, or do not play an important role, in the context of, say, Segal algebras. We only mention the behaviour under the canonical mapping $T_H\colon L^1(G)\to L^1(G/H)$ of these spaces (cf. [10], 3.4), or the existence of compact multipliers between certain spaces of this type on a non-compact group (see [12], in contrast to the situation for Segal algebras, cf. [18], Proposition 2.2).

The paper is organized as follows. § 2 describes some notations and the basic tools needed in § 3: I) a partition $\mathfrak{P}=(P_n)_{n\geq 1}$; II) a solid Banach function space B on G, III) a solid sequence space X. In § 3 the spaces $F(\mathfrak{P},B,X)$ are defined and general properties of these spaces are derived. In particular, sufficient conditions are stated implying that $B \cap F(\mathfrak{P}, B, X)$ is a Banach convolution algebra. In § 4 these results are applied and, in some cases even extended. In the final section connections with various spaces considered in the literature are discussed.

2. Terminology and Auxiliary Material

As far as possible the terminology in REITER's book [20] will be used; some further notations are taken from [9]. Troughout this paper G denotes a *non-compact*, locally compact group with a fixed left Haar measure dx. If G is not specified the group operation is written multiplicatively. As usual it will be convenient to speak — by abuse of language — of "*measurable functions*", identifying two functions coinciding l. a. e. (locally almost everywhere). $L^p(G)$, $1 \leqslant p \leqslant \infty$, has the usual meaning. The values p and p', $1 \leqslant p \leqslant \infty$, will always be related by $1/p + 1/p' = 1$. $\mathcal{K}(G)$ denotes the space of all continuous functions on G with compact support (supp). $C^0(G)$ can be identified with the closure of $\mathcal{K}(G)$ in $L^\infty(G)$.

The left (right) *translation operators* $L_y(R_y)$, $y \in G$, are defined by

$$L_y f(x) := f(y^{-1}x), \; R_y f(x) := f(xy^{-1})\Delta^{-1}(y),$$

Δ being the Haar module on G. For a measurable set $M \subseteq G$, $|M|$ denotes its Haar measure.

Constants are denoted by $C, K, C_1, \ldots$. The same symbol may refer to different constants at various occurrences.

For the construction of the spaces defined below we need some auxiliary material. Except for a few additional assumptions this will be very similar to that one considered in [9], pp. 138—140.

I) First we need an appropriate partition $\mathfrak{P} = (P_n)_{n \geqslant 1}$ of the group G. We obtain it in the following way: Consider a sequence $(Q_n)_{n \geqslant 0}$ of open subsets of G satisfying Q1)—Q3):

$Q1)$ $Q_0 = \emptyset, \bigcup_{n=1}^{\infty} Q_n = G;$

$Q2)$ $Q_n \cdot Q_n \subseteq Q_{n+1}$ for $n \geqslant 1;$

$Q3)$ $Q_n \neq G$ for all $n \geqslant 1$.

$Q2)$ is the essential property; $Q3)$ excludes trivial cases, in particular it makes it necessary to consider only non-compact groups. Without loss of generality we may assume that $1 \in Q_1$, and thus $Q_n \subseteq Q_{n+1}$ for $n \geqslant 1$ according to $Q\,2$). The required partition $\mathfrak{P} = (P_n)_{n \geqslant 1}$ is now given by

$$P_n := Q_n \setminus Q_{n-1} \text{ for } n \geqslant 1\,. \tag{1}$$

The characteristic function of P_n is denotes by ψ_n, that of $G \setminus Q_n$ by χ_n.

Examples. The *dyadic decomposition* $\mathfrak{P}_2$ of $G = \mathbb{R}^m$ or $\mathbb{Z}^m$, given by

$$P_1 := \{x \mid |x| \leqslant 1\},\ P_n = \{x \mid 2^{n-1} \leqslant |x| < 2^n\} \quad \text{for } n \geqslant 2, \tag{2}$$

is certainly the most natural example. Nevertheless it has to be emphasized that neither compactness nor symmetry properties of the sets $Q_n, n \geqslant 1$, are required. The following example is more typical for the general situation:

Let $\sigma = (s_1, s_{-1}, s_2, \dots, s_m, s_{-m})$ be a finite sequence of real numbers satisfying $2 \leqslant |s_{\pm i}| \leqslant \infty$ for $1 \leqslant i \leqslant m$. Then $\mathfrak{P}_\sigma$ denotes the partition derived from the sequence $(Q_n)_{n \geqslant 1}$:

$$Q_n := \{x = (x_1, \dots, x_m), -s_{-i}^n \leqslant x_i \leqslant s_i^n, i = 1, 2, \dots, m\}, n \geqslant 1\,. \tag{3}$$

For $s_1 = s_{-1} = \dots = s_{-m} = s$ we call $\mathfrak{P}_\sigma = \mathfrak{P}_s$ *isotropic*, otherwise *anisotropic*. If $P_n = P_n^{-1}$ for $n \geqslant 1$, $\mathfrak{P}$ is called *symmetric*. Besides these systems of the form $\mathfrak{P} = \mathfrak{P}_\sigma$ the partitions derived from $Q_n := \{x \mid |x| \leqslant 2^{r2^n}\}$ for $n \geqslant 1$, $r > 0$, or any related sequence would be admissible as well. If G is a compactly generated group, $G = \bigcup_{k=1}^{\infty} U^k$, then for any $l \in \mathbb{N}$, $l \geqslant 2$, the sequence $Q_n := U^{l^{n-1}}$ satisfies $Q\,1$)—$Q\,3$) (cf. also [9], pp. 138—139, where the notation is different).

II) Throughout this paper X stands for a *shift-invariant, solid BK-space* (cf. [9], p. 139), i. e. X is a space of sequences $x = (x_n) = (x_n)_{n \geqslant 1}$ satisfying

$X\,1$) $(X, \|\ \|_X)$ is a Banach space;

$X\,2$) X is an ℓ^∞-module (with respect to coordinatewise multiplication), such that

$X\,3$) $\|xy\|_X \leqslant \|x\|_X\,\|y\|_\infty$ for all $x \in X$, $y \in \ell^\infty$;

$X\,4$) X contains all "finite" sequences and $\|(1, 0, 0, \dots)\|_X = 1$;

$X\,5$) D: $(x_1, x_2, \dots) \mapsto (0, x_1, \dots)$ satisfies $DX \subseteq X$;

$X\,7$) G: $(x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$ satisfies $GX \subseteq X$.

It follows from the conditions above that D and G define bounded linear operators on X, the norm of which will be written as $\|D\|$ and $\|G\|$ respectively. Sometimes we need property $X6$):

$X6$) The subspace of all 'finite' sequences is dense in X.

Let now a tripel $X=(X_1,X_2,X_3)$ of such BK-spaces be given. We call X an *S-tripel*, if there exists $C>0$ such that the bilinear mapping $S: X_1\times X_2\to X_3$, given by

$$S(x,y):=(z_n), z_n:=\sum_{k=n}^{\infty} x_k y_k \tag{4}$$

satisfies

$$\|S(x,y)\|_{X_3}\leqslant C\|x\|_{X_1}\|y\|_{X_2} \quad \text{for all } x\in X_1, y\in X_2. \tag{5}$$

The following two properties will be of relevance later on:

$X8$) (X,X,X) is an S-tripel;

$X9$) (ℓ^∞,X,X) is an S-tripel, or equivalently

$$\|z\|_X\leqslant C\|x\|_X, \text{ for } z_n:=\sum_{k=n}^{\infty}|x_k|, x\in X. \tag{6}$$

We write $x\sim y$ or $x_n\sim y_n$ if there exists $K>0$ such that $K^{-1}x_n\leqslant y_n\leqslant Kx_n$ for $n\geqslant 1$. The most important examples of such BK-spaces are weighted sequence spaces $X_w:=\{x\,|\,xw\in X\}$, with the norm $\|x\|_{X_w}:=\|xw\|_X$, derived from rearrangement-invariant BK-spaces X, such as the spaces ℓ^p, $1\leqslant p\leqslant\infty$, c^o (the space of sequences converging to zero), or more generally, Lorentz- or Orlicz sequence spaces. It is clear that $X_w=X_{w_1}$ iff $w\sim w_1$.

In our examples we confine our attention mainly to the family of spaces $\ell^p_{a,b}$, $1\leqslant p\leqslant\infty$, $a,b\in\mathbb{R}$, given by

$$\ell^p_{a,b}:=\{x\,|\,(x_n 2^{an} n^b)_{n\geqslant 1}\in\ell^p\}, \tag{7}$$

together with $c^o_{a,b}:=\{x\,|\,(x_n 2^{an} n^b)_{n\geqslant 1}\in c^o\}$. For shortness of notation we write ℓ^p_b instead of $\ell^p_{0,b}$, if no confusion can arise, or ℓ^q_a instead of $\ell^q_{a,0}$.

Note that a comparison with the notation used in [9] shows: $X^q_s=\ell^q_{0,s}$, $X^o_s=c^o_{0,s}$ (p. 140), and $X^q_\alpha=\ell^q_{\alpha,0}$ (p. 160). The following theorem gives information concerning properties $X8$) and $X9$) for these spaces:

Theorem 2.1. *Let q, $1\leqslant q\leqslant\infty$, be given.*

A) $\ell^q_{a,b}$ and $c^o_{a,b}$ satisfy $X9$) (and hence $X8$) for $a>0$, $b\in\mathbb{R}$;

B) $(\ell^q_{b_1},\ell^{q'}_{b_2},\ell^r_{b_3})$ is an S-tripel for $b_1+b_2\geqslant b_3+1/r$.

C) ℓ^q_b *satisfies* $X\,8)$ *for* $1 \leqslant q \leqslant 2$ *and* $b \geqslant 1/q$, *and for* $2 \leqslant q \leqslant \infty$, $b > 1/q'$. *Furthermore* c^0_b *satisfies* $X\,8)$ *for* $b > 1$.

Remark 2.1. It can be shown that at least the results A) and C) are sharp, i. e. the conditions on the parameters $a, b, b_1, \dots$ cannot be weakened.

The proof of Theorem 2.1 is based on two lemmata: In order not to interrupt the presentation we postpone their proofs to the end of this section. For the following definition cf. [9], p. 136—137.

III) $A, B, B_1, \dots$ shall be *solid BF-spaces* on G, i. e. Banach spaces of measurable functions which are $L^\infty(G)$-modules with respect to pointwise multiplication. The norm of a BF-space is unique up to equivalence. The intersection $B_1 \cap B_2$ of two solid BF-spaces is again a solid BF-space, endowed with the canonical norm

$$\|f\|_{B_1 \cap B_2} := \|f\|_{B_1} + \|f\|_{B_2}. \tag{8}$$

If B is left (right) translation invariant, then $\|L_y\|_B (\|R_y\|_B)$ denotes the norm of the translation operator on B.

The basic examples of solid BF-spaces are the spaces $L^p(G)$, or, more generally, the Lorentz spaces $L(p,q)$ (see [22]) and Orlicz spaces (see [5]). For a given solid BF-space B and a positive (continuous) function w on G, B_w denotes the space $\{f \mid fw \in B\}$. We write $L^p_{a,b}(\mathbb{R}^m)$ for the spaces $L^p_{w_{a,b}}(\mathbb{R}^m)$, with

$$w_{a,b}(x) := (1 + |x|)^a \log^b (e + |x|), a \geqslant 0, \ b \in \mathbb{R}. \tag{9}$$

A tripel $B = (B_1, B_2, B_3)$ of solid BF-spaces is called a *Banach convolution tripel* (*BCT*) if $f * g$ is well defined and if there exists $C_1 > 0$ such that

$$\|f * g\|_{B_3} \leqslant C_1 \|f\|_{B_1} \|g\|_{B_2} \quad \text{for all } f \in B_1, g \in B_2. \tag{10}$$

A BF-space A is called a *Banach convolution algebra* (*BCA*) *if* (A, A, A) is a BCT, and B is called a *left* (*right; twosided*) *Banach convolution module* over A if (A, B, B) (or; and (B, A, B)) is a BCT. Note that the associativity of convolution is guaranteed by Fubini's theorem, since $f * g$ exists iff $|f| * |g|$ exists, the spaces under consideration being solid.

Given a solid BK-space X and a BF-space B, $X(B)$ denotes the space of all B-valued sequences $(f_n)_{n \geqslant 1} \subseteq B$ such that $(\|f_n\|_B)_{n \geqslant 1} \in X$. The expression

$$\|(f_n)\|_{X(B)} := \|(\|f_n\|_B)_{n \geqslant 1}\|_X \tag{11}$$

defines a norm on $X(B)$ for which $X(B)$ is complete.

The terminology and the results concerning the real and the complex method of interpolation are taken from [2] and [6]. We conclude this section with the proof of Theorem 2.1.

Lemma 2.2. *Let* $X = \ell^q_w, 1 \leqslant q \leqslant \infty$, *be given. Then* X *satisfies* $X9$) *if the sequence* $(w_n)_{n \geqslant 1}$ *satisfies the following conditions:*

i) $\sup\limits_n w_n \sum\limits_{k=n}^{\infty} w_k^{-1} = C_1 < \infty$;

ii) $\sup\limits_n w_n^{-1} \sum\limits_{k=1}^{n} w_k = C_2 < \infty$.

Proof. Let $x \in \ell^\infty_w$ be given. Then the inequality

$$\|z\,w\|_\infty \leqslant \sup_n w_n \sum_{k=n}^{\infty} |x_k| \leqslant \sup_k |x_k|\, w_k \sup_n w_n \sum_{k=n}^{\infty} w_k^{-1} \leqslant \|x\,w\|_\infty C_1$$

shows that i) implies that ℓ^∞_w satisfies $X9$). For $x \in \ell^1_w$ assumption ii) implies

$$\|z\,w\|_1 \leqslant \sum_{n=1}^{\infty} w_n \sum_{k=n}^{\infty} |x_k w_k|\, w_k^{-1} = \sum_{k=1}^{\infty} |x_k w_k|\, w_k^{-1} \sum_{j=1}^{k} w_j \leqslant$$

$$\leqslant \|x\,w\|_1 \sup_k w_k^{-1} \sum_{j=1}^{k} w_j \leqslant \|x\,w\|_1 C_2,$$

i. e. ℓ^1_w satisfies $X9$). For $1 < q < \infty$ the result now follows by complex interpolation (see [2], 4.4.1 and 5.5.3).

Remark 2.2. It should be observed that the properties i) and ii) are preserved if w is multiplied by another sequence w^1 satisfying $w^1_{n+1} \geqslant w^1_n$ for all $n \geqslant n_0$.

Lemma 2.3. *Let* $(w_n)_{n \geqslant 1}$ *be an increasing sequence. Then* $(\ell^q_b, \ell^{q'}_w, \ell^r_w)$ *is an S-tripel for* $1 \leqslant q \leqslant \infty$, and $b \geqslant 1/r$. *In particular* ℓ^q_b *satisfies* $X8$) *for* $1 \leqslant q \leqslant 2$, $b \geqslant 1/q$.

Proof. We suppose throughout that $x \geqslant 0$, $y \geqslant 0$ holds. Let $x \in \ell^1$, $y \in \ell^\infty_w$, be given. Then

$$\|z\,w\|_\infty = \sup_n w_n \sum_{k \geqslant n} x_k y_k \leqslant \|x\|_1 \|y\,w\|_\infty .$$

This shows that $(\ell^1, \ell^\infty_w, \ell^\infty_w)$ is an S-tripel. On the other hand $(\ell^1_{0,1}, \ell^\infty_w, \ell^1_w)$ is an S-tripel, since

$$\|z\,w\|_1 = \sum_{n=1}^{\infty} w_n \sum_{k=n}^{\infty} x_k y_k \leqslant \sum_{j=1}^{\infty} \Big(\sum_{n=1}^{j} w_n \Big) x_j y_j \leqslant \sum_{j=1}^{\infty} j\, w_j x_j y_j \leqslant \|y\,w\|_\infty \sum_{j=1}^{\infty} j\, x_j$$

for $x \in \ell^1_{0,1}$, $y \in \ell^\infty_w$. Complex interpolation ([2], 4.4.1 and 5.5.3) implies that $(\ell^1_b, \ell^\infty_w, \ell^r_w)$ is an S-tripel for $b \geqslant 1/r$. But this is equivalent to the assertion that $(\ell^\infty_b, \ell^1_w, \ell^r_w)$ constitutes an S-tripel. Further complex interpolation, now applied to the last two assertions, implies that $(\ell^q_b, \ell^{q'}_w, \ell^r_w)$ is an S-tripel for $1 \leqslant q \leqslant \infty$, and $b \geqslant 1/r$. The special case $r = q$, $w_n = n^b$, shows that $(\ell^q_b, \ell^{q'}_b, \ell^q_b)$ is an S-tripel for $b \geqslant 1/q$. This implies that ℓ^q_b satisfies $X\,8)$ for $1 \leqslant q \leqslant 2$, $b \geqslant 1/q$.

Proof of Theorem 2.1. A) It is easy to check that the sequence $w_n = 2^{an}$, $a > 0$, satisfies i) and ii) of Lemma 2.2. Consequently $w'_n = 2^{an} n^b = 2^{a'n}[2^{(a-a')n} n^b]$ satisfies these conditions for $a > 0$, $b \in \mathbb{R}$ (choose $0 < a' < a$ and cf. with remark 2.2). That $c^o_{a,b}$ satisfies $X\,9)$ for these indices follows from the result for $\ell^\infty_{a,b}$, since $c^o_{a,b}$ satisfies $X\,6)$ and S takes "finite sequences" into finite sequences.

B) follows readily from Lemma 2.3.

C) For $1 \leqslant q \leqslant 2$, $b \geqslant 1/q$ this is Lemma 2.3. Since $(\ell^1, \ell^\infty_b, \ell^\infty_b)$ is an S-tripel for $b \geqslant 0$ and $(\ell^1, \ell^\infty_b, \ell^1)$ for $b \geqslant 1$, it follows that $\ell^1 \cap \ell^\infty_b$ satisfies $X\,8)$ for $b \geqslant 1$. In particular, ℓ^∞_b satisfies $X\,8)$ for $b > 1$. Combined with Lemma 2.3, taking there $q = 2$, the complex method of interpolation yields $X\,8)$ for ℓ^q_b, $b > 1/q'$, $2 \leqslant q \leqslant \infty$.

3. Basic Results

Definition. Let $\mathfrak{P} = (P_n)_{n \geqslant 1}$, $(\psi_n)_{n \geqslant 1}, X$, and B be given as described in § 2, I—III). Then we define

$$F(\mathfrak{P}, B, X) := \{f \mid (f\psi_n)_{n \geqslant 1} \in X(B)\}, \tag{12}$$

$$\|f\|_F := \|(f\psi_n)_{n \geqslant 1}\|_{X(B)}, \tag{13}$$

$$F_0(\mathfrak{P}, B, X) := B \cap F(\mathfrak{P}, B, X), \|f\|_{F_0} := \|f\|_B + \|f\|_F. \tag{14}$$

Again, we write simply F or $F(B, X)$, occasionally.

Remark 3.1. In many cases F coincides with F_0, and the norms $\| \ \|_F$ and $\| \ \|_{F_0}$ are equivalent, for example, if $X \subseteq \ell^1$.

Theorem 3.1. *A)* $F = (F(\mathfrak{P}, B, X), \| \ \|_F)$ *and hence* F_0 *is a solid BF-space on* G.

B) If X *satisfies* $X\,5)$ *and* $X\,7)$, *and if* B *is left (right) invariant, then so are* F *and* F_0. *Furthermore, there exists* $C > 1$ *such that*

$$\|L_y\|_F \leqslant \|L_y\|_B\, C^{n+1} \text{ for } y^{-1} \in P_n;\ n \geqslant 1. \tag{15}$$

C) If $\mathscr{K}(G)$ is dense in B, and if X satisfies $X6)$, then $\mathscr{K}(G)$ is dense in F and in F_0.

Proof. A) it is clear that $\| \, \|_F$ defines a norm on $F(B,X)$. The completeness of $(F(B,X), \| \, \|_F)$ follows directly from the fact that an absolutely convergent series $\sum_{k=1}^{\infty} f_k$ with $\sum_{k=1}^{\infty} \|f_k\|_F \leqslant C < \infty$ defines some $f = \sum_{k=1}^{\infty} f_k$ with $\|f\|_F \leqslant C$.

B) Suppose that B is left invariant, and let $f \in F(B,X)$, $y \in G$ be given. Then $y^{-1} \in P_n$ for some $n \geqslant 1$. For $k > n$ $Q2)$ implies

$$L_{y^{-1}} \psi_k \leqslant \psi_{k-1} + \psi_k + \psi_{k+1}, \quad \text{and}$$

$$\|(L_y f)\psi_k\|_B \leqslant \|L_y\|_B \|f L_{y^{-1}} \psi_k\|_B \leqslant$$
$$\leqslant \|L_y\|_B (\|f\psi_{k-1}\|_B + \|f\psi_k\|_B + \|f\psi_{k+1}\|_B).$$

For $k \leqslant n$ assumptions $Q2)$ (cf. (17) below) and $X7)$ imply

$$\|(L_y f)\psi_k\|_B \leqslant \|L_y(\sum_{j=1}^{n+1} f\psi_j)\|_B \leqslant \|L_y\|_B \sum_{j=1}^{n+1} \|f\psi_j\|_B \leqslant \|L_y\|_B \sum_{j=0}^{n} \|G\|^j \|f\|_F.$$

Combining these two inequalities we obtain by $X4)$

$$\|L_y f\|_F \leqslant \|L_y\|_B [(c_n \sum_{j=0}^{n} \|G\|^j) + \|G\| + \|D\| + 1]$$

for all $y \in P_n^{-1}$, with $c_n := \|(1,1,\ldots 1,0,0\ldots)\|_X$ (cf. [9], p. 139). Since $c_n \leqslant \sum_{k=0}^{n=1} \|D\|^k$, it is obvious that the required estimate can be given (e. g. $C = \|D\| + \|G\| + 1$).

C) It follows from $X6)$ that for any $f \in F(B,X)$ there exists $n_0 \in \mathbb{N}$ such that $\|f - f\chi_n\|_F < \varepsilon$ for $n \geqslant n_0$. On the other hand the shift invariance of X implies that the norms $\| \, \|_B$ and $\| \, \|_F$ are equivalent when restricted to $B_{Q_n}: \{g = | g \in B, g\chi_n = 0\}$, for any given $n \in \mathbb{N}$. Thus, by assumption, there exists $k \in \mathscr{K}(G)$ such that $\|f\chi_n - k\|_F < \varepsilon$. This implies $\|f - k\|_F < 2\varepsilon$, and the proof is complete.

The decisive step towards the main results is contained in the following theorem:

Theorem 3.2. *A) Let a tripel (F_0^1, F_0^2, F_0^3), $F_0^i = F_0(\mathfrak{P}, B_i, X_i)$, $i = 1,2,3$ be given such that*

a) (B_1, B_2, B_3) is a Banach convolution tripel;

b) (X_1, X_2, X_3) *is an S-tripel; and*

c) $X_1 + X_2 \subseteq X_3$.

Then (F_0^1, F_0^2, F_0^3) *constitutes a Banach convolution tripel.*

B) *Let B be a two-sided Banach convolution module over a Banach convolution algebra A. Then* $A \cap F(\mathfrak{P}, B, X)$ *is a Banach convolution algebra, if X satisfies* $X9$).

Corollary 3.3. *Let A be a BCA, and let X be a BK-space satisfying* $X8$). *Then* $F_0(\mathfrak{P}, B, X)$ *is a Banach convolution algebra.*

Proof of Theorem 3.1. *Step I.* Let $f \in F_0^1$, $g \in F_0^2$ be given. Since $|f * g| \leqslant |f| * |g|$ holds and since $F_0^i, i = 1, 2, 3$ are solid BF-spaces, we may suppose without loss of generality $f \geqslant 0$, $g \geqslant 0$. We start by considering the expression

$$(f * g)\psi_n = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} (f\psi_k * g\psi_m)\psi_n = \sum_{k=1}^{n-2} \cdots + \sum_{k=n-1}^{n+1} \cdots + \sum_{k=n+2}^{\infty} \cdots \tag{16}$$

It is clear that $P_k P_m \cap P_n = \emptyset$ implies $(f\psi_k * g\psi_m)\psi_n = 0$, and therefore it turns out that the inner sum $\sum_{m=1}^{\infty}$ can be replaced by a finite sum in all three cases respectively. In fact, $Q2$) implies

$P_i P_j \subseteq Q_{j+1}$ for $j \geqslant i$, and

$P_i P_j \subseteq Q_i (G \setminus Q_{j-1}) \subseteq G \setminus Q_{j-2}$ for $j \geqslant i + 2$. Thus

$$P_i P_j \cap P_l = \emptyset \text{ implies } j - 1 \leqslant l \leqslant j + 1 \text{ for } j \geqslant i + 2 \text{ or } l \geqslant i + 2. \tag{17}$$

Consequently,

$$\begin{aligned} \sum_{k=1}^{n-2} (f\psi_k * g)\psi_n &\leqslant \sum_{k=1}^{n-2} (f\psi_k) * g(\psi_{n-1} + \psi_n + \psi_{n+1}) \leqslant \\ &\leqslant f * g(\psi_{n-1} + \psi_n + \psi_{n+1}); \end{aligned} \tag{18}$$

$$\sum_{k=n-1}^{n+1} (f\psi_k * g)\psi_n \leqslant f(\psi_{n-1} + \psi_n + \psi_{n+1}) * g; \tag{19}$$

$$\sum_{k=n+2}^{\infty} (f\psi_k * g)\psi_n \leqslant \sum_{k=n+2}^{\infty} f\psi_k * g(\psi_{k-1} + \psi_k + \psi_{k+1}). \tag{20}$$

If we set $r = (r_k) := (\|f\psi_k\|_{B_1})$, $s = (s_k) := (\|g\psi_k\|_{B_2})$ and $t = (t_k) := (\|(f * g)\psi_k\|_{B_3})$, we have

$$\|f\|_{F_1} = \|r\|_{X_1}, \|g\|_{F_2} = \|s\|_{X_2} \text{ and } \|f * g\|_{F_3} = \|t\|_{X_3}.$$

By assumption a) (18)—(20) together with (16) imply

$$\begin{aligned} t_n \leqslant K_1 \|f\|_{B_1} (s_{n-1} + s_n + s_{n+1}) + K_1 (r_{n-1} + r_n + r_{n+1}) \|g\|_{B_2} + \\ + K_1 \sum_{k=n+2}^{\infty} r_k (s_{k-1} + s_k + s_{k+1}) \quad \text{for } n \geqslant 1, \text{ i. e.} \end{aligned} \tag{21}$$

$$\begin{aligned} t \leqslant K_1 \|f\|_{B_1} (D s + s + G s) + K_1 (D r + r + G r) \|g\|_{B_2} + \\ + K_1 G^2 S (r, D s + s + G s) . \end{aligned} \tag{22}$$

Assumption c) implies $r, s \in X_3$ and $\|r\|_{X_3} \leqslant C_1 \|r\|_{X_1}, \|s\|_{X_3} \leqslant C_2 \|s\|_{X_2}$. Using b) we derive from (22)

$$\begin{aligned} \|t\|_{X_3} \leqslant K_1 C_2 \|f\|_{B_1} \|Ds + s + Gs\|_{X_2} + K_1 C_1 \|D r + r + G r\|_{X_1} \|g\|_{B_2} + \\ + K_1 K_2 \| G^2 r\|_{X_1} \| G^3 s + G^2 s + G s\|_{X_2} . \end{aligned} \tag{23}$$

Since D and G are bounded operators on X_i, $i = 1,2$, we can find some $K_3 > 0$ such that

$$\|t\|_{X_3} \leqslant K_3 (\|f\|_{B_1} \|s\|_{X_2} + \|r\|_{X_1} \|g\|_{B_2} + \|r\|_{X_1} \|s\|_{X_2}) . \tag{24}$$

This implies that there exists $K_4 > 0$ such that

$$\begin{gathered} \|f * g\|_{F_0^3} \leqslant K_4 \|f\|_{F_0^1} \|g\|_{F_0^2} \\ \text{for all } f \in F_0 (B_1, X_1), g \in F_0 (B_2, X_2) . \end{gathered} \tag{25}$$

Step II. In order to prove B) it will be sufficient to give an estimate for $\|f * g\|_{F(B,X)}$ for $f, g \in A \cap F(B,X)$. Let r, s, t have the same meaning as above, with $B_1 = B_2 = B_3 = B$. We now use the inequality

$$(f * g) \psi_n \leqslant f * g (\psi_{n-1} + \psi_n + \psi_{n+1}) + \sum_{k=n-2}^{\infty} f \psi_k * g . \tag{26}$$

Then

$$t_n \leqslant \|f\|_A (s_{n-1} + s_n + s_{n+1}) + \sum_{k=n-2}^{\infty} r_k \|g\|_A , \tag{27}$$

i. e.

$$t \leqslant \|f\|_A (D s + s + G s) + D^2 S (r, \|g\|_A) . \tag{28}$$

By X 9) this gives

$$\|t\|_X \leqslant \|f\|_A K_5 \|s\|_X + K_6 \|r\|_X \|g\|_A . \tag{29}$$

It thus becomes clear that there exists $K > 0$ such that

$$\|f * g\|_A + \|f * g\|_F \leqslant K\,(\|f\|_A + \|f\|_F)\,(\|g\|_A + \|g\|_F) \quad \text{for all } f, g \in A \cap F(B, X). \tag{30}$$

This completes the proof of B).

Remark 3.2. i) The connection between part B) and Theorem 2.2 of [9] will be discussed in section 5 (see Theorem 5.1).

ii) Theorem 3.2 and Corollary 3.3, together with Theorem 2.1 would allow us to write down already a number of concrete examples of new Banach convolution algebras. Since for some of these examples results are available that go beyond those that can be derived immediately, we prefer to give a more detailed discussion in the following section.

The remaining part of this section is devoted to an exposition of general properties shared by the spaces $F(\mathfrak{P}, B, X)$ on a locally compact group G. For the sake of shortness, B_1 denotes in the sequel of this section one of the spaces $F(B, X)$, $F_0(B, X)$, or $A \cap F(B, X)$. Suppose that B_1 is a translation invariant Banach convolution algebra containing $\mathcal{K}(G)$ as a dense subalgebra (cf. 3.1.C). Then we have, among others, the following results (cf. [9], pp. 154—155, a proof of Theorem C can be found in [8]):

Theorem 3.4. *B_1 possesses multiple, two-sided approximate units in $K(G)$ and the closed left (right) invariant subspaces coincide with the closed left (right) ideals of B_1.*

If furthermore the underlying group is Abelian we have (cf. [9], pp. 155—158):

Theorem 3.5. *Suppose that L_y is an isometry on B. Then B_1 satisfies BD′), and $\hat{B}_1$ is of type F. Then the only multiplicative linear functionals on B_1 are those of the form $f \to \hat{f}(x_0)$, $x_0 \in \hat{G}$. The topology of $\hat{G}$ is the weakest among all topologies for which all $\hat{f} \in \hat{B}_1$ are continuous functions on $\hat{G}$. Therefore the space of regular maximal ideals of B_1 can be identified with the dual group $\hat{G}$. Moreover we have*

$$\lim_{n \to \infty} \|f * f * \ldots * f\|_{B_1}^{1/n} = \|\hat{f}\|_\infty \quad (n\text{-fold convolution}).$$

Further, the functions $f \in B_1$ such that $\hat{f}$ has compact support are dense in B_1. Therefore $\hat{B}_1$ is a Wiener algebra in the sense of [20], II, § 2.4. *In particular, any closed ideal I of B_1 contains all functions*

$f \in B_1$ with $\operatorname{supp} \hat{f} \subseteq G \setminus \operatorname{cosp} I$. *If further $\mathfrak{P}$ is symmetric, and A and B are rearrangement-invariant, then B_1 has the (weak) factorization property if and only if B_1 has bounded approximate units.*

Proof. First of all one has to verify that B_1 satisfies BD'), i. e. that $\sum_{n=1}^{\infty} n^{-2} \log |w(x^n)| < \infty$ for all $x \in G$, with $w(x) := \max(1, \|L_x\|_{B_1})$. Since $\|L_x\|_A = \|L_x\|_B = 1$ for all $x \in G$ it is no problem to verify that this relation holds, using (15) (cf. [9]). That $\hat{B}_1$ is of type F follows now from Lemma 4.1 of [9]. The other assertions are therefore a consequence of the corresponding results in DOMAR's paper [7].

The results concerning weak factorizations follow from the converse to Cohen's factorization theorem for self-adjoint group algebras (cf. [11], Theorem 3), the symmetry of $\mathfrak{P}$, together with the rearrangement-invariance of A and B implying that B_1 is invariant under the involution $f \mapsto f^*$, $f^*(x) := \overline{f(-x)}$, of $L^1(G)$.

For later use we state the following result on duality and on complex interpolation.

Theorem 3.6. *A)* $F(\mathfrak{P}, B, X)' = F(\mathfrak{P}, B', X')$ *if X satisfies $X6$). In particular, $F(\mathfrak{P}, B, X)$ is a reflexive space if B and X are reflexive.*

B) $(F(\mathfrak{P}, B_1, X_1), F(\mathfrak{P}, B_2, X_2))_{[\theta]} = F(\mathfrak{P}, B_*, X_*)$ *for* $0 < \theta < 1$, *with* $B_* := (B_1, B_2)_{[\theta]}$, *and* $X_* := (X_1, X_2)_{[\theta]}$.

Proof. A) If X satisfies $X6$), then X has absolutely continuous norm. Thus X' coincides with the Köthe dual X^a of X (cf. [26], § 72). The proof of assertion A) is now essentially the same as of Theorem 2 of [16] and is therefore omitted. Since any solid, reflexive BK-space has absolutely continuous norm (cf. [26], § 73), it satifies $X6$) and the additional assertion follows.

B) We observe that $F(\mathfrak{P}, B, X)$ can be considered as a retract of $X(B)$ (cf. [2], Definition 6.4.1), given by $\mathscr{S}f := (f\psi_j)_{j \geq 1}$, and $\mathscr{P}((f_j)_{j \geq 1}) := \sum_{j=1}^{\infty} f_j$. According to Theorem 6.4.2 of [2] the interpolation of the spaces $F_i, i = 1, 2$ is reduced to the interpolation of the corresponding spaces $X_i(B_i)$ of vector-valued sequences, which in turn may be carried out on the pairs (X_1, X_2) and (B_1, B_2) separately (see [2], Theorem 5.6.3).

Remark 3.3. i) A more elementary proof of part B) can be given, using the characterization of $(F_1, F_2)_{[\theta]}$ as $F_1^{1-\theta} F_2^{\theta}$ (cf. [6], § 13.5, or [2], 5.8.1).

ii) For the sake of correctness let us mention that the ambiguity of the symbol $\mathfrak{P}_2$, denoting either the partition given by (2) or the isotropic partition $\mathfrak{P}_s$, with $s=2$ is of course irrelevant, since the corresponding spaces $F(\mathfrak{P}_2, B, X)$ are equal.

4. Applications, Concrete Examples

In this section we show that the method introduced in section 3, combined with complex interpolation, enables us to prove that a number of spaces which are closely related to the spaces $F(\mathfrak{P}, B, X)$ are Banach convolution algebras. In view of the vast variety of possible examples we restrict our attention mainly to the spaces $B = L^p(G)$ or $B = L^1 \cap L^p(G)$, $1 \leqslant p \leqslant \infty$, and to the spaces $X = \ell^q_{a,b}$, $1 \leqslant q \leqslant \infty$, $a \geqslant 0$, $b \in \mathbb{R}$. We thus have to look for appropriate conditions for the indices involved which ensure that the spaces under consideration are convolution algebras. Many of the results derived are best possible, i. e. the conditions on the parameters cannot be weakened, at least for the dyadic decomposition of $\mathbb{R}^m$. We begin with a Lemma which gives a general necessary condition.

Lemma 4.1. *Let a space* $F = F(\mathfrak{P}_\sigma, B, X)$ *or* $F = F_0(\mathfrak{P}_\sigma, B, X)$ *on* $G = \mathbb{R}^m$, *with* $B = L^p(G)$ *or* $B = L^1 \cap L^p(G)$, $1 \leqslant p \leqslant \infty$, *be given, which is a BCA. Then* F *is contained in* $L^1(G)$.

Proof. Suppose the contrary. Then there exists $f \neq 0$, $f \in F$, $f \notin L^1(\mathbb{R}^m)$, $f \geqslant 0$. Thus, a sequence $(K_n)_{n \geqslant 1}$ of compact subsets of $\mathbb{R}^m$ can be found such that

$$\int_{K_n} f(y)\, dy \geqslant n, \quad n = 1, 2, \ldots . \tag{31}$$

Given $j, n \in \mathbb{N}$ let us define the sets

$$P_j^+ := \{x \mid x \in P_j, x_i \geqslant 0, i = 1, 2 \ldots, m\} \tag{32}$$

$$\text{and } S(n, j) := \{x \mid x \in P_j^+, x - K_n \subset P_j^+\}. \tag{33}$$

We assert that there exists a sequence $(j_n)_{n \geqslant 1} \subset \mathbb{N}$ such that

$$|S(n, j_n)| \geqslant 6^{-m} |P_{j_n}^+|, \quad n = 1, 2, \ldots . \tag{34}$$

In fact, suppose that $K_n \in [-r_n, r_n]^m$. For $\sigma = (s_1, s_{-1}, \ldots, s_{-m})$ we have

$$|P_j^+| = \prod_{i=1}^m s_i^j - \prod_{i=1}^m s_i^{j-1} \leqslant \prod_{i=1}^m s_i^j \tag{35}$$

If we choose j_n large enough, we have

$$12\, r_n \leqslant s_i^{j-1} \text{ for } i = 1, 2, \ldots, m; j \geqslant j_n. \tag{36}$$

Observing that $s_i \geqslant 2$ we have $\frac{5}{6} s_i - \frac{7}{6} \geqslant s_i/6$. Combined with (36) and (35) this allows us to prove (34):

$$|S(n,j)| \geqslant \prod_{i=1}^{m} (s_i^j - 2\, r_n) - \prod_{i=1}^{m} (s_i^{j-1} + 2\, r_n) \geqslant \prod_{i=1}^{m} \tfrac{5}{6}\, s_j^i - \prod_{i=1}^{m} \tfrac{7}{6}\, s_i^{j-1} \geqslant$$
$$\geqslant 6^{-m} \prod_{i=1}^{m} s_i^j \geqslant 6^{-m}\, |P_j^+| \text{ for } j \geqslant j_n.$$

If ψ_j^+ denotes the characteristic function of P_j^+ we can give the following estimate (cf. (31)):

$$f * \psi_j^+ (x) \geqslant \int_{K_n} f(y)\, \psi_j^+ (x-y)\, dy \geqslant n \text{ for } x \in S(n,j), j \geqslant j_n. \tag{37}$$

For $1 \leqslant p \leqslant \infty$ (37) implies

$$\begin{aligned} \|(f * \psi_j^+)\, \psi_j\|_p &\geqslant n\, |S(n,j_n)|^{1/p} \geqslant n \cdot 6^{-m/p}\, |P_j^+|^{1/p} \geqslant \\ &\geqslant n\, 6^{-m/p}\, \|\psi_j^+\|_p \text{ for } j \geqslant j_n. \end{aligned} \tag{38}$$

Since (38) holds true for any $n \geqslant 1$ it follows that F cannot be a *BCA*, and the proof is complete.

Corollary 4.2. *A) Suppose that G is connected and that L_y is an isometry on B. Then $F(B,X)$ is not a BCA if $X \not\subseteq \ell^1$, in particular, if $X = \ell_b^q$, $0 \leqslant b \leqslant 1/q'$, $1 < q \leqslant \infty$.*

B) Let $\mathfrak{P} = \mathfrak{P}_2$ be the dyadic decomposition of $\mathbb{R}^m$. Then $F(L^p, \ell^q_{a,b})$ is not a BCA for $a < m/p'$, $b \in \mathbb{R}$, or $a = m/p'$, $b \leqslant 1/q'$.

Proof. A) Take any $f_0 \in \mathcal{K}(G)$ with $\operatorname{supp} f_0 = K$. Then, according to Lemma 3.1 of [9], there exists $k_0 \in \mathbb{N}$ and $(y_k)_{k \geqslant 1} \subseteq G$ such that $y_k K \subseteq P_{k_0+k}$ for $k \geqslant 1$. Let now $(x_k)_{k \geqslant 1}$ be any sequence which belongs to X, but not to ℓ_1. Then the function f, defined by

$$f := \sum_{k \geqslant 1} x_k L_{y_k} f_0, \tag{39}$$

does not belong to $L^1(G)$, but

$$\|f \psi_n\|_B = \|x_{n-k_0} L_{y_{n-k_0}} f_0\|_B = x_{n-k_0} \|f_0\|_B \quad \text{for } n > k_0.$$

Thus f belongs to $F(B,X)$ and Lemma 4.1 applies.

B) Let $(x_n)_{n \geqslant 1}$ be a sequence in $\ell_b^q \setminus \ell^1$ for $b = 1/q'$. Since $\|\psi_n\|_p \sim 2^{mn/p}$, the function $f := \sum x_n \psi_n$ belongs to $F(L^p, \ell^q_{a,b})$, but not to $L^1(\mathbb{R}^m)$ for the indices indicated.

The two theorems below are the main result of this section.

Theorem 4.3. *Let* $1 \leqslant p, q \leqslant \infty$, $a \geqslant 0$, $b \in \mathbb{R}$, G *unimodular, be given.*

A) $L^1 \cap F(\mathfrak{P}, L^p, \ell^q_{a,b})$ *is a BCA for* $a > 0$, *or* $a = 0$, $p \leqslant q$, $b \geqslant 0$.

B) $F(\mathfrak{P}_2, L^p, \ell^q_{a,b})$ *is a BCA on* $G = \mathbb{R}^m$ *or* $\mathbb{Z}^m$ *if and only if* $a > m/p'$ *or* $a = m/p'$ *and* $b > 1/q'$.

Proof. A) For $a > 0$ the space $X = \ell^q_{a,b}$ satisfies $X\,9)$ according to 2.1.A, and the result is a consequence of 3.2.B. The case $a = 0$, $b \geqslant 0$ can be treated in the following way. For $p = q$ the space $L^1 \cap F(\mathfrak{P}, L^p, \ell^p_b)$ coincides with $L^1 \cap L^p_{w_b}$, w_b being the strictly positive function on G given by $w_b(y) := n^b$ for $y \in P_n$, $n \geqslant 1$. It is no problem to verify that w_b is a weakly subadditive function (cf. [4]), i. e. that there exists $C_b > 0$ such that

$$w_b(y_1 y_2) \leqslant C_b\big(w(y_1) + w(y_2)\big) \text{ for all } y_1, y_2 \in G. \tag{40}$$

This implies

$$|f * g|\, w_b \leqslant C_b(|f|\, w_b * |g| + |f| * |g|\, w_b) \text{ for all } f, g \in L^1 \cap L^p_{w_b}(G). \tag{41}$$

It is an immediate consequence of (41) that $L^1 \cap L^p_{w_b}$ is a *BCA* for $b \geqslant 0$. For $p = 1$, $q = \infty$, we have to consider $F := L^1 \cap F(L^1, \ell^\infty_b) = F(L^1, \ell^1 \cap \ell^\infty_b)$. 2.1.B implies that $(\ell^1 \cap \ell^\infty_b, \ell^1 \cap \ell^\infty_b, \ell^\infty_b)$ is an S-tripel for $b \geqslant 0$, and thus $F * F \subset F(L^1, \ell^\infty_b)$ according to 3.2.A. Since the inclusion $F * F \subset L^1$ is obvious one verifies without difficulty that F is a *BCA*.

The general result of A) follows from these two special cases by means of complex interpolation (cf. 3.6.B). In fact, one has

$$L^1 \cap F(L^p, \ell^q_b) = \big(L^1 \cap F(L^{p_1}, \ell^{p_1}_b), L^1 \cap F(L^1, \ell^\infty_b)\big)_{[\theta]} \tag{42}$$

for $b \geqslant 0$, $p < q$, if one takes $p_1 = 1 + 1/q$, and $\theta = 1 + 1/q - 1/p$.

B) In view of part A) and Lemma 4.1 it will be sufficient to prove that $F(\mathfrak{P}_2, L^p, \ell^q_{a,b}) \subset L^1(\mathbb{R}^m)$ for $a = m/p'$, $b > 1/q'$ (and hence for $a > m/p'$, $b \in \mathbb{R}$). Since $|P_n| \sim 2^{mn}$ for $n \to \infty$, Hölder's inequality implies

$$\|f \psi_n\|_1 \leqslant \|f \psi_n\|_p \|\psi_n\|_{p'} \leqslant C \cdot 2^{mn/p'} \|f \psi_n\|_p, n \geqslant 1. \tag{43}$$

Therefore

$$F(\mathfrak{P}_2, L^p, \ell^q_{a,b}) \subset F(\mathfrak{P}_2, L^1, \ell^q_b) \subset F(\mathfrak{P}_2, L^1, \ell^1) = L^1(\mathbb{R}^m)$$

for $a = m/p'$ and $b > 1/q'$, and the proof of B) is complete.

Remark 4.1. i) Further Banach algebras of the form $F(L^1, X_1) \cap \cap F(L^p, X_2)$ are easily derived from the results given above.

ii) The special case $p = q$, $b = 0$, $G = \mathbb{R}$ of Theorem 4.3. B gives the Banach algebras considered by WERMER ([25], the proof given there is completely different).

iii) The result of 4.3 remains true of one replaces F by F_0.

iv) That the condition $p \leqslant q$ in 4.3. A cannot be dropped in general is shown by the following example.

Counterexample 1. Let $1 \leqslant q < \infty$ be given. Then $L^1 \cap F(\mathfrak{P}_2, L^\infty, \ell^q_b)$ is not a *BCA* on $G = \mathbb{R}^m$, for any $b > 0$.

Proof. We give the proof for $m = 1$, but essentially the same argument is applicable to the general case. Denote the characteristic function of $[-2^{-n-1} n^b, 2^{-n-1} n^b]$ by c_n, $n \geqslant 1$, and set

$$f_n := L_{2^n} \sum_{k=0}^{2^n - 1} L_{k+1/2}\, c_n . \tag{44}$$

Then $\|f_n\|_1 = n^b$ and $\|f_n\|_\infty = 1$. Thus $\|f_n^*\| = \|f_n\| = \|f_n\|_1 + + n^b \|f_n\|_\infty = 2 n^b$. Consider now $f_n * f_n^*$. Then $f_n * f_n^*(0) = \|f_n\|_2^2 = n^b$, and further $f_n * f_n^*(k) \geqslant n^b/2$ for $k \leqslant 2^{n-1}$, $k \in \mathbb{N}$. This implies

$$\|f_n * f_n^*\| \sim n^b/2 \, \Big(\sum_{k=1}^{n-1} k^{bq}\Big)^{1/q} \sim n^{2b+1/q} \sim n^{1/q} \|f_n\| \, \|f_n^*\| . \tag{45}$$

The assertion is thus proved.

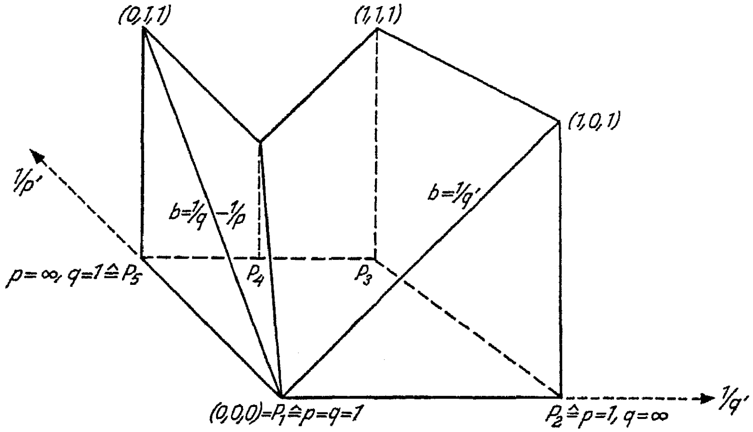
Theorem 4.4. *Let* $1 \leqslant p, q \leqslant \infty$, $a \geqslant 0$, $b \in \mathbb{R}$ *be given.*

A) $F(L^1 \cap L^p, \ell^q_{a,b})$ *is a BCA for* $a > 0$, $b \in \mathbb{R}$, *or* $a = 0$, *and* $b > b_0$, $b_0 := \max(1 - 1/q, 1/q - 1/p)$.

B) $F_0(L^1 \cap L^p, \ell^q_{a,b})$ *is a BCA for* $a > 0$, $b \in \mathbb{R}$, *and for* $a = 0$, $b \geqslant \max(0, 1/q - 1/p)$, *if G is unimodular.*

Proof. Step I. For $a > 0$ part A) is a simple consequence of 4.3.A (cf. remark 4.1.i). Let us therefore consider the case $a = 0$. Figure I may serve as an illustration of the result as well as a tool for the proof, in the sense that we shall be able to take the final result directly from the figure instead of presenting long calculations.

The figure has to be interpreted as follows: $F(\mathfrak{P}, L^1 \cap L^p, \ell^q_b)$ is a *BCA* for $1 \leqslant p$, $q \leqslant \infty$, $b \geqslant 0$, as long as the tripel $(1/q', 1/p', b)$ $(\triangleq (p, q, b))$ is to be found above the threedimensional body sketched below.



Step II. Let us start with the observation that the scale $(L^1 \cap L^p(G))_{1 \leqslant p \leqslant \infty}$ is closed under complex interpolation; more precisely,

$$(L^1 \cap L^{p_1}, L^1 \cap L^{p_2})_{[\theta]} = L^1 \cap L^{p_*},$$

$$1/p_* = (1 - \theta)/p_1 + \theta/p_2, \text{ for } 0 < \theta < 1.$$

This, together with the interpolation formula 3.6.B implies that the set of all triples $(1/q', 1/p', b)$ with the property that $F(L^1 \cap L^p, \ell^b_b)$ is a Banach convolution algebra is convex in $\mathbb{R}^3$. Having the above figure in mind we see that it will be sufficient to find the optimal indices b at the points $P_1, P_2, \ldots, P_5$.

Step III. Consider the question at P_1, i. e. for $p = q = 1$. Then $F(L^1, \ell^1_b) = L^1 \cap F(L^1, \ell^1_b) = L^1_{w_b}$ is a Beurling algebra (cf. 4.2.A). For the line $\overline{P_2 P_3 P_4 P_5}$, i. e. for $p = \infty$, $1 \leqslant q \leqslant \infty$, and for $1 \leqslant p \leqslant \infty$, $q = \infty$, the result follows from Corollary 3.3, combined with Theorem 2.1.C. In view of step II the proof of A) is now complete.

Step IV. We have to prove B). For a > 0 this is the same result as A). For $a = 0$ the result is once more derived by means of complex interpolation. It will therefore be sufficient to check the result for $p \leqslant q$, and for $p = \infty$, $q = 1$. For $p \leqslant q$ $F(L^p, \ell^q_b)$ coincides with $F_0(L^p, \ell^q_b)$. It is therefore possible to write

$$F_0(L^1 \cap L^p, \ell^q_b) = [L^1 \cap F(L^1, \ell^q_b)] \cap [L^1 \cap F(L^p, l\ell^q_b)].$$

It now follows from 4.2.A that this space is a *BCA*. That the space $F_0(L^1 \cap L^\infty, \ell^1_b)$ is a *BCA* for $b \geqslant 1$ follows from 3.2.A combined with 2.1.C. This completes the proof of B).

Remark 4.2. It follows from Corollary 4.2 that the condition $b > 1/q'$ is also necessary in order to have A) for $a = 0$. Furthermore, the counterexample below shows that at least for $p = \infty$ the condition $b \geqslant 1/q - 1/p$ is necessary as well.

Counterexample 2. For $b < 1/q$ we can define a sequence $(f_n)_{n \geqslant 1} \subset \subset F_0(L^1 \cap L^\infty, \ell^q_b)$ such that $\|f_n\|_{F_0} = \|f_n^*\|_{F_0} \leqslant C$, but $\|f_n * f_n^*\|_F$ is unbounded.

Proof. Denote the characteristic function of $[-2^{-n-1}, 2^{-n-1}]$ by c_n. Set

$$f_n := n^{-b} L_{2^n} \sum_{k=0}^{2^n-1} L_{(k+1)/2}\, c_n\,. \tag{46}$$

Then $\|f_n\|_1 = \|f_n\|_2^2 = \|f_n\|_\infty = n^{-b}$ for $n \geqslant 1$, i. e. $\|f_n\|_{F_0} \leqslant 4$ for $n \geqslant 1$. If d_n denotes the triangular function $c_n * c_n$, we have

$$f_n * f_n^* = n^{-2b} \sum_{j=-2^n}^{2^n} (2^n - |j|)\, L_j\, d_n\,. \tag{47}$$

Consequently $f_n * f_n^*(j) \geqslant n^{-2b}/2$ for $j \in \mathbb{N}$, $|j| \leqslant 2^{n-1}$. This implies

$$\|f_n * f_n^*\|_{F(L^\infty, \ell^q_b)} \geqslant n^{-2b}/2 \Big(\sum_{k=1}^{n-1} k^{bq} \Big)^{1/q} \sim C\, n^{-2b} n^{b+1/q}\,. \tag{48}$$

Since this expression tends to infinity for $b < 1/q$ the proof is complete.

Remark 4.3. i) For the general case, i. e. for $1 \leqslant q < p < \infty$ we have no counterexample showing that the condition $b > 1/q - 1/p$ is necessary as well, but one can 'show' that such an example is all but obvious. Therefore this question has to be left open.

ii) There is another relation which breaks down for $b < 1/q - 1/p$: Let $1 \leqslant p \leqslant \infty$, and a weight function w on G (cf. [20], VI § 7) be given. Consider now the translation invariant BF-space $L^p_w(G)$. Then it is easily shown that

$$\|L_{y_n} k\|_{L^p_w} \sim \|L_{y_n}\|_{L^p_w} \text{ for any } k \in \mathscr{K}(G),\ (y_n)_{n \geqslant 1},\ y_n \to \infty\,. \tag{49}$$

Relation (49) is still true if one only assumes that $L^p_w(G)$ is a translation invariant BF-space. It is an interesting fact that (49) is not true for $F = F(\mathfrak{P}_2, L^p, \ell^q_b)$, if $b < 1/q - 1/p$. In fact, one has $\|L_{2^n} k\|_F \sim n^b$ for $k \in K(G)$, but $\|L_{2^n}\|_F \sim n^{1/q - 1/p}$.

For any weight function w, e. g. for $w = w_{a,b}$, one has the relation $L^p_w * L^{p'}_w \subset L^\infty_w$, as a consequence of the inequality

$$|f * g|\, w \leqslant |f|\, w * |g|\, w\,. \tag{50}$$

The following theorem shows that the method given above yields better results:

Theorem 4.5. *Let G be unimodular. Then we have:*

A) $F_0(L^p, \ell^q_{a,b}) * F_0(L^r, \ell^{q'}_{a,b}) \subset F_0(L^s, \ell^{q'}_{a,b})$ for $1/s = 1/p + 1/r - 1$, *if* $1 \leqslant q \leqslant 2$, *and* $a > 0$, $b \in \mathbb{R}$, *or* $a = 0$ *and* $b \geqslant 1/q'$.

B) $L^p_{a,b} * L^{p'}_{a,b} \subset F_0(L^\infty, \ell^{p'}_{a,b})$ *for* $1 \leqslant p \leqslant 2$, $a > 0$, *or* $a = 0$, $b \geqslant 1/p'$.

Proof. According to Young's inequality, (L^p, L^r, L^s) constitutes a BCT on a unimodular group for $1/p + 1/r \geqslant 1 + 1/s \geqslant 0$. It follows from 2.1.B that $(\ell^q_b, \ell^{q'}_b, \ell^{q'}_b)$ constitutes an S-tripel for $b \geqslant 1/q'$. Further, $\ell^q_b + \ell^{q'}_b \subset \ell^{q'}_b$ for $1 \leqslant q \leqslant 2$. It follows that the same is true for $a > 0$, $b \in \mathbb{R}$. It is now possible to apply Theorem 3.2.A in order to derive part A). The second part is only a special case of A), with $q = p$, $r = p'$, and $s = \infty$.

5. Relations to Other Spaces, Equivalent Norms, Inclusions

First of all we discuss the relations to the spaces $\Lambda(A, B, X) = A \cap \Lambda(B, X)$ introduced in [9].

Theorem 5.1. *Let B, X be given, and suppose that $\Lambda(B, X)$ and $F(B, X)$ are defined by means of the same increasing sequence $(Q_n)_{n \geqslant 0} \subset G$. Then $\Lambda(B, X) \subset F_0(B, X)$. If further X satisfies $X9$), we have $\Lambda(B, X) = F_0(B, X) = F(B, X)$.*

Proof. It follows from the assumptions that $\psi_n \leqslant \chi_{n-1} \leqslant \sum_{k=n-1}^{\infty} \psi_k$ for $n \geqslant 1$. Therefore

$$\|f\psi_n\|_B \leqslant \|f\chi_{n-1}\|_B \leqslant \sum_{k=n-1}^{\infty} \|f\psi_k\|_B \text{ for } f \in B,\ n \geqslant 1. \tag{51}$$

The inclusion for general X is now obvious. Since any X satisfying $X9$) must be contained in ℓ^1 we have $F = F_0 \subset B$ in this case (cf. Remark 3.1), i. e. $\|f\chi_0\|_B < \infty$. An application of $X9$) to the second half of (51) gives the inclusion $F(B, X) \subset \Lambda(B, X)$, and the proof is complete.

Remark 5.1. In contrast to the above result, which implies $F(L^p, \ell^q_{a,b}) = \Lambda(L^p, \ell^q_{a,b})$ for $a > 0$, it can be shown that for $G = \mathbb{R}^m$ $L^1 \cap \Lambda(L^p, \ell^q_b)$ is a proper subspace of $L^1 \cap F(L^p, \ell^q_b)$ for $1 \leqslant p$, $q \leqslant \infty$, $b > 0$ (except the case $p = q = \infty$).

The connection with the spaces $K^a_{p,q}(\mathbb{R}^m)$ introduced by HERZ [15] is as follows.

Theorem 5.2. *Let* $G = \mathbb{R}^m$, $a > 0$, $1 \leqslant p$, $q \leqslant \infty$ *be given. Then*

A) $F_0(\mathfrak{P}_2, L^p, \ell^q_a) = L^p \cap K^a_{p,q}$;

B) $F_0(\mathfrak{P}_2, L^p, \ell^q_a)$ *is a dense subspace of* $K^a_{p,q}$ *for* $1 \leqslant q < \infty$, *and it coincides with the space of all* $f \in L^p_{\mathrm{loc}}(\mathbb{R}^m)$, *such that* $L_y f \in K^a_{p,q}$ *for all* $y \in \mathbb{R}^m$.

Proof. A) It follows from Theorem 2 of [17] (or from [14], Theorems 5.1 and 3.7.i) that

$$K^a_{p,q}(\mathbb{R}^m) = \{f \mid f \in L^p_{\mathrm{loc}}(\mathbb{R}^m), [\sum_{n=-\infty}^{\infty} 2^{anq} (\int_{S_n} |f(x)|^p \, dx)^{q/p}]^{1/q} < \infty\} \quad (52)$$

with $S_n := \{x \mid 2^{n-1} \leqslant |x| \leqslant 2^n\}$ for $n \in \mathbb{Z}$. Consequently $L^p \cap K^a_{p,q} \subset$ $\subset F_0(L^p, \ell^q_a)$. On the other hand we have for $a > 0$

$$\sum_{n=-\infty}^{0} 2^{anq} (\int_{S_n} |f(x)|^p \, dx)^{q/p} \leqslant K \|f\|_p^p \text{ for } f \in L^p(\mathbb{R}^m).$$

This gives the converse inclusion.

B) Since $F_0(L^p, \ell^q_a)$ is a translation invariant subspace of $L^p(\mathbb{R}^m)$ it will be sufficient to point out that f must belong to $L^p(\mathbb{R}^m)$ if f and $L_y f$ belong to $K^a_{p,q}$ for some y, $|y| > 4$. In fact, equation (52) implies that $f\chi_2 \in L^p$ for any $f \in K^a_{p,q}$, while $L_y(f(\psi_1 + \psi_2)) \leqslant (L_y f)\chi_2$ imlies $f(\psi_1 + \psi_2) \in L^p$. Consequently, $f = f\psi_1 + f\psi_2 + f\chi_2$ belongs to $L^p(\mathbb{R}^m)$. Since the functions with compact support not containing the origin are dense in $K^a_{p,q}$ for $1 \leqslant q < \infty$ (by (52), cf. also [13], § 3 vi), it follows that F is dense in $K^a_{p,q}$.

The above result already suggests that some of the spaces $F(B,X)$ can be interpreted as real interpolation spaces for a pair of weighted L^p-spaces. This is, in fact, true:

Theorem 5.3. *Let* $G = \mathbb{R}^m$ *or* $\mathbb{Z}^m$, $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$, $a > 0$, $0 < \theta < 1$ *be given. Then*

$$F(\mathfrak{P}_2, L^p, \ell^q_a) = (L^p_{a_1}, L^p_{a_2})_{\theta, q} \quad \textit{for } a = (1-\theta) a_1 + \theta a_2.$$

Proof. For $a_1 = 0$ this result follows from Theorem 3.7.i) of [14] (choose $r = 2$). If a_i, $i = 1, 2$ are given, such that $a = (1-\theta) a_1 +$ $+ \theta a_2$, Theorem 5.4.1 of [2] implies $L^p_{a_i} = (L^p, L^p_{a_3})_{\theta_i, p}$ for any a_3, $a_3 > \max(a_1, a_2)$, with $\theta_i = a_i / a_3$. The general result follows now from the reiteration theorem ([2], Theorem 3.11.5).

We use the above Theorem in order to identify the Beurling spaces $b^{ap}_q(\mathbb{Z})$ considered in [1] with spaces of type $F(B,X)$.

Corollary 5.4. *For* $a > 0$ *we have* $b_q^{ap}(\mathbb{Z}) = F(\mathfrak{P}_2, l^q(\mathbb{Z}), \ell_a^p)$.

Proof. This result follows from Theorem 5.3, using the characterization of $b_q^{ap}(\mathbb{Z})$ as $(l^p(\mathbb{Z}), l_k^p(\mathbb{Z})_{\theta,q})$, with $k \in \mathbb{N}$, $0 < \theta = a/k < 1$, and $l_k^p(\mathbb{Z}) := \{(x_n)_{n\in\mathbb{Z}} \mid (x_n(1+|n|)^k)_{n\in\mathbb{Z}} \in l^p(\mathbb{Z})\}$, given in [1], p. 46.

Remark 5.2. It is worth mentioning that the results of § 4 and § 5 imply that the spaces $L^1 \cap L^p \cap K_{p,q}^a(\mathbb{R}^m)$ or $\ell^1 \cap b_q^{ap}(\mathbb{Z})$ are Banach convolution algebras for $a > 0$, $1 \leqslant p$, $q \leqslant \infty$ (the last mentioned result implies for example Theorem 4.4 of [1]). Beyond these special cases the results derived in § 4 give as well convolution results for anisotropic versions of these Beurling-Herz spaces on $G = \mathbb{R}^m$ or $\mathbb{Z}^m$, for spaces with other "weights" (ℓ_b^q instead of ℓ_a^q for example), as well as for spaces on more general groups.

It has to be mentioned that in many cases the "discrete" method of defining $F_0(\mathfrak{P}, B, X)$ can be replaced by a "continuous" method, involving integrals over $\mathbb{R}^+$ instead of finite sums. Since the proof of this result is only a routine matter, we state the result without proof.

Proposition 5.5. *Let* $1 \leqslant p, q < \infty$, *and a strictly positive, increasing function* w *on* $[0, \infty)$ *be given, satisfying* $w(2x) \leqslant K w(x)$ *for all* $x > 0$. *Set* $w_n := w(n)$, *and* $S(t) := \{x \mid x \in \mathbb{R}^m, t \leqslant |x| \leqslant 2t\}$. *Then the expression*

$$\|f\|_p + \Big[\int_1^\infty w^q(\log(t+1)) \Big(\int_{S(t)} |f(x)|^p\, dx\Big)^{q/p} (1/t)\, dt\Big]^{1/q} \tag{53}$$

defines an equivalent norm on $F_0(\mathfrak{P}_2, L^p(\mathbb{R}^m), \ell_w^q)$.

The case $p = \infty$ or $q = \infty$ requires only obvious modifications. In particular, for $X = \ell_{a,b}^\infty$ or $X = c_{a,b}^0$ a useful characterization of $F(L^p(\mathbb{R}^m), X)$ arises. For example

$$\|f\| := \|f\|_p + \sup_{t \geqslant 1} t^a \Big(\int_{t < |x| < 2t} |f(x)|^p\, dx\Big)^{1/p}$$

gives a complete norm for $F_0(\mathfrak{P}_2, L^p, \ell_a^\infty)$, $a \geqslant 0$.

The following proposition gives a short summary of the most important inclusions.

Proposition 5.6. *Let* $G = \mathbb{R}^m$ *or* $\mathbb{Z}^m$, $1 \leqslant p_i, q_i \leqslant \infty$, $a_i, b_i \in \mathbb{R}$, $i = 1, 2$ *be given. Then*

i) $F(\mathfrak{P}, L^p, \ell_{a_1,b_1}^{q_1}) \subset F(\mathfrak{P}, L^p, \ell_{a_2,b_2}^{q_2})$ *for a)* $a_1 > a_2$, *or b)* $a_1 = a_2$, *and* $b_1 > b_2 + \max(0, 1/q_2 - 1/q_1)$;

ii) $F(\mathfrak{P}, L^1 \cap L^{p_1}, \ell_{a,b}^q) \subset F(\mathfrak{P}, L^1 \cap L^{p_2}, \ell_{a,b}^q)$ *for* $p_1 \geqslant p_2$;

iii) $F(\mathfrak{P}_2, L^{p_1}, \ell^{q_1}_{a_1,b_1}) \subset F(\mathfrak{P}_2, L^{p_2}, \ell^{q_2}_{a_2,a_2})$ *for* $p_2 < p_1$, *if* $a_1 > a_2 + m(1/p_2 - 1/p_1)$.

Proof. The assertions are either obvious or a consequence of the generalized Hölder inequality. It is left to the reader to carry out the details.

At the end of this paper we discuss the dependence of $F(\mathfrak{P}, B, X)$ on the partition $\mathfrak{P}$ involved in the construction.

Theorem 5.7. *Let* $1 \leqslant p_i, q_i \leqslant \infty$, $a_i, b_i \geqslant 0$, $i = 1, 2$, *and* $\sigma = (s_1, s_{-1}, \dots, s_{-m})$, $\varrho = (r_1, r_{-1}, \dots, r_{-m})$, $1 < s_i \leqslant \infty$, $1 < r_i \leqslant \infty$ *be given. Then*

$$F(\mathfrak{P}_\sigma, L^{p_1}, \ell^{q_1}_{a_1,b_1}) = F(\mathfrak{P}_\varrho, L^{p_2}, \ell^{q_2}_{a_2,b_2})$$

iff $p_1 = p_2$, $q_1 = q_2$, $b_1 = b_2$, *and* $s_i = \infty$ *iff* $r_i = \infty$, $1 \leqslant |i| \leqslant m$, *and*

i) $\varrho = \sigma^d$ (*i. e.* $r_i = s_i^d$ *for some* $d > 0$, $1 \leqslant |i| \leqslant m$), *and* $a_2 = d a_1$; *or*

ii) $a_1 = a_2 = 0$, $p_1 = p_2 = q_1 = q_2$.

Proof. The proof is divided into several steps. For simplicity denote the two spaces by F^1 and F^2, and write $\mathfrak{P}_\sigma =: (P^1_n)_{n \geqslant 1}$, $\mathfrak{P}_\varrho =: (P^2_n)_{n \geqslant 1}$. At first we prove that the conditions stated are necessary in order to have $F^1 = F^2$.

Step I. First of all, $F^1 = F^2$ obviously implies that, for any given i, s_i is finite iff r_i is finite. Therefore we may assume in the sequel that we have $1 < s_i < \infty$, $1 < r_i < \infty$ for $1 \leqslant |i| \leqslant m$. Since $p_1 \neq p_2$ implies $L^{p_1}(K) \neq L^{p_2}(K)$ for any compact subset $K \subset G$, it is clear that $p_1 = p_2 =: p$ is a necessary condition.

Step II. If $F^1 = F^2$, then the corresponding norms are equivalent; in particular, there exists $C > 0$ such that

$$C^{-1} \|L_y k\|_{F^1} \leqslant \|L_y k\|_{F^2} \leqslant C \|L_y k\|_{F^1} \text{ for all } y \in G, k \in \mathscr{K}(G). \quad (54)$$

Now, for any i, $1 \leqslant |i| \leqslant m$, there exists $d_i > 0$ such that $r_i = s_i^{d_i}$. Define the sequence $(y_n)_{n \geqslant 1}$ by $y_n := (0, \dots, s_i^n, 0, \dots)$. Then we have for $k \in \mathscr{K}(G)$

$$2^{a_1 n} n^{b_1} \sim \|L_{y_n} k\|_{F^1} \sim \|L_{y_n} k\|_{F^2} \sim 2^{a_2 n d_i} (n d_i)^{b_2} \quad (55)$$

This implies $a_1 = d_i a_2$ for all i, and $b_1 = b_2$. Thus, either $a_1 = a_2 = 0$, or $d_i = a_1/a_2 =: d > 0$ for all i, $1 \leqslant |i| \leqslant m$.

Step III. In both cases we obtain by renorming, and choosing a suitable subsequence of $(L_{y_n} k)_{n \geqslant 1}$, a sequence $(g_n)_{n \geqslant 1} \subset \mathscr{K}(G)$ with the properties

i) $\operatorname{supp} g_n \cap P^i_k \neq \emptyset \Rightarrow \operatorname{supp} g_j \cap P^i_k = \emptyset$ for $n \neq j$, and

ii) $1 \leqslant \|g_n\|_{F^i} \leqslant C$ for $i = 1, 2$.
These two conditions imply

$$\|\sum_{n=1}^{k} g_n\|_{F^i} \sim k^{1/q_i} \text{ for } i = 1, 2, \text{ i. e. } q_1 = q_2 =: q .$$

Step IV. It remains to show that also in the case $a_1 = a_2 = 0$ $F^1 = F^2$ implies $\varrho = \sigma^d$, if $p \neq q$. Observing the configuration along the x_i-axis one sees that for $k \in \mathbb{N}$ there exists $k_i \in \mathbb{N}$ such that

$$k/d_i \leqslant k_i \leqslant (k+1)/d_i + 1, \text{ and } P^1_{k_i} \cap P^2_k \supset V \neq \emptyset, \ V \text{ open.} \tag{56}$$

Suppose that not all d_i are equal. We may assume $d_1 > d_2$. Then

$$k_2 - k_1 \geqslant k\,(1/d_2 - 1/d_1) - 2 \geqslant k\,\delta \text{ for } k \geqslant k_0 \tag{57}$$

and some $\delta > 0$. Since P^2_k is connected this implies that there are at least $k_2 - k_1 + 1$ sets P^1_j such that $P^1_j \cap P^2_k \supset y_j + U$ for some open set $U \neq \emptyset$. Note that we have $C_1^{-1} k \leqslant j \leqslant C_1 k$ for some $C_1 > 0$, $k \geqslant 1$. Choose now any $k_1 \in \mathscr{K}(G)$ with $\operatorname{supp} k_1 \subset U$, and define $h_k :=$ $= \sum_{j=1}^{\delta k} L_{y_j} k_1$. Since $\operatorname{supp} h_k \subseteq P^2_k$, but $\operatorname{supp} L_{y_j} k_1 \subseteq P^1_j$, we have

$$\|h_k\|_{F_1} = k^b \|h_k\|_p \sim k^b (\delta k)^{1/p} \|k_1\|_p, \text{ but}$$

$$\|h_k\|_{F_2} = (\sum_{j=1}^{k\delta} j^{bq} \|k_1\|_p^q)^{1/q} \sim k^b (\delta k)^{1/q} \|k_1\| .$$

This yields a contradiction if $p \neq q$.

Step V. We have to show that the conditions stated are also sufficient in order to imply $F_1 = F_2$. Assume therefore $p_1 = p_2 =: p$, $q_1 = q_2 =: q$, $b_1 = b_2 =: b$.

At first let us consider the case $a_1 = a_2 = 0$, $p = q$. Then $F_1 = L^1 \cap L^p_{w_1}$, and $F_2 = L^1 \cap L^p_{w_2}$ for suitably defined weight functions w_1, w_2 (cf. the proof of 4.3.A). It is not difficult to verify that these two weight functions are equivalent; thus $F_1 = F_2$.

We conclude the proof by showing that $\varrho = \sigma^d$, $a_1 = a_2 d$, for some $d > 0$ implies $F_1 = F_2$. For reasons of symmetry it is sufficient to show that $F_2 \subseteq F_1$. The assumptions imply that

$$P^2_k \subseteq \bigcup_{j \in I_k} P^1_j , \ I_k := \{j \mid k/d \leqslant j \leqslant (k+1)/d + 1\} \tag{58}$$

Therefore

$$2^{a_2 k} k^b \|f \psi^2_k\|_p \leqslant \sum_{j=k/d}^{k/d+1+d^{-1}} 2^{a_2 dj} (dj)^b \|f \psi^1_j\|_p . \tag{59}$$

Taking in account that $l^q_{a_2,b}$ is shift-invariant and that any given $j \in \mathbb{N}$ appears for at most $d+1$ different values of k, (56) implies

$$\|f\|^q_{F_2} < C_1 \sum_{k=1}^{\infty} \sum_{j} 2^{a_1 j q} j^{bq} \|f \psi^1_j\|^q_p \leqslant C_2 \|f\|^q_{F_1} \text{ for } f \in F_2 . \tag{60}$$

This completes the proof of theorem 5.7.

Remark 5.3. In view of the above Theorem one may restrict the considerations to $\mathfrak{P}_2$ if one is only interested in the isotropic case. Furthermore, the restriction $s_i \geqslant 2$, $1 \leqslant |i| \leqslant m$, imposed in § 2, can be replaced by the condition $s_i > 1$.

For $G = \mathbb{R}$, $a = 0$, the situation is still simpler.

Corollary 5.8. *On* $G = \mathbb{R}$ *we have* $F(\mathfrak{P}_\sigma, \ell^q_b) = F(\mathfrak{P}_\varrho, \ell^q_b)$ *for* $1 \leqslant q \leqslant \infty$, $b \geqslant 0$.

Proof. The norms of $F(\mathfrak{P}_{s_1})$, $F(\mathfrak{P}_{s_{-1}})$, $F(\mathfrak{P}_{r_1})$, $F(\mathfrak{P}_{r_{-1}})$ are equivalent to each other by 5.7. Write $f = f_1 + f_2$, with $f_1(x) = 0$ for $x < 0$, and $f_2(x) = 0$ for $x > 0$. Then $\|f_1\|_F + \|f_2\|_F$ defines an equivalent norm for any of the spaces $F(\mathfrak{P}, B, X)$ listed above. Combining these arguments one derives the result.

Concluding Remarks. 1) Besides the decomposition method presented in this paper there is at least one different type of decompositions of a locally compact group G into sets of 'equal seize', which can also be used for the construction of a family of Banach convolution algebras. These "weighted algebras of Wiener-type" will be treated in a subsequent paper.

2) Finally, we mention that the essential part of the results given in the present paper can also be derived for a more general class of Banach convolution algebras B of measurable functions satisfying certain smoothness conditions (e. g. spaces of continuous functions, or the spaces $B^s_{p,q}(\mathbb{R}^m)$, for appropriate indices s, p, q; for details see Volume II of Triebel's book: [23], Theorem 2.3.8). In this case one has to replace the characteristic functions of P_n by a suitable sequence of smooth functions $(\psi_n)_{n \geqslant 1}$, satisfying

$$\psi_n(x) = 1 \text{ for } x \in P_n, \ \operatorname{supp} \psi_n \subseteq P_{n-1} \cap P_n \cap P_{n+1}, n \geqslant 1 \tag{61}$$

such that for some $C > 0$

$$\|f \psi_n\|_B \leqslant C \|f\|_B \text{ for any } f \in B . \tag{62}$$

With these assumptions Corollary 3.3 can be proved in this more general context for any $X \subseteq \ell^1$ satisfying $X 8$). It is left to the interested reader to carry out the necessary modifications in the proof. The most interesting Banach convolution algebras arising in this way are certain spaces of Kudrjavcev-type as treated by H. TRIEBEL [24].

3) For the sake of completeness let us mention that there are also results concerning the behaviour of the spaces $F(B,X)$ under the canonical mapping $T_{H,q}: L^1(G) \to L^1(G/H)$ (see [10], in particular Theorem 3.4).

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