Contents

Robustness of Sampling in Sobolev Algebras

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ABSTRACT
It is the purpose of this paper to feature the link between the theory of minimal norm interpolation over lattices by elements from *Sobolev algebras* $\mathcal{H}_s(\mathbb{R}^d)$ with what is known as the theory of spline-type (or principal shift invariant) spaces. As an extremely useful tool allowing to establish various kinds of robustness results we will present the so-called Wiener Amalgam spaces $W(B, \ell^p)$, with general (smooth) local components and global ℓ^p behavior. For this reason a summary of their most important properties, including convolution relations and the behavior under the Fourier transform, is presented.
The discussion of projection and minimal norm interpolation operators is not restricted to the pure Hilbert space setting for which these concepts were developed originally. Among others we show L^p-stability of the (for $p = 2$ orthogonal) projection from L^p onto the corresponding spline-type spaces with ℓ^p-coefficients.
As a main result (which can be formulated in several different concrete ways) we show that for $s > d/2$ the mapping $f \mapsto Q_{s,a}(f)$, from f to the minimal norm interpolation in $\mathcal{H}_s$ over the lattice $a\mathbb{Z}^d, a > 0$, depends continuously on the input parameters (s, a). It also extends to certain fractional L^p-Sobolev spaces consisting of continuous functions in L^p. In this modified setting the outcome of this procedure depends continuously on (s, a) in the L^p-sense. Moreover, the mapping is robust against small jitter errors. Wiener amalgam spaces turn out to be very useful, both for a precise formulation and in the proofs of such results.

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1 Introduction

Sobolev spaces over $\mathbb{R}^d$, derived by differentiability conditions in the L^2-sense, play an important role in many areas of analysis. Using the Fourier transform they can be defined for fractional (positive and even negative) orders $s \in \mathbb{R}$, and will be denoted by $\mathcal{H}_s(\mathbb{R}^d)$. For $s > d/2$ they are embedded into the space of continuous and bounded functions over $\mathbb{R}^d$, due to Sobolev's embedding theorem; and hence they are reproducing kernel Hilbert spaces. Since ordinary translation operators act isometrically on $\mathcal{H}_s(\mathbb{R}^d)$ there is a single function $\varphi_s \in \mathcal{H}_s(\mathbb{R}^d)$ which completely describes the reproducing kernel through the identity

$$f(x) = \langle f, T_x \varphi_s \rangle_{\mathcal{H}_s} \quad \forall f \in \mathcal{H}_s,\ x \in \mathbb{R}^d \,, \tag{1.1}$$

where T_x denotes the translation by x. The spaces $\mathcal{H}_s(\mathbb{R}^d)$ are even Banach algebras (we name them Sobolev algebras) with respect to pointwise multiplication in this case.

Given Sobolev's embedding it makes sense to ask the question whether for a given lattice $a\mathbb{Z}^d \lhd \mathbb{R}^d$ and for an arbitrary function $f \in \mathcal{H}_s(\mathbb{R}^d)$ the samples $(f(ak))_{k\in\mathbb{Z}^d}$ are in $\ell^2(\mathbb{Z}^d)$, and, on the other hand, whether for arbitrary sequences $\left(d_k\right)_{k\in\mathbb{Z}^d}$ it is always possible to interpolate exactly those data by some function f, which thus satisfies $f(ak) = d_k$. The answers to both questions turn out to be affirmative; and there are indeed infinitely many different such functions in the latter case. In order to make the problem unique one may require therefore to choose the minimal norm interpolation for the given data sequence. If we consider the mapping from $\mathcal{H}_s(\mathbb{R}^d)$ into itself which maps f to the minimal norm interpolator for the sampling sequence $(f(ak))_{k\in\mathbb{Z}^d}$ this turns out to be a well defined and bounded linear operator on $\mathcal{H}_s(\mathbb{R}^d)$, which we denote by $Q_{s,a}$. The understanding of the robustness properties of this family of operators is at the center of this paper.

It turns out that a detailed analysis requires the description of certain spline type spaces. Indeed, as one of the remarkable facts, the family $(T_{ak}\varphi_s)_{k\in\mathbb{Z}^d}$ is a Riesz basis with respect to both the standard L^2 and the $\mathcal{H}_s$ scalar product. The properties of the corresponding biorthogonal bases play a crucial role for the analysis, implying, for example, continuity with respect to L^p-norms, or robustness against jitter error for the sampling process. Extended use of so-called Wiener amalgam spaces is made, and therefore a separate section describes this useful family of Banach spaces.

Using them we can also show that the result of the minimal norm interpolation operator depends not only continuously on the sampled function $f \in \mathcal{H}_s(\mathbb{R}^d)$, but also on the smoothness parameter $s > d/2$ and the lattice constant $a > 0$ for the sampling lattice.

We start this paper by giving a short survey of the relevant subclass of Wiener amalgam spaces, as well as a summary of basic facts about Sobolev algebras.

2 Wiener Amalgam Spaces $W(B, \ell^p)$

Although various kinds of Wiener amalgam spaces have been used in connection with sampling theory, in particular in connection with qualitative error analysis, there are few reliable references for this useful family of function spaces. In the present section we want to give a short self-contained summary of those facts about Wiener amalgam spaces which are of interest in connection with respect to the analysis of minimal norm interpolation in Sobolev algebras.

In order to avoid unnecessary complication in the presentation we do not introduce Wiener amalgams with general weighted spaces, but emphasize that amalgam spaces (with global ℓ^p components) already allow to use a variety of local norms. There are quite strong local norms, such as that of bounded measures (allowing to handle sums of point measures) as well as norms rather sensitive norms, such as local $Lip\,(\alpha)$-conditions, describing local smoothness.

The main idea of Wiener Amalgam spaces is to control the *global behavior* (in terms of global ℓ^p summability) of the *local properties* of a measure or a function f. This local property, in turn, is measured by some local B-norm. Typical examples will be given in a moment. We start by recalling the relevant definitions.

First we describe a sufficiently large class of norms which can be used to measure the local quality of a function.

Definition 2.1. A Banach space $(B, \|\ \|_B)$ of tempered distributions is called *localizable*, if the following three properties are verified:

(a) $(B, \|\ \|_B)$ is a *Banach space of tempered distributions*, continuously embedded into $\mathcal{S}'(\mathbb{R}^d)$, endowed with the weak*-topology;

(b) $(B, \|\ \|_B)$ is isometrically translation invariant, i.e.,

$$\|T_z f\|_B \;=\; \|f\|_B \quad \forall\, z \in \mathbb{R}^d,\ f \in B.$$

(c) $\mathcal{D} \cdot B \subseteq B$, that is, $\mathcal{D} = C_c^\infty$ operates on B via pointwise multiplication.

It is essentially a consequence of the closed graph theorem (for a fixed $\varphi \in \mathcal{D}$) and property (c) above that for each $\varphi \in \mathcal{D}(\mathbb{R}^d)$ there exists $C = C_\varphi > 0$ such that

$$\|f \cdot T_x\varphi\|_B \le C_\varphi\ \|f\|_B \ \ \forall\, x \in \mathbb{R}^d. \tag{2.2}$$

For $(B, \|\ \|_B)$ as above the space B_{loc} of tempered distributions belonging locally to B can be defined as

$$B_{loc} \;=\; \{\ \sigma \ \big|\ \sigma \in \mathcal{S}'(\mathbb{R}^d)\ ,\ \varphi \cdot \sigma \in B \ \text{ for all } \ \varphi \in \mathcal{D}\ \}\,.$$

Definition 2.2. Assume that $(B, \| \ \|_B)$ is a localizable Banach space of distributions. Then for $f \in B_{loc}$ the *local control function* with respect to the B-norm, using some non-zero *window* $\varphi \in \mathcal{D}$ is given as

$$C(f, B, \varphi) : x \mapsto \|f \cdot T_x \varphi\|_B. \tag{2.3}$$

Using these control functions we are able to generate different Banach spaces of distributions by the following continuous and selective method:

Definition 2.3. Let a localizable Banach space $(B, \| \ \|_B)$ and $p \in [1, \infty]$ be given. Then the *Wiener Amalgam space* $W(B, L^p)$ is given by

$$W(B, L^p)(\mathbb{R}^d) \ = \ \{ \ f \in B_{loc} \, \big| \, C(f, B, \varphi) \in L^p(\mathbb{R}^d) \} \, . \tag{2.4}$$

with the associated norm

$$\|f\|_{W(B,L^p)} := \|C(f, B, \varphi)\|_p \, . \tag{2.5}$$

One of the first things to be verified is that the above definition is independent of the particular window φ. Indeed, any non-zero test function can be used as a window, defining the same space and an equivalent norm. At this point essentially two properties of the global component $L^p(\mathbb{R}^d)$ are used: it is invariant under translations, and it is a *solid* space, i.e., $f \in L^p(\mathbb{R}^d)$ and $|g(x)| \leq |f(x)|$ for some measurable function g imply $g \in L^p(\mathbb{R}^d)$ and $\|g\|_p \leq \|f\|_p$.

Typical examples of Wiener Amalgam spaces are the classical amalgam spaces $W(L^q, L^p)$ and $W(C, L^q)$ for $1 \leq p, q \leq \infty$. We will see many more amalgam spaces throughout this paper. As a first observation let us formulate the following statement.

Lemma 2.1. *$W(B, L^p)(\mathbb{R}^d)$ is a Banach space with respect to its natural norm given by (2.5), continuously embedded into the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ (with its weak*-topology).*

Proof. We only give a hint concering completeness: One just has to verify that absolutely convergent series in $W(B, \ell^p)$ are also distributionally convergent, with a limit in $W(B, \ell^p)$ satisfying the obvious norm estimate. □

There is an equivalent characterization of Wiener Amalgam spaces replacing the continuous control function by a discretized version. The ideal case of a discrete control function is based on regular partitions of unity.

Definition 2.4. A family $\Psi = \{T_{\alpha k} \psi\}_{k \in \mathbb{Z}^d}$ is a *bounded uniform partition of unity* (BUPU) of translates along $\alpha \mathbb{Z}^d$, for some $\alpha > 0$, if

(a) ψ as a bounded function with compact support,

(b) $\sum_{k \in \mathbb{Z}^d} T_{\alpha k} \psi(x) = 1 \quad \forall x \in \mathbb{R}^d$.

We say that the BUPU has size δ if $\text{supp}(\psi) \subseteq B_\delta(0) = \{x \in \mathbb{R}^d \mid \|x\| \leq \delta\}$. The BUPU is called *smooth* if $\psi \in \mathcal{D}(\mathbb{R}^d)$, i.e., if ψ is infinitely differentiable.

The following theorem describes the equivalence of the (more elegant) continuous description with the (more practical) discrete description.

Theorem 2.1. *Let $\Psi = \{T_{ak}\psi\}_{k\in\mathbb{Z}^d}$ be a smooth BUPU on $\mathbb{R}^d$. Then*

$$W(B, L^p) = W(B, \ell^p) := \left\{ f \in B_{loc} \;\middle|\; k \mapsto \|f \cdot T_{\alpha k}\psi\|_B \in \ell^p(\mathbb{Z}^d) \right\}. \quad (2.6)$$

Moreover, the $\ell^p(\mathbb{Z}^d)$-norm of $k \mapsto (C(f, B, \psi)(\alpha k))_{k\in\mathbb{Z}^d}$, i.e., for $p < \infty$,

$$\Big(\sum_{k\in\mathbb{Z}^d} \|f \cdot T_{\alpha k}\psi\|_B^p \Big)^{1/p}, \quad (2.7)$$

defines an equivalent norm on $W(B, \ell^p)$

As a matter of fact there are degrees of freedom in both versions. In the first case the “window” φ may be chosen arbitrarily. In most cases functions with non-compact support, such as Schwartz functions φ, or from spaces of the form $W(C^{(k)}, \ell^1)$ can be used. On the other hand we have many possible BUPUs in the case of the discrete description, and free choice of α. It will be convenient to work with $\alpha = 1$, for example, and in any case it is recommended to think of one fixed norm from the large equivalence class of norms for a given space. It is in this sense that we understand absolute constants $C > 0$ to be interpreted.

We also observe that in both cases property (2.2) is crucial. Therefore the smoothness condition can be dropped in both versions as long as the local component is just an L^p space, and bounded, measurable functions φ resp. ψ will do. In particular, we may use the “trivial” BUPU obtained by starting from the indicator function of the unit cube $Q_0 = [0, 1)^d \subseteq \mathbb{R}^d$ as long as $B = L^p(\mathbb{R}^d)$ or $B = M(\mathbb{R}^d)$, the space of bounded Radon measures on $\mathbb{R}^d$, i.e., the dual to $C_0(\mathbb{R}^d)$, with sup--norm. Thus for the Wiener Amalgam space $W(L^\infty, \ell^p)$, $1 \leq p \leq \infty$, a convenient choice of a natural norm is

$$\|f\|_{W(L^\infty,\ell^p)} = \Big(\sum_{k\in\mathbb{Z}^d} \sup\nolimits_{z\in k+Q_0} |f(z)| \Big)^{1/p}. \quad (2.8)$$

The space $W(C_0, \ell^p)$ is a closed subspace of $W(L^\infty, \ell^p)$, consisting exactly of the continuous functions in $W(L^\infty, \ell^p)$ for the case $p < \infty$.

The following continuous embedding relations follow easily from the discrete characterization of Wiener amalgam spaces.

1. $W(B, \ell^p) \hookrightarrow W(B, \ell^r)$ if and only if $p \leq r$.
2. If $B^1_{loc} \hookrightarrow B^2_{loc}$ then $W(B^1, \ell^p) \hookrightarrow W(B^2, \ell^p)$ for $1 \leq p \leq \infty$.
3. $W(B, \ell^1) \hookrightarrow B \hookrightarrow W(B, \ell^\infty)$.

4. $L^p = W(L^p, \ell^p) \hookrightarrow W(L^1, \ell^p)$.
5. $W(C_0, \ell^p) \hookrightarrow L^p \cap C_o$.

Note that properties (1) and (2) are true even for more general amalgam spaces, while (3) follows from the fact that $f = \sum_{k\in\mathbb{Z}^d} f\psi_k$ and the validity of (2.2), which in turn was deduced from the isometric translation invariance of $(B, \| \, \|_B)$.

One of the most useful "true" Wiener Amalgam spaces is the space

$$S_o = W(\mathcal{F}L^1, \ell^1),$$

also known as *Feichtinger's Algebra*. Nowadays it is often described as the modulation space $M^1(\mathbb{R}^d)$ or $M_0^{1,1}(\mathbb{R}^d)$ in the literature. We refer to [40], [53] for general references concerning $S_o(\mathbb{R}^d)$, [34] for descriptions of the relevance of this space for Gabor analysis, and [25] for an elementary description of that space.

We also deal with the Wiener Amalgam space $W(M, \ell^p)$, $1 \le p \le \infty$. For them the "trivial" BUPUs using the indicator functions of the standard cubes are still admissible, so that (we write down only the case $p < \infty$) one has as one of the equivalent and very convenient norms for those spaces the following expression:

$$\|\mu\|_{W(M,\ell^p)} = \Big(\sum_{k\in\mathbb{Z}^d} |\mu(k + Q_0)| \Big)^{1/p}. \tag{2.9}$$

Using amalgams we can express more precisely the fact that both the convolution operation and the Fourier transform preserve, resp., switch local and global properties, i.e., allow – at least at a formal level – to handle them in a *coordinate-wise* manner.

Let us first introduce the following definition, see, for instance, [45].

Definition 2.5. A Banach space $(B, \| \, \|_B)$ of locally integrable functions is called a *homogeneous Banach space* on $\mathbb{R}^d$ if it satisfies

1. $\|T_x f\|_B = \|f\|_B \quad \forall f \in B, x \in \mathbb{R}^d$;
2. $\|T_x f - f\|_B \to 0$ for $x \to 0, \quad \forall f \in B$.

It is easy to show that any localizable Banach space B of Radon measures for which $\mathcal{D}(\mathbb{R}^d)$ is dense in $(B, \| \, \|_B)$, is a homogeneous Banach space, e.g., $B = L^p(\mathbb{R}^d)$ for $1 \le p < \infty$.

It is a well known consequence of Definition 2.5 that the convolution of bounded measures $\mu \in M(\mathbb{R}^d)$ with elements from a homogeneous Banach space exists as a vector-valued integral and satisfies

$$\|\mu * f\|_B \le \|\mu\|_M \|f\|_B \quad \forall \mu \in M(\mathbb{R}^d), f \in B. \tag{2.10}$$

This observation motivates the following theorem which is a special version of Young's inequality for Wiener Amalgam spaces.

Theorem 2.2. [23] *Let B be a homogeneous Banach space. Then, for $p, q, r \geq 1$ with $1/p + 1/q = 1 + 1/r$, we have*

$$W(M, \ell^p) * W(B, \ell^q) \subset W(B, \ell^r) \,. \tag{2.11}$$

Good illustrations for this convolution theorem are the following examples:

$$W(M, \ell^p) * W(C_0, \ell^1) \quad \subset \quad W(C_0, \ell^p) \tag{2.12}$$

$$W(M, \ell^p) * W(\mathcal{F}L^1, \ell^1) \quad \subset \quad W(\mathcal{F}L^1, \ell^p) \tag{2.13}$$

$$W(M, \ell^p) * W(C^{(k)}, \ell^1) \quad \subset \quad W(C^{(k)}, \ell^p) \,, \tag{2.14}$$

where $\mathcal{F}$ is the Fourier transform operator (with any normalization). Since $L^2 * L^2 = \mathcal{F}L^1$, the same arguments as in Theorem 2.11 yield

$$W(L^2, \ell^1) * W(L^2, \ell^1) \; \subset \; W(\mathcal{F}L^1, \ell^1) \; \hookrightarrow W(C_0, \ell^1). \tag{2.15}$$

Besides the convolution relations which can be interpreted as working in the local and global component independently, a similar statement can be made for pointwise relations. Instead of giving a formal general statement, let us give a lemma which covers concrete cases of interest.

Lemma 2.2.

$$W(C_0, \ell^p) \cdot W(M, \ell^\infty) \subseteq W(M, \ell^p); \tag{2.16}$$

$$W(C_0, \ell^2) \cdot W(M, \ell^2) \subseteq W(M, \ell^1)(\mathbb{R}^d) = M(\mathbb{R}^d). \tag{2.17}$$

Next, we describe how the Fourier transform $\mathcal{F}$ acts on amalgams.

Theorem 2.3. [26] *For $1 \leq q \leq p \leq \infty$ we have*

$$\mathcal{F}W(\mathcal{F}L^p, \ell^q) \hookrightarrow W(\mathcal{F}L^q, \ell^p) \,. \tag{2.18}$$

When combining this result with the classical Hausdorff-Young theorem we immediately obtain for $1 \leq p, q \leq 2$ that

$$\mathcal{F}W(L^p, \ell^q) \hookrightarrow W(L^{q'}, \ell^{p'}) \tag{2.19}$$

where $1/p + 1/p' = 1/q + 1/q' = 1$. In particular, we have

$$\mathcal{F}[W(L^2, \ell^1)] \hookrightarrow W(\mathcal{F}L^1, \ell^2) \; \hookrightarrow W(C_0, \ell^2). \tag{2.20}$$

For the discussions later on in this paper, and in order to derive basic results related to sampling theory, the following lemma will turn out to be of high relevance:

Lemma 2.3. *The following estimates hold true.*

(1) *For any* $a_0 > 0$ *there exists* $C(a_0) > 0$ *such that*

$$\Big(\sum_{k\in\mathbb{Z}^d} |f(ak)|^p\Big)^{1/p} \;\le\; C(a_0)\, \|f\|_{W(C_0,\ell^p)} \quad \forall f \in W(C_0,\ell^p). \tag{2.21}$$

for all $a \ge a_0$ *and* $1 \le p \le \infty$*, with the usual modification for* $p = \infty$*.*

(2) *For* $g \in W(L^\infty, \ell^1)$*, there exists* $C = C(g)$ *such that for all* $p \in [1,\infty]$

$$\big\| (\langle f, T_k g\rangle_{L^2})_{k\in\mathbb{Z}^d} \big\|_{\ell^p(\mathbb{Z}^d)} \;\le\; C\, \|f\|_{L^p} \qquad \forall f \in L^p(\mathbb{R}^d)\,. \tag{2.22}$$

(3) *For* $a_0 > 0$ *there exists a constant* $C = C(a_0) > 0$ *such that for any* $\varphi \in W(L^\infty, \ell^1)$*, any* $a \ge a_0$*, and* $1 \le p \le \infty$*,*

$$\|\sum_{k\in\mathbb{Z}^d} c_k T_{ak}\varphi\|_{W(L^\infty,\ell^p)} \;\le\; C\, \|\mathbf{c}\|_{\ell^p} \|\varphi\|_{W(L^\infty,\ell^1)} \quad \forall \mathbf{c} \in \ell^p. \tag{2.23}$$

Proof. For (1) we assume that the norm on $W(C_0,\ell^p)$ is given by (2.8). For any $k \in \mathbb{Z}^d$ there are at most $(1/a+1)^d$ points of $a\mathbb{Z}^d$ in $k+Q_0$. Hence $a \ge a_0$ yields

$$\Big(\sum_{k\in\mathbb{Z}^d} |f(ak)|^p\Big)^{1/p} \;\le\; \big(\frac{1}{a_0}+1\big)^d\, \|f\|_{W(C_0,\ell^p)} \qquad \forall f \in L^p.$$

The second part is Lemma 2.10 in [4], but it can also be derived directly from equation (2.12), followed by (2.21). The argument given for (1) combined with the proof of Lemma 2.9 in [4] gives (2.23), but it can also be seen as a consequence of the L^∞-variant of equation (2.12) by interpreting the left hand side as convolution of $\mu = \sum_{k\in\mathbb{Z}^d} c_k T_{ak}\delta_k \in W(M,\ell^p)$ with $\varphi \in W(L^\infty,\ell^1)$. □

Another important property of most Wiener Amalgam spaces is their invariance under dilations.

Lemma 2.4. *Assume that* $(B, \|\ \|_B)$ *is invariant under arbitrary dilations, i.e., for any* $a > 0$ *the mapping* $f \mapsto D_a f$*, with*

$$D_a f(x) = f(x/a) \quad \forall x \in \mathbb{R}^d, \tag{2.24}$$

maps B *boundedly into itself. Then the same is true for* $W(B,\ell^p)$*.*

Corollary 2.1. *Each of the spaces* $B = W(L^r,\ell^p)$ *or* $B = W(C_0,\ell^p)$ *with* $r,p \in [1,\infty]$ *is dilation invariant. Moreover, for* $r,p \in [1,\infty)$ *the mapping* $a \mapsto D_a f$ *is continuous, for each* $f \in B$*.*

Corollary 2.2. *The mapping* $(f,a) \mapsto \big(f(ak)\big)_{k\in\mathbb{Z}^d}$ *is continuous from* $W(C_0,\ell^1)(\mathbb{R}^d) \times (0,\infty)$*, and hence also from* $S_o(\mathbb{R}^d) \times (0,\infty)$ *into* $\ell^1(\mathbb{Z}^d)$*.*

Proof. Since $S_o(\mathbb{R}^d) = W(\mathcal{F}L^1,\ell^1) \hookrightarrow W(C_0,\ell^1)$ the result is a simple consequence of Lemma 2.3.1, for $p=1$ and Corollary 2.1. □

3 Generalities on Spline-type Spaces

In this section we will summarize a few basic facts about spline-type spaces. Although most of the results are valid for general locally compact groups we restrict ourselves to $\mathbb{R}^d$. We denote by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\xi} dx, \qquad \xi \in \mathbb{R}^d \tag{3.25}$$

the *Fourier transform* of an absolutely integrable function f, with the understanding that by Plancherel's theorem it extends to a unitary isomorphism of the Hilbert space $L^2(\mathbb{R}^d)$ into itself.

Definition 3.1. A sequence (h_k) in a separable Hilbert space $\mathcal{H}$ is a *Riesz basis* for its closed linear span if for two constants $0 < D_1 \leq D_2 < \infty$,

$$D_1 \|c\|_{\ell^2}^2 \leq \Big\| \sum_k c_k h_k \Big\|_{\mathcal{H}}^2 \leq D_2 \|c\|_{\ell^2}^2, \qquad \forall \mathbf{c} \in \ell^2 . \tag{3.26}$$

The concept of Riesz bases is nicely described in [66], see also [18]. In particular, a sequence (h_k) is a *Riesz basis* if and only if the corresponding *Gram matrix*, whose entries are the scalar products $(\langle h_k, h_{k'} \rangle)_{k,k'}$ is invertible on the corresponding ℓ^2-space.

Definition 3.2. Given $a > 0$, and a function $\varphi \in L^2(\mathbb{R}^d)$ we call $V_{\varphi,a}$ the *spline type space* generated by the pair (φ, a) if the family $\big(T_{ak}\varphi\big)_{k\in\mathbb{Z}^d}$ is a Riesz basis for $V_{\varphi,a}$, its closed linear span in L^2.

Clearly, $V_{\varphi,a}$ is translation invariant with respect to the shifts from the discrete subgroup $a\mathbb{Z}^d \lhd \mathbb{R}^d$. Therefore such spaces, generated from a single *atom*, are often called principal shift invariant spaces.

Due to the assumed Riesz basis property of $\big(T_{ak}\varphi\big)_{k\in\mathbb{Z}^d}$, we have

$$V_{\varphi,a} = \Big\{ \sum_{k\in\mathbb{Z}^d} c_k T_{ak}\varphi \ \Big|\ \mathbf{c} \in \ell^2(\mathbb{Z}^d) \Big\} .$$

In the present case it is easily verified that the corresponding Gram matrix is a circulant matrix over $\mathbb{Z}^d$, via the simple calculation

$$\langle T_{ak}\varphi, T_{ak'}\varphi \rangle = \varphi * \varphi^*(a(k'-k)) \tag{3.27}$$

where $\varphi^*(x) = \overline{\varphi(-x)}$. Using a standard Fourier argument, it is therefore clear that the invertibility question can be settled positively by verifying that the function

$$\Phi^{(a)}(\xi) = \sum_{k\in\mathbb{Z}^d} |\hat{\varphi}(\xi - a^{-1}k)|^2 \tag{3.28}$$

is essentially bounded and essentially bounded away from zero, i.e., there exist positive constants D_1, D_2, such that

$$D_1 \ \leq\ \Phi^{(a)}(\xi) \ \leq\ D_2 \qquad \text{a. e.} \quad \xi \in \mathbb{R}^d . \tag{3.29}$$

Note that $a^{-1}\mathbb{Z}^d$ is the orthogonal subgroup to $a\mathbb{Z}^d$.

Since the usual Sobolev spaces are invariant with respect to dilations we may - without loss of generality - restrict our discussion of the general case to the special case $a = 1$, and suppress the index a for most of the section. We will re-introduce this parameter when we discuss the continuous dependence on the lattice constant a in Section 7.

It is well known that the dual basis (biorthogonal system) within the span of a Riesz basis can be obtained through suitable linear combinations of the elements of the original basis, the coefficients being taken from the inverse Gram matrix. For the case $\left(T_k\varphi\right)_{k\in\mathbb{Z}^d}$ it is further well known (see [4] for generalities and a large number of references, or [50] for an early reference), that the dual basis is given by the collection of $\mathbb{Z}^d$-translates of the function φ^d defined by

$$\widehat{\varphi^d}(\xi) \;=\; \hat{\varphi}(\xi)/\Phi(\xi), \quad \xi \in \mathbb{R}^d. \tag{3.30}$$

Then, every $f \in V_\varphi$ enjoys the representations

$$f = \sum_{k\in\mathbb{Z}^d} \langle f, T_k\varphi^d\rangle T_k\varphi = \sum_{k\in\mathbb{Z}^d} \langle f, T_k\varphi\rangle T_k\varphi^d\,. \tag{3.31}$$

Due to the symmetric role of the atom φ and the dual atom φ^d in this formula, the overall quality of this expansion depends on the properties of both φ and φ^d, and therefore it is of great interest to investigate the question to which extent good properties of φ (such as decay conditions) are automatically shared by φ^d. The following theorem improves on a corresponding statement in [30], where the case $B = S_0$ has been discussed. Using weights, much stronger results can be given which, however, will not be needed in the present context.

Theorem 3.1. *Let $(B, \|\ \|_B)$ be some isometrically translation invariant Banach space, continuously embedded into $W(L^2, \ell^1)$. Assume that $\varphi \in B$ generates a Riesz basis for its closed linear span in $L^2(\mathbb{R}^d)$. Then the dual atom φ^d also belongs to the same space B.*

Proof. The Hausdorff Young principle for Wiener Amalgam spaces as described in the previous section implies that if $\varphi \in W(L^2, \ell^1)$, then $\hat{\varphi} \in W(\mathcal{F}L^1, \ell^2)$ which in turn gives $|\hat{\varphi}|^2 \in W(\mathcal{F}L^1, \ell^1)$. It follows that the periodic function Φ has an absolutely convergent Fourier series expansion. By Wiener's inversion theorem, see [53], also $1/\Phi$ has an absolutely convergent Fourier transform. This implies that for some ℓ^1-sequence c we have

$$\varphi^d = \sum_{k\in\mathbb{Z}^d} c_k T_k\varphi \tag{3.32}$$

which in turn implies that $\varphi^d \in B$ whenever $\varphi \in B$, due to the assumed isometric translation invariance of B. □

Example 3.1. Besides $B = W(L^2, \ell^1)$ and $S_o(\mathbb{R}^d)$, other typical examples of spaces B as in the above theorem are the spaces $W(C_0, \ell^1)$ and $W(C^{(k)}, \ell^1)$, for $k \geq 1$.

Remark 3.1. We emphasize that in contrast to the classical spline case on $\mathbb{R}$, the atom φ generating a Riesz basis in L^2 for the spline-type space V_φ is, in general, not of compact support and convergence of sums of the form $\sum_{k\in\mathbb{Z}^d} c_k T_k \hat{\varphi}$ in $W(C_0, \ell^2)$, for coefficients $c \in \ell^2(\mathbb{Z}^d)$, has to be controlled using amalgam estimates such as (2.12).

Another observation, bringing some interesting Wiener amalgam spaces into the game, is contained in the following statement.

Theorem 3.2. *Assume that $\big(T_k\varphi\big)_{k\in\mathbb{Z}^d}$ with $\varphi \in W(C^{(k)}, \ell^1)$ is a Riesz basis for V_φ in $L^2(\mathbb{R}^d)$. Then, on V_φ, the norms of $L^2(\mathbb{R}^d)$ and $W(C^{(l)}, \ell^2)$ are equivalent, for any $l \in \{0, 1, 2, \ldots, k\}$. In other words, V_φ sits continuously and closed in $W(C^{(l)}, \ell^2)$ for any $l \in \{0, 1, 2, \ldots, k\}$.*

Proof. Since $C^{(l)}_{loc} \hookrightarrow L^2_{loc}$ for any nonnegative integer l, the norm estimate

$$\|f\|_{L^2} \;\leq\; C_l \|f\|_{W(C^{(l)},\ell^2)} \qquad \forall f \in W(C^{(l)}, \ell^2) \tag{3.33}$$

with some $C_l > 0$ is always satisfied. Due to the properties of a Riesz bases, $f \in V_\varphi$ if and only if

$$f = \sum_{k\in\mathbb{Z}^d} c_k T_k \varphi = \big(\sum_{k\in\mathbb{Z}^d} c_k \delta_k\big) * \varphi \tag{3.34}$$

for the sequence $\mathbf{c} = \big(c_k\big)_{k\in\mathbb{Z}^d} = (\langle f, T_k\varphi^d\rangle)_{k\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$ and the ℓ^2-norm of the coefficient sequence c is equivalent to the L^2-norm of f.

The right-hand side of (3.34) is convergent in the $W(C^{(k)}, \ell^2)$-sense, which follows from the convolution and embedding relations for $0 \leq l \leq k$:

$$W(M, \ell^2) * W(C^{(k)}, \ell^1) \subseteq W(C^{(k)}, \ell^2) \hookrightarrow W(C^{(l)}, \ell^2) \hookrightarrow L^2(\mathbb{R}^d)\,. \tag{3.35}$$

The last relation also implies the estimate

$$\begin{aligned} \|f\|_{W(C^{(l)},\ell^2)} &= \|\mu * \varphi\|_{W(C^{(l)},\ell^2)} \;\leq\; C(l) \|\mu\|_{W(M,\ell^2)} \|\varphi\|_{W(C^{(k)},\ell^1)} \\ &= C(l,\varphi)\; \|c\|_{\ell^2}, \qquad 0 \leq l \leq k \end{aligned}$$

where $\mu = \sum_{k\in\mathbb{Z}^d} c_k \delta_k$ and $f = \mu * \varphi$. The proof is complete. □

Remark 3.2. The situation above indicates that it will be possible to combine sampling of derivatives with amalgam techniques.

4 Sobolev Spaces $\mathcal{H}_s^2$

Sobolev Spaces in the L^2-context are Hilbert spaces that can be identified with the classes of those tempered distributions whose Fourier transform belongs to some weighted L^2-space. This concept allows for the extension to fractional Sobolev spaces defined for any $s \in \mathbb{R}$ by

$$\mathcal{H}_s^2(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \;\middle|\; \hat{f} w_s \in L^2(\mathbb{R}^d) \right\} \quad \text{with} \quad w_s(\xi) = (1 + |\xi|^2)^{s/2}.$$

$\mathcal{H}_s^2(\mathbb{R}^d)$ is a Hilbert space with respect to the natural scalar product

$$\langle f, g \rangle_s = \langle \hat{f} w_s, \hat{g} w_s \rangle_{L^2}. \tag{4.36}$$

We recall the Sobolev Lemma, cf. [54], which says that

$$\mathcal{H}_s^2(\mathbb{R}^d) \hookrightarrow \mathcal{C}^{(k)}(\mathbb{R}^d) \qquad \text{for} \quad s > d/2 + k\,. \tag{4.37}$$

Within the context of Wiener Amalgam spaces the case $k = 0$ can actually be derived quite easily using amalgam methods. Observe first that

$$L^2_{w_s}(\mathbb{R}^d) = W(\mathcal{F}L^2, \ell^2_{w_s}) \hookrightarrow W(\mathcal{F}L^2, \ell^1),$$

because $\ell^2_{w_s}(\mathbb{Z}^d) \hookrightarrow \ell^1(\mathbb{Z}^d)$ due to the Cauchy Schwartz inequality since $w_{-s} \in \ell^2(\mathbb{Z}^d)$ if and only $s > d/2$. Then (2.20) implies

$$\mathcal{H}_s^2(\mathbb{R}^d) \hookrightarrow W(\mathcal{F}L^1, \ell^2) \hookrightarrow W(C_0, \ell^2) \subseteq L^2(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)\,.$$

From now on we always assume $s > d/2$. In this case $\mathcal{H}_s = \mathcal{H}_s^2(\mathbb{R}^d)$ is an isometrically translation invariant reproducing kernel Hilbert space whose kernel is given by

$$\Phi_s(x, y) = \varphi_s(x - y) = T_y(\mathcal{F}^{-1} w_{-2s})(x),$$

where the inverse Fourier transform is in the classical sense because w_{-2s} is integrable whenever $s > d/2$. Obviously, φ_s is an even function, and

$$f(y) = \langle f, \Phi_s(\cdot, y) \rangle_s = \langle f, T_y \varphi_s \rangle_s \qquad \forall f \in \mathcal{H}_s(\mathbb{R}^d),\; y \in \mathbb{R}^d. \tag{4.38}$$

For $s \geq 0$, the weight functions w_s are *weakly subadditive*, that is,

$$w_s(\xi + \omega) \leq C_s\big(w_s(\xi) + w_s(\omega)\big) \quad \forall\, \omega, \xi \in \mathbb{R}^d, \tag{4.39}$$

for some $C_s > 0$, which in turn implies the pointwise estimate for all ξ:

$$|h_1 {*} h_2(x) w(\xi)| \;\leq C_s\big(|h_1| {*} |h_2| w + |h_1| w {*} |h_2|\big)(\xi) \quad \forall\, h_1, h_2 \in L^2_{w_s}. \tag{4.40}$$

It follows easily from this estimate that $L^2_{w_s} \cap L^1(\mathbb{R}^d)$ is a Banach convolution algebra, but since $L^2_{w_s} \hookrightarrow L^1(\mathbb{R}^d)$ for $s > d/2$ this is just $L^2_{w_s}(\mathbb{R}^d)$, up to equivalence of norms. This in turn implies that $\mathcal{H}_s$ is a Banach algebra for pointwise multiplication.

Since $L^2_{w_s}$ coincides with $W(L^2, \ell^2_{w_s})$ (cf. [42]), Theorem 2.3 for weighted spaces, cf. [26], yields

$$\mathcal{H}_s = \mathcal{F}^{-1}[L^2_{w_s}] = \mathcal{F}^{-1}[W(L^2, \ell^2_{w_s})] = W(\mathcal{F}^{-1}L^2_{w_s}, \ell^2) = W(\mathcal{H}_s, \ell^2). \tag{4.41}$$

Hence, $\mathcal{H}_s$ is a so-called l^2-puzzle, cf. [59], and we may apply amalgam techniques whenever it is convenient.

In order to see that the reproducing kernel φ_s is in $S_o(\mathbb{R}^d)$ we follow the arguments of [51], where it is shown that the short time Fourier transform of φ_s with respect to the Gaussian window function g is

$$V_g\varphi_s(x,\xi) = \frac{2^d\pi^{d-s}}{\Gamma(s)} \iint_0^\infty u^{-d/2+s-1} e^{-(\frac{4\pi^3 t^2}{u} + \frac{u}{4\pi} + 2\pi i t\xi)} T_x\overline{g}(t)\; dudt \tag{4.42}$$

which implies the estimate

$$\|\varphi_s\|_{S_o} = \|V_g\varphi_s\|_{L^1} \le C \int_0^\infty u^{-d/2+s-1}(u+4\pi^2)^{d/2} e^{-\frac{u}{4\pi}}\, du\,; \tag{4.43}$$

the last expression being finite whenever $s > d/2$. We thus have shown the following:

Lemma 4.1. *For $s > d/2$ the reproducing kernel function φ_s is in $S_o(\mathbb{R}^d)$.*

Remark 4.1. One may replace w_s by more general weight functions w with similar properties. The arising reproducing kernel Hilbert spaces are *Harmonic Hilbert spaces* as discussed in [20], see also [37].

Since for $s > d/2$ we have $\varphi_s \in S_o$, both $\hat{\varphi}_s = w_{-2s}$ and $|\hat{\varphi}_s|^2 = w_{-4s}$ are in S_o, which in turn implies that both

$$\Psi^{(s)}(\xi) = \sum_{k\in\mathbb{Z}^d} \hat{\varphi}_s(\xi - k)\,,$$

$$\Phi^{(s)}(\xi) = \sum_{k\in\mathbb{Z}^d} |\hat{\varphi}_s(\xi - k)|^2$$

are periodic functions which have an absolutely convergent Fourier series expansions, and are bounded above and away from zero, i.e., there exist pairs of positive constants (A_s, B_s) and (A'_s, B'_s) such that

$$A_s \le \Psi^{(s)}(\xi) \le B_s \qquad (\xi \in \mathbb{R}^d)\,, \tag{4.44}$$

$$A'_s \le \Phi^{(s)}(\xi) \le B'_s \qquad (\xi \in \mathbb{R}^d)\,. \tag{4.45}$$

These observations lead us to the following result:

Theorem 4.1. *For $s > d/2$, the family $\big(T_k\varphi_s\big)_{k\in\mathbb{Z}^d}$ is a Riesz basis for its linear span $V_s = V_{\varphi_s,1}$ in both L^2 and $\mathcal{H}_s$. The functions φ_s^d and φ_s^L defined via their Fourier transforms as*

$$\hat{\varphi}_s^d(\xi) = \frac{\hat{\varphi}_s(\xi)}{\Phi^{(s)}(\xi)}\,, \qquad \hat{\varphi}_s^L(\xi) = \frac{\hat{\varphi}_s(\xi)}{\Psi^{(s)}(\xi)}, \qquad \xi \in \mathbb{R}^d \tag{4.46}$$

generate the dual basis for L^2 and $\mathcal{H}_s$, respectively.

Proof. Because of (4.44), the L^2-case is already described in Section 3. For $\mathcal{H}_s$, we verify that $\big(T_k\varphi\big)_{k\in\mathbb{Z}^d}$ satisfies (3.26). For $\mathbf{c} \in \ell^2(\mathbb{Z}^d)$, we have

$$\begin{aligned}
&\Big\| \sum_{k\in\mathbb{Z}^d} c_k T_k \varphi_s \Big\|^2_{\mathcal{H}_s} = \\
&= \int_{\mathbb{R}^d} \Big| \sum_{k\in\mathbb{Z}^d} c_k e^{2\pi i k\xi} \hat{\varphi}_s(\xi)\Big|^2 w_s^2(\xi)\, d\xi \\
&= \sum_{n\in\mathbb{Z}^d} \int_{n+[0,1]^d} \Big| \sum_{k\in\mathbb{Z}^d} c_k e^{2\pi i k\xi}\Big|^2 \hat{\varphi}_s(\xi)^2\, w_s^2(\xi)\, d\xi \\
&= \int_{[0,1]^d} \Big| \sum_{k\in\mathbb{Z}^d} c_k e^{2\pi i k\xi}\Big|^2 \Psi^{(s)}(\xi)\, d\xi\,.
\end{aligned}$$

Because of (4.45) it follows that $\big(T_k\varphi_s\big)_{k\in\mathbb{Z}^d}$ satisfies (3.26). Hence, $\big(T_k\varphi_s\big)_{k\in\mathbb{Z}^d}$ is a Riesz basis for its linear span in $\mathcal{H}_s$ and since

$$\langle T_j\varphi_s, T_k\varphi_s^L\rangle_{\mathcal{H}_s} = \delta_{jk}$$

$\mathbb{Z}^d$-translates of the function φ_s^L generate the $\mathcal{H}_s$-dual basis. □

Corollary 4.1. *For $s > d/2$, the spline type space*

$$V_s = \Big\{ \sum_{k\in\mathbb{Z}^d} c_k T_k\varphi_s \;\Big|\; \mathbf{c} \in \ell^2(\mathbb{Z}^d)\Big\} \tag{4.47}$$

is a closed subspace of L^2 and $\mathcal{H}_s$.

As a consequence of this characterization of biorthogonal Riesz bases we find that the orthogonal projection Q_s from $\mathcal{H}_s$ onto V_s is given by

$$Q_s f = \sum_{k\in\mathbb{Z}^d} f(k) T_k \varphi_s^L, \qquad f \in \mathcal{H}_s\,. \tag{4.48}$$

Since φ_s is an interpolating function (Lagrange interpolator), that is,

$$\varphi_s^L(k) = \langle \varphi_s^L, T_k\varphi_s\rangle_{\mathcal{H}_s} \;=\; \delta_{0k} \qquad k \in \mathbb{Z}^d\,, \tag{4.49}$$

we immediately see that

$$Q_s f(k) = f(k) \qquad (k \in \mathbb{Z}^d)\,. \tag{4.50}$$

Note that all the statements hold true for more general lattices $a\mathbb{Z}^d \lhd \mathbb{R}^d$, $a > 0$. In this case,

$$\Psi^{(s,a)}(\xi) = \sum_{k\in\mathbb{Z}^d} \hat{\varphi}_s(\xi - a^{-1}k) \tag{4.51}$$

and the orthogonal projection $Q_{s,a}$ from $\mathcal{H}_s$ onto $V_{s,a}$ takes the form

$$Q_{s,a} f = \sum_{k\in\mathbb{Z}^d} f(ak) T_{ak} \varphi^L_{s,a} \qquad f \in \mathcal{H}_s \tag{4.52}$$

now with $\hat{\varphi}^L_{s,a}(\xi) = \hat{\varphi}_s(\xi)/\Phi^{(s,a)}(\xi)$ for $\xi \in \mathbb{R}^d$.

We now summarize the above results in a slightly different form.

Theorem 4.2. *For every lattice $a\mathbb{Z}^d \lhd \mathbb{R}^d$, the minimal norm interpolation problem in $\mathcal{H}_s$ for a given data sequence $\left(d_k\right)_{k\in\mathbb{Z}^d}$ has the unique solution*

$$h = \sum_{k\in\mathbb{Z}^d} d_k T_{ak} \varphi^L_{s,a}\,,$$

where $\varphi^L_{s,a}$ is the Lagrange interpolator in $V_{s,a}$ (corresponding to the unit vector $\mathbf{e}_0 \in \ell^2(\mathbb{Z}^d)$). Moreover, the sum is unconditionally convergent in the $\mathcal{H}_s$-norm.

5 L^p-Theory

In order to obtain L^p-variants, $1 \le p \le \infty$, for the Hilbert space results stated in the previous sections, we first recall two important results for spline type space (cf. [4], Theorem 3.1 or [30]) which are both essentially based on Lemma 2.3.

Theorem 5.1. *Assume that $\left(T_k\varphi\right)_{k\in\mathbb{Z}^d}$ is a Riesz basis for its closed linear span V_φ in $L^2(\mathbb{R}^d)$, for some $\varphi \in W(C_0, \ell^1)$. Then it is also an unconditional basis for its closed linear span $V^p_\varphi(\mathbb{R}^d)$ within $L^p(\mathbb{R}^d)$, for $1 \le p \le \infty$, that is, the synthesis mapping*

$$c \mapsto \sum_{k\in\mathbb{Z}^d} c_k T_k \varphi$$

extends to an isomorphism between $\ell^p(\mathbb{Z}^d)$ and $V^p_\varphi(\mathbb{R}^d)$, for $p \in [1,\infty)$, and between $c_0(\mathbb{Z}^d)$ and V^∞_φ. In particular, there are constants A_p, B_p such that

$$A_p \|c\|_{\ell^p} \le \Big\| \sum_{k\in\mathbb{Z}^d} c_k T_k \varphi \Big\|_{L^p} \le B_p \|c\|_{\ell^p} \qquad \forall\; \mathbf{c} \in \ell^p(\mathbb{Z}^d) \tag{5.53}$$

is satisfied for all $1 \le p \le \infty$. *Furthermore*

$$V_\varphi^p(\mathbb{R}^d) = \Big\{ \sum_{k\in\mathbb{Z}^d} c_k T_k \varphi \,\Big|\, c \in \ell^p(\mathbb{Z}^d)) \Big\} \tag{5.54}$$

and the $W(L^\infty, \ell^p)$ *and the* L^p*-norms are equivalent on those spaces, for* $1 \le p < \infty$.

This result is closely related to the fact that the orthogonal projection from $L^2(\mathbb{R}^d)$ onto V_φ extends in a natural way to projections from $L^p(\mathbb{R}^d)$ onto $V_\varphi^p(\mathbb{R}^d)$, for $1 \le p \le \infty$.

Theorem 5.2. *Assume that* $\varphi \in W(C_0, \ell^1)$ *and that* $\big(T_k\varphi\big)_{k\in\mathbb{Z}^d}$ *is a Riesz basis for its closed linear span* V_φ *within* $L^2(\mathbb{R}^d)$. *Then the dual atom* φ^d, *generating the biorthogonal Riesz basis* $\big(T_k\varphi^d\big)_{k\in\mathbb{Z}^d}$ *to* $\big(T_k\varphi\big)_{k\in\mathbb{Z}^d}$ *also belongs to* $W(C_0, \ell^1)$ *and therefore the operator*

$$P : f \mapsto \sum_{k\in\mathbb{Z}^d} \langle f, T_k\varphi^d \rangle_{L^2} T_k \varphi \tag{5.55}$$

defines a continuous projection from $L^p(\mathbb{R}^d)$ *onto* $V_\varphi^p(\mathbb{R}^d)$ *for* $1 \le p \le \infty$.

Theorem 5.1 says that the $W(C_0, \ell^p)$- and the L^p-norms are equivalent on the spline type space V_φ^p with $\varphi \in W(C_0, \ell^1)$. We even have uniform estimates for the constants for these norm-equivalences, for all $1 \le p \le \infty$.

Proposition 5.1. *Let* V_φ^p *a spline type space with* $\varphi \in W(C_0, \ell^1)$. *Then, there exists a constant* $c > 0$ *such that for all* $p \in [1, \infty]$,

$$c\, \|f\|_{W(C_0,\ell^p)} \;\le\; \|f\|_{L^p} \;\le\; \|f\|_{W(C_0,\ell^p)} \qquad \forall f \in V_\varphi^p . \tag{5.56}$$

Proof. The upper estimate is always satisfied. For the lower estimate we recall that any $f \in V_\varphi^p$ is of the form

$$f = \sum_{k\in\mathbb{Z}^d} \langle f, T_k\varphi^d \rangle_{L^2}\; T_k\varphi$$

where φ^d denotes the L^2 dual atom of φ and by Theorem 3.1 $\varphi^d \in W(C_0, \ell^1)$. Applying (2.22), we see that $\mathbf{c} = (\langle f, T_k\varphi^d \rangle)$ is in ℓ^p. Hence,

$$f = \big(\sum_{k\in\mathbb{Z}^d} c_k \delta_k \big) * \varphi$$

and from the convolution relation (2.12) we obtain the lower estimate. □

Let $\varphi \in W(C_0, \ell^1)$ generates a Riesz basis of $a\mathbb{Z}^d$-translates for all a within some set $A \subset \mathbb{R}^+$. As a consequence of (2.23), the linear map

$$Q_a : f \mapsto \sum_{k\in\mathbb{Z}^d} f(ak) T_{ak} \varphi \tag{5.57}$$

is bounded from $W(C_0,\ell^p)$ to $V^p_{\varphi,a}$ for all $a \in A$. For $a=1$ we skip the index and simply write $Q = Q_1$.

Proposition 5.2. *Assume that $\varphi \in W(C_0,\ell^1)$ generates a Riesz basis of $a\mathbb{Z}^d$-translates for all a in some $A \subset \mathbb{R}^+$ with minimal $a_0 > 0$. Then there exists $C = C(a_0) > 0$ such that for all $a \in A$ and $1 \le p \le \infty$*

$$\|Q_a f\|_{W(L^\infty,\ell^p)} \;\le\; C\;\|f\|_{W(C_0,\ell^p)} \qquad \forall f \in W(C_0,\ell^p).$$

Proof. The claim follows from (2.21) and (2.23) in Lemma 2.3. □

The operator Q_a which coincides with the minimal norm interpolation for the dual element φ_s^L of φ_s in $\mathcal{H}_s$, can be extended to a family of Sobolev spaces or potential spaces $\mathcal{H}^p_s$ which we define for $s > 0$, as in [56], by

$$\mathcal{H}^p_s = \mathcal{J}_s\big(L^p(\mathbb{R}^d)\big),$$

where $\mathcal{J}_s$ is the Bessel potential $\mathcal{J}_s(f) = G_s * f$, and G_s is the kernel of $(1-\Delta)^{-s/2}$. For $s = 0$, we have $\mathcal{H}^p_s = L^p$. Properties of G_s can be found in [56]. Here we just state properties of $\mathcal{H}^p_s$ with corresponding references.

1. $\mathcal{H}^p_s$ endowed with the norm

   $$\|G_s * f\|_{\mathcal{H}^p_s} = \|f\|_{L^p}$$

   is a Banach space, continuously embedded into $L^p = \mathcal{H}^p_0$ which follows from the norm estimate for all $1 \le p \le \infty$ and $s \ge 0$, see [56], that

   $$\|f\|_{L^p} \le \|f\|_{\mathcal{H}^p_s} \qquad \forall f \in \mathcal{H}^p_s\,.$$

2. $\mathcal{H}^p_s$ coincides with the classical Sobolev space $W^{s,p}$ for all positive integers s and $1 < p < \infty$, [1].

3. $\mathcal{H}^p_s$ is an algebra under pointwise multiplication if and only if $\mathcal{H}^p_s \hookrightarrow C$ that is if and only if $s > d/p$, [56].

4. $\mathcal{H}^p_s \hookrightarrow C^\alpha$ for all $s > d/p + \alpha$ with $\alpha \ge 0$ where C^α denotes the Hölder space of order α, see [60].

The following connection to Wiener Amalgam spaces is given by the so-called *localization principle* of the Triebel-Lizorkin spaces $F^s_{p,q}$, which contain the fractional Sobolev spaces as special case, cf. [61], section 2.4.7.

Lemma 5.1. *For any $1 < p < \infty$ and $s \in \mathbb{R}$ we have*

$$\mathcal{H}^p_s = W(\mathcal{H}^p_s, \ell^p)\,,$$

with equivalence of the corresponding norms.

Combined with the classical embedding theorem into C^α we obtain an interesting embedding result (both locally and globally):

Lemma 5.2. *Let $p \in (1, \infty)$ be given, and $s_0 > p/d + \alpha$ for some $\alpha \geq 0$. Then, $\mathcal{H}_s^p \hookrightarrow W(C^\alpha, \ell^p)$ and there exists $C = C(p, \alpha, s_0) > 0$ such that*

$$\|f\|_{W(C^\alpha,\ell^p)} \ \leq \ C \ \|f\|_{\mathcal{H}_s^p} \qquad \forall f \in \mathcal{H}_s^p, \tag{5.58}$$

for all $s \geq s_0$.

Proof. For $1 < p < \infty$, the continuous embedding $\mathcal{H}_{s_0}^p(\mathbb{R}^d) \hookrightarrow C^\alpha$ for $s_0 > d/p + \alpha$ is a special case of Theorem 2.8.1 in [60]. Furthermore, it can easily be seen from the definition of $\mathcal{H}_s^p$ that

$$\mathcal{H}_s^p \hookrightarrow \mathcal{H}_{s_0}^p \quad \text{and} \quad \|f\|_{\mathcal{H}_{s_0}^p} \leq \|f\|_{\mathcal{H}_s^p} \quad \text{if} \quad s \geq s_0\,.$$

Therefore, we obtain

$$\|f\|_{C^\alpha} \ \leq \ C(p, \alpha, s_0)\, \|f\|_{\mathcal{H}_{s_0}^p} \ \leq \ C(p, \alpha, s_0)\, \|f\|_{\mathcal{H}_s^p} \qquad \forall f \in \mathcal{H}_s^p$$

if $s \geq s_0 > d/p + \alpha$. The estimate (5.58) follows using Lemma 5.1. □

We now see that the operator Q_a is well defined from $\mathcal{H}_s^p$ into $W(L^\infty, \ell^p)$ for $1 < p < \infty$ and $s > p/d$, and we can state a result that is similar to Proposition 5.2 except for the uniformity with respect to p.

Proposition 5.3. *Assume that $\varphi \in W(C_0, \ell^p)$ generates a Riesz basis of $a\mathbb{Z}^d$-translates for all a in some $A \subset \mathbb{R}^+$ with minimal $a_0 > 0$. For $1 < p < \infty$ and $s_0 > d/p$, there exists $C = C(p, a_0, s_0) > 0$ such that*

$$\|Q_a f\|_{W(L^\infty,\ell^p)} \leq C \|f\|_{\mathcal{H}_s^p} \qquad \forall f \in \mathcal{H}_s^p \tag{5.59}$$

for all $s \geq s_0$ and $a \in A$.

Proof. As a summary of the above results, we now have for $f \in \mathcal{H}_s^p$

$$\|Q_a f\|_{W(L^\infty,\ell^p)} \leq C \|f(ak)\|_{\ell^p} \leq C(a_0) \|f\|_{W(C_0,\ell^p)} \leq C(p, a_0, s_0) \|f\|_{\mathcal{H}_s^p}$$

for all $s \geq s_0 > d/p$ and $a \in A$. □

6 Changing the Smoothness Parameter

In this section we discuss the influence of the parameter s of the weight function w_s and state some stability results of the minimal norm interpolation in the form of continuous dependence on s.

We first consider the S_o-norm. In Section 4, we have seen that $\varphi_s \in S_o$ for $s > d/2$. Using (4.42) and (4.43), dominated convergence gives

$$\varphi_s \xrightarrow{S_o} \varphi_{s_0} \qquad s \searrow s_0 \ > \ d/2\,.$$

The result is true for both the L^2-dual φ_s^d and the $\mathcal{H}_s$-dual φ_s^L atom of φ_s.

Lemma 6.1.

$$\begin{array}{l} \varphi_s^L \xrightarrow{S_o} \varphi_{s_0}^L \\ \varphi_s^d \xrightarrow{S_o} \varphi_{s_0}^d \end{array} \qquad s \searrow s_0 \ > \ d/2\,.$$

Proof. We only prove the statement for φ_s^L since for φ_s^d we can apply exactly the same arguments. Because $\varphi_s \in S_o$, we know that

$$\sum_{k\in\mathbb{Z}^d} \hat{\varphi}_s(\xi - k) = \sum_{k\in\mathbb{Z}^d} b_k^{(s)} e^{-2\pi i k\xi} = \hat{b}^{(s)}$$

with

$$b^{(s)} = \big(\varphi_s(k)\big) \in \ell^1(\mathbb{Z}^d)\,.$$

From the estimate (2.21) of Lemma 2.3 we obtain

$$\sum_{k\in\mathbb{Z}^d} |b_k^{(s)} - b_k^{(s_0)}| \le C\|\varphi_s - \varphi_{s_0}\|_{S_o}\,,$$

hence

$$b^{(s)} \xrightarrow{\ell^1} b^{(s_0)} \qquad s \searrow s_0 \ > \ d/2\,.$$

By Wiener's Lemma, there exist $c^{(s)} \in \ell^1(\mathbb{Z}^d)$ such that

$$\hat{c}_k^{(s)} = 1/\hat{b}_k^{(s)} \qquad \forall k \in \mathbb{Z}^d \quad \text{and} \quad \varphi_s^L = \sum_{k\in\mathbb{Z}^d} c_k^{(s)} T_k \varphi_s\,.$$

Because the mapping $b \to b^{-1}$ is continuous in any Banach algebra, in particular in $\ell^1(\mathbb{Z}^d)$ with respect to convolution, it follows that

$$c^{(s)} \xrightarrow{\ell^1} c^{(s_0)} \qquad s \searrow s_0 \ > \ d/2\,.$$

Hence, $c^{(s)}$ is uniformly bounded and the norm estimate

$$\big\|\varphi_s^L - \varphi_{s_0}^L\big\|_{S_0} = \big\|\sum c_k^{(s)} T_k\varphi_s - \sum c_k^{(s_0)} T_k\varphi_{s_0}\big\|_{S_0} =$$

$$\big\|\sum c_k^{(s)} T_k\varphi_s - \sum c_k^{(s)} T_k\varphi_{s_0} + \sum c_k^{(s)} T_k\varphi_{s_0} - \sum c_k^{(s_0)} T_k\varphi_{s_0}\big\|_{S_0} \le$$

$$\le \ C\big\|c^{(s)}\|_{\ell^1}\big\|\big\|\varphi_s - \varphi_{s_0}\big\|_{S_0} + C\big\|\varphi_{s_0}\big\|_{S_0}\big\|c^{(s)} - c^{(s_0)}\big\|_{\ell^1}$$

implies that

$$\varphi_s^L \xrightarrow{S_0} \varphi_{s_0}^L \qquad s \searrow s_0 \ > \ d/2\,.$$

□

Remark 6.1. For $s \ge s_0$ we have $\mathcal{H}_s \hookrightarrow \mathcal{H}_{s_0}$. In particular, φ_s is in $\mathcal{H}_{s_0}$ for all $s \ge s_0 > d/2$. Since

$$|w_{-2s}(x) - w_{-2s_0}(x)|^2 w_{2s_0}(x) \le w_{-s_0}(x) \qquad \forall x \in \mathbb{R}^d ,$$

dominated convergence gives

$$\varphi_s \xrightarrow{\mathcal{H}_{s_0}} \varphi_{s_0} \qquad s \searrow s_0 \;>\; d/2 . \tag{6.60}$$

Similarly, we show that the dual atom φ_s^L and the interpolation

$$Q_s f = \sum_{k \in \mathbb{Z}^d} f(k) T_k \varphi_s^L \qquad (f \in \mathcal{H}_{s_0})$$

enjoy the same convergence behaviour, i.e., depend continuously on s for $s \searrow s_0$.

Theorem 6.1. *If $s \searrow s_0 \;>\; d/2$, then we have*

(i) $\|\varphi_s^L - \varphi_{s_0}^L\|_{\mathcal{H}_{s_0}} \to 0$,

(ii) $\|Q_s f - Q_{s_0} f\|_{\mathcal{H}_{s_0}} \to 0$ *for any* $f \in \mathcal{H}_{s_0}$.

Proof. We know that

$$\sum \hat{\varphi}_s(\xi - k) \;\le\; \sum \hat{\varphi}_{s_0}(\xi - k) \;\le\; B_{s_0} .$$

Thus,

$$\Big|\frac{\hat{\varphi}_{s_0}(\xi)}{\sum \hat{\varphi}_{s_0}(\xi - k)} - \frac{\hat{\varphi}_s(\xi)}{\sum \hat{\varphi}_s(\xi - k)}\Big|^2 \le \frac{1}{B_{s_0}^2}\Big|\hat{\varphi}_{s_0} - \hat{\varphi}_s\Big|^2 .$$

Hence,

$$\|\varphi_s^L - \varphi_{s_0}^L\|_{\mathcal{H}_{s_0}} \;=\; \|\hat{\varphi}_{s_0}^L - \hat{\varphi}_s^L\|_{L^2_{s_0}} \;\le\; B_{s_0}^{-2}\|\varphi_s^L - \varphi_{s_0}^L\|_{\mathcal{H}_{s_0}}$$

and by (6.60)

$$\varphi_s^L \xrightarrow{\mathcal{H}_{s_0}} \varphi_{s_0}^L \qquad s \searrow s_0 \;>\; d/2 .$$

Since $(f(k))$ is in ℓ^2 for any $f \in \mathcal{H}_s = W(\mathcal{H}_s, \ell^2)$, the estimate

$$\begin{aligned}
&\Big\|\sum_{|k| \le N} f(k) T_k \varphi_s^L - \sum_{|k| \le N} f(k) T_k \varphi_{s_0}^L\Big\|_{\mathcal{H}_{s_0}} \le \\
&= \int \Big|\sum_{|k| \le N} f(k) e^{2\pi i k \xi}\Big|^2 \Big|\hat{\varphi}_s^L(\xi) - \varphi_{s_0}^L(\xi)\Big|^2 w_{s_0}^2(\xi) d\xi \\
&\le \sum_{k \in \mathbb{Z}^d} |f(k)|^2 \big\|\varphi_s^L - \varphi_{s_0}^L\big\|_{\mathcal{H}_{s_0}}
\end{aligned}$$

holds uniformly for all N which implies

$$\sum f(k) T_k \varphi_s^L \xrightarrow{\mathcal{H}_{s_0}} \sum f(k) T_k \varphi_{s_0}^L \qquad s \searrow s_0 \;>\; d/2 .$$

□

7 Changing the Sampling Lattice

In this section we want to discuss the influence of the lattice constant $a > 0$ on dual atoms, as well as on the the resulting minimal norm interpolation, by showing continuous dependence of the Lagrange interpolator on the lattice constant $a > 0$. Again, Wiener amalgams are the appropriate tool to formulate the corresponding statements. Recall that we write φ_a^d for the generator of the biorthogonal Riesz basis of $(T_{ak}\varphi)_{k\in\mathbb{Z}^d}$.

Theorem 7.1. *For every $\varphi \in W(L^2, \ell^1)$ the set of all parameters $a > 0$ such that $\left(T_{ak}\varphi\right)_{k\in\mathbb{Z}^d}$ defines a Riesz basis for its closed linear span $V_s(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d)$, is an open subset $\mathcal{O}_\varphi \subseteq (0,\infty)$. Moreover,*

1. *the mapping $a \mapsto \varphi_a^d$ is continuous from $\mathcal{O}_\varphi$ into $W(L^2, \ell^1)$,*
2. *for any homogeneous Banach space $(B, \| \|_B)$ in $W(L^2, \ell^1)$, we have $\varphi_a^d \in B$ for all $a \in \mathcal{O}_\varphi$, and $a \mapsto \varphi_a^d$ is continuous from $\mathcal{O}_\varphi$ into $(B, \| \|_B)$.*

We can even state a somewhat stronger result, emphasizing the jointly continuous dependence on both the “atom” and the lattice constant.

Theorem 7.2. *Define a subset of $W(C_0, \ell^1) \times (0,\infty)$, by*

$$\mathcal{O}\left(W(C_0,\ell^1)\right) = \{\ (\varphi, a)\ \big|\ \left(T_{ak}\varphi\right)_{k\in\mathbb{Z}^d}\ \text{ is a Riesz basis in } L^2(\mathbb{R}^d)\ \}.$$

Then $\mathcal{O}\left(W(C_0,\ell^1)\right)$ is open, and the mapping $(\varphi, a) \mapsto \varphi_a^d$ is continuous from $\mathcal{O}\left(W(C_0,\ell^1)\right)$ into $W(C_0, \ell^1)$.

Proof. Since the arguments are quite analogous to the proof of Theorem 6.1, making use of the inversion of the corresponding Gram matrix as in Theorem 3.1, we only give an outline of the key arguments required for the proof of Theorem 7.2, making clear that the $W(L^2, \ell^1)$-context stated in Theorem 7.1 can be derived by the same arguments.

Let as assume that $(\varphi_0, a_0) \in \mathcal{O}\left(W(C_0,\ell^1)\right)$, i.e., we have invertibility of the Gram-matrix associated to the system $(T_{a_0k}\varphi)_{k\in\mathbb{Z}^d}$. Since this is a circulant matrix whose entries are just the samples over $a_0\mathbb{Z}^d$ of $\varphi * \varphi^* \in W(L^2, \ell^1) * W(L^2, \ell^1) \hookrightarrow W(C_0, \ell^1)$, we only have to invoke that Lemma 2.2 to guarantees that, for (φ, a) close enough to (φ_0, a_0) in $W(L^2, \ell^1) \times (0,\infty)$, the $\ell^1(\mathbb{Z}^d)$ convolution kernels representing those Gram matrices are close to each other in the ℓ^1-sense.

Since the set of all invertible elements is open in the Banach convolution algebra $\ell^1(\mathbb{Z}^d)$ and the mapping associating with each invertible element its inverse is continuous in the ℓ^1-sense we may conclude that we will use “similar” coefficients to synthesize the dual atom with respect to $a\mathbb{Z}^d$ from shifts of φ along $a\mathbb{Z}^d$ if the lattice constants a and a_0 are close to each other. However, for $a \neq a_0$ the difference between ak and a_0k becomes arbitrarily large (for $|k|$ large) and thus we cannot just use the standard jitter error argument (as used in section 8). The claimed continuity result is shown below in Lemma 7.1. □

Lemma 7.1. *Assume that $(B, \| \|_B)$ is a homogeneous Banach space. Then the mapping $R : (\alpha, a, \varphi) \mapsto \sum_{k\in\mathbb{Z}^d} \alpha(k) T_{ak}\varphi$ is continuous from $\ell^1(\mathbb{Z}^d) \times (0,\infty) \times B$ into B.*

Proof. Let $(\alpha, a, \varphi) \to (\alpha_0, a_, \varphi_0)$ in $\ell^1(\mathbb{Z}^d) \times (0,\infty) \times B$. We are going to verify that $R(\alpha, a, \varphi)$ is in a given ε-neighborhood in B of $R(\alpha_0, a_0, \varphi)$ provided that all parameters are δ-close to their limit, for some $\delta > 0$.

Without restriction we assume a priori that convergence takes place in some δ_1-neighborhood of $(\alpha_0, a_0, \varphi_0)$, for some $\delta_1 \leq 1$. Since $\|T_{ak}\varphi\|_B = \|\varphi\|_B$ for all $a > 0$ and $\varphi \in B$, the following estimate is obvious:

$$\begin{aligned} &\|R(\alpha, a_0, \varphi) - R(\alpha_0, a_0, \varphi_0)\|_B \;\leq \\ &\leq\; \|R(\alpha - \alpha_0, a_0, \varphi)\|_B \;+ \|R(\alpha_0, a_0, \varphi - \varphi_0)\|_B \\ &\leq\; \|\alpha_0 - \alpha\|_{\ell^1(\mathbb{Z}^d)}\; \|\varphi\|_B + \|\alpha_0\|_{\ell^1(\mathbb{Z}^d)}\; \|\varphi - \varphi_0\|_B \\ &\leq \delta_1 \cdot (1 + \|\varphi_0\|_B + \|\alpha_0\|_{\ell^1(\mathbb{Z}^d)}) \;\leq\; \varepsilon/3 \end{aligned} \tag{7.61}$$

for δ_1 sufficiently small. This shows that the estimate can be reduced the the case where (α_0, φ_0) is fixed. As in the above estimate we may replace the sequence α_0, if necessary, by another sequence with only a finite number of non-zero coordinates (at the cost of a possible approximation error of the order $< \varepsilon/3$). For the case of finite, hence bounded, subsets of $\mathbb{Z}^d$ one has of course uniform convergence of $\|T_{ak}\varphi_0 - T_{a_0k}\varphi_0\|_B \to 0$, according to the definition of a homogeneous Banach space. Hence we can choose $\delta > 0$ such that $\|R(\alpha_0, a, \varphi_0) - R(\alpha_0, a_0, \varphi_0\|_B < \varepsilon/3$ for $|a - a_0| < \delta \leq \delta_1$, and the proof is complete. □

We have seen that a central part of the proof of Theorem 7.2 was Wiener's inversion theorem, which guarantees the invertibility of the Gram matrix for the family $\left(T_{ak}\varphi\right)_{k\in\mathbb{Z}^d}$, as an operator on $\ell^1(\mathbb{Z}^d)$. The entry point to this argument was the fact that for $\varphi \in W(C_0, \ell^1)$ or at least $W(L^2, \ell^1)$ its autocorrelation function belongs to $S_o(\mathbb{R}^d)$ and therefore its samples over $\mathbb{Z}^d$ are in $\ell^1(\mathbb{Z}^d)$. For the case of the Sobolev scalar product $\langle f, g\rangle_{\mathcal{H}_s}$ the situation is similar and in a way easier: Due to the reproducing property of φ_s we find that the entries of the Gram matrix for the same system of translates are just the sampling values of φ_s, known to belong to $S_o(\mathbb{R}^d)$ if only $s > d/2$, cf. Lemma 4.1 and equation (4.38).

The following result is a corollary to the (proof of the) above theorems:

Corollary 7.1. *Let $s > d/2$ be given. Then,*

(a) *the mapping $a \mapsto \varphi^d_{s,a}$ is continuous from $(0,\infty)$ into $S_o(\mathbb{R}^d)$,*

(b) *the mapping $a \mapsto \varphi^L_{s,a}$ is continuous from $(0,\infty)$ into $S_o(\mathbb{R}^d)$.*

The relevance of Corollary 7.1.b for minimal norm interpolation in Sobolev algebras is described in the following theorem.

Theorem 7.3. *For each $p \in [1,\infty)$ the mapping $(s,a,f) \mapsto Q_{s,a}(f)$ is continuous from $(d/2,\infty) \times (0,\infty) \times W(C_0,\ell^p)$ into $W(C_0,\ell^p)$.*

For the proof of this theorem we will need another continuity statement, similar to Lemma 7.1.

Lemma 7.2. *Let $p \in [1,\infty)$ be given. Then the mapping*

$$R : (\alpha, a, \varphi) \mapsto \sum_{k \in \mathbb{Z}^d} \alpha(k) T_{ak} \varphi$$

is continuous from $\ell^p(\mathbb{Z}^d) \times (0,\infty) \times W(C_0,\ell^1)$ into $W(C_0,\ell^p)$.

Proof. By Lemma 2.3.3. we know that over compact subsets of the parameter a within $(0,\infty)$ we have uniform boundedness of the bilinear synthesis $(\alpha,\varphi) \mapsto R(\alpha,a,\varphi)$ from $\ell^p(\mathbb{Z}^d) \times W(C_0,\ell^1)$ into $W(C_0,\ell^p)$. As in the proof of Lemma 7.1 we may reduce the discussion to the second variable, keeping (α_0,φ_0) fixed, even with φ_0 in any dense subspace of $W(C_0,\ell^1)$, such as $C_c(\mathbb{R}^d)$, and α_0 with only finitely many non-zero coordinates. The rest of the proof is then similar to that of the previous lemma, using the density of "finite" sequences in $\ell^p(\mathbb{Z}^d)$ for $p < \infty$. □

Proof. (Theorem 7.3) According to Corollary 7.1 the mapping $(s,a) \mapsto \varphi^L_{s,a}$ is continuous with values in $S_o(\mathbb{R}^d)$. In combination with Lemma 2.3.1 and Lemma 7.2 the claim is verified. □

In view of Lemma 5.1 the following statement is an immediate consequence of Theorem 7.3:

Corollary 7.2. *For every pair (p,s_1) with $s_1 > d/p$ the mapping $(a,s,f) \mapsto Q_{s,a}(f)$ is continuous from $(0,\infty) \times (d/2,\infty) \times \mathcal{H}^p_{s_1}(\mathbb{R}^d)$ into $W(C_0,\ell^p)$, hence, in particular, into $L^p(\mathbb{R}^d)$ as well as $C_0(\mathbb{R}^d)$.*

Remark 7.1. By means of complex interpolation or by means of a direct proof making use of the corresponding Wiener amalgam convolution relation (2.11) the following analogue to the two statements about the continuity of R can be obtained, which contains the previous ones as limiting case: Let $p \in (1,\infty)$ be given. Then the mapping $R : (\alpha,a,\varphi) \mapsto \sum_{k\in\mathbb{Z}^d} \alpha(k) T_{ak}\varphi$ is continuous from $\ell^r(\mathbb{Z}^d) \times (0,\infty) \times W(C_0,\ell^q)$ into $W(C_0,\ell^p)$ provided that $1/p \geq 1 - 1/r - 1/q$.

8 Jitter Stability

In this section we discuss various forms of jitter which may occur in the context of minimal norm interpolation along some lattice. The first problem coming to mind is related to the following situation: What happens to the minimal norm interpolation operator in case that, instead of the exact sampling values $f(ak)_{k\in\mathbb{Z}^d}$, sampling values taken at nearby points, i.e., values $(f(ak+\eta_k))_{k\in\mathbb{Z}^d}$, are used ? Can we control the error on the output side in terms of the maximal jitter error $\|\eta\|_\infty = \ sup_k|\eta_k|$? We will show that the resulting minimal norm interpolator is still well defined and that maximal reconstruction error (for all possible families of jitter errors) tends to zero when η becomes small. For our proof we will make use of properties of the oscillation function.

Definition 8.1. For a function f on $\mathbb{R}^d$ and any $\delta > 0$, the function

$$x \mapsto \operatorname{osc}_\delta(f)(x) = \sup_{|y|\le\delta} |T_y f(x) - f(x)| = \sup_{y\in B_\delta(x)} |f(x) - f(x+y)|$$

is called the δ-*oscillation* of f.

The following lemma summarizes basic facts:

Lemma 8.1. *Let $p \in [1,\infty)$ be given. Then*

(1.) *$f \in W(C_0,\ell^p)$ implies $\operatorname{osc}_\delta(f) \in W(C_0,\ell^p)$, and*

$$\|\operatorname{osc}_\delta(f)\|_{W(C_0,\ell^p)} \ \le\ 2\|f\|_{W(C_0,\ell^p)}.$$

(2.) *$\|\operatorname{osc}_\delta(f)\|_{W(C_0,\ell^p)} \to 0$ as $\delta \to 0$, for every $f \in W(C_0,\ell^p)$.*

(3.) *Let $\varphi \in W(C_0,\ell^1)$ generate the spline type space V^p_φ. Then $\|\operatorname{osc}_\delta(f)\|_{W(C_0,\ell^p)} \to 0$ as $\delta \to 0$, uniformly over the unit ball of V^p_φ in $W(C_0,\ell^p)$, i.e., $\forall\ \eta > 0\ \ \exists\ \delta > 0$ such that*

$$\|\operatorname{osc}_\delta(f)\|_{W(C_0,\ell^p))} \ \le\ \eta\cdot\|f\|_{W(C_0,\ell^p)} \ \ \forall f \in V^p_\varphi. \tag{8.62}$$

Proof. The proof for (1) is an easy exercise. Note that $\operatorname{osc}_\delta(f)$ is continuous. Claim (3) as well as claim (2) for $p = 1$ can be found in [4], Lemma 6.3. (the corresponding result for spaces of band-limited functions is given as Lemma 6.3 in [36]). The arguments for $p > 1$ in (2) are essentially the same as for the case $p = 1$. □

Corollary 8.1. *Let $f \in W(C_0,\ell^p)$. For any given $a_0 > 0$ and $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $a \ge a_0$ on has:*

$$\|(\operatorname{osc}_\delta(f)(ak))_{k\in\mathbb{Z}^d}\|_{\ell^2(\mathbb{Z}^d)} = \Big(\sum_{k\in\mathbb{Z}^d} |\operatorname{osc}_\delta(f)(ak)|^2\Big)^{-1/2} \le \varepsilon.$$

Proof. The statement follows from Lemma 8.1(2) and inequality (2.21). □

We have seen in Section 5 that for any $\varphi \in W(C_0, \ell^1)$ the operator

$$Q : f \mapsto \sum_{k \in \mathbb{Z}^d} f(ak)\; T_{ak}\varphi \tag{8.63}$$

is well defined, mapping $W(C_0, \ell^p)$ boundedly into itself, for every $p \in [1, \infty]$. It even maps $W(C_0, \ell^p)$ into $V^p_\varphi(\mathbb{R}^d)$, if only $(T_{ak}\varphi)_{k\in\mathbb{Z}^d}$ is a Riesz basis for its closed linear span within $L^2(\mathbb{R}^d)$. Indeed, by Theorem 5.1 the $W(C_0, \ell^p)$-norm is equivalent to the L^p-norm on V^p_φ which in turn is equivalent to the ℓ^p-norm of the sequence of coefficients of functions in V^p_φ.

Our first goal is to discuss the influence of imprecise sampling on this operator, i.e., the effect of using, instead of the exact samples at ak, only values at nearby points. Thus for fixed $a > 0$ and $\varphi \in W(C_0, \ell^1)$, we define for any sequence $\big(\eta_k\big)_{k\in\mathbb{Z}^d}$ in $\mathbb{R}^d$,

$$Q_\eta : f \mapsto \sum_{k \in \mathbb{Z}^d} f(ak + \eta_k)\; T_{ak}\varphi \tag{8.64}$$

which is again a bounded operator from $W(C_0, \ell^p)$ into $V^p_\varphi(\mathbb{R}^d)$, by essentially the same argument as for Q, if only $\|\eta\|_\infty = \sup_k |\eta_k| \le \delta < \infty$. We only have to argue that the pointwise estimate

$$|f(ak + \eta_k)| \le |f(ak)| + \mathrm{osc}_\delta f(ak) \in \ell^p(\mathbb{Z}^d) \tag{8.65}$$

holds true as a consequence of Lemma 8.1(1) in conjunction with Lemma 2.3(3).

Our first claim concerns the fact that Q_η is close to Q in the sense of the strong operator topology on $W(C_0, \ell^p)$, if only $\|\eta\|_\infty$ is small enough.

Theorem 8.1. *Let $p \in [1, \infty)$ be given. Then for every $f \in W(C_0, \ell^p)$*

$$\big\|Qf - Q_\eta f\big\|_{W(C_0,\ell^p)} \to 0 \quad \text{for} \quad \|\eta\|_\infty \to 0\,. \tag{8.66}$$

Proof. We make use of estimate (2.23) in Lemma 2.3(3):

$$\big\|Qf - Q_\eta f\big\|_{W(C_0,\ell^p)} \;\le\; C \cdot \|\varphi\|_{W(C_0,\ell^1)} \big\|\big(f(k) - f(k+\eta_k)\big)_{k\in\mathbb{Z}^d}\big\|_{\ell^p} \;\le$$

$$\le C \cdot \|\varphi\|_{W(C_0,\ell^1)} \big\|\mathrm{osc}_\delta(f)(k)\big\|_{\ell^p} \;\le\; C \cdot \|\varphi\|_{W(C_0,\ell^1)} \cdot \|\mathrm{osc}_\delta(f)\|_{W(C_0,\ell^p)}\,.$$

The claim now follows from Lemma 8.1. □

Of course, the above theorem applies to the particular choice $\varphi = \varphi^L_s$, $s > d/2$. Then Q is a quasi-interpolation operator. For $p = 2$ we can even give a stronger statement:

Corollary 8.2. *For $f \in \mathcal{H}_s$, we have*

$$\big\|Q_s f - Q_{s,\eta} f\big\|_{\mathcal{H}_s} \to 0 \qquad \text{as} \quad \|\eta\|_\infty = \delta \to 0\,.$$

Proof. This immediately follows from Theorem 8.1 and the fact that the norms of $\mathcal{H}_s$ and $W(C_0, \ell^p)$ are equivalent on V_s. □

In order to verify uniform convergence over certain subspaces of $W(C_0, \ell^p)$ we will need statements like the following ones:

Proposition 8.1. *Let $a > 0$. Assume that $(T_{ak}\varphi)$ is a Riesz basis. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\|\text{osc}_\delta f\|_{W(C_0,\ell^p)} \;\leq\; \varepsilon \cdot \|f\|_{W(C_0,\ell^p)} \quad \forall f \in V^p_\varphi. \tag{8.67}$$

Proof. For $f = \sum_{k\in\mathbb{Z}^d} c_k T_{ak}\varphi$ the following pointwise estimate is valid:

$$\text{osc}_\delta(f)(x) \leq \sum_{k\in\mathbb{Z}^d} |c_k| T_{ak}\text{osc}_\delta(\varphi)(x) \tag{8.68}$$

which yields via Lemma 2.3, estimate (2.23),

$$\|\text{osc}_\delta(f)(x)\|_{W(C_0,\ell^p)} \;\leq\; C(a)\;\|\mathbf{c}\|_{\ell^p} \cdot \|\text{osc}_\delta(\varphi)\|_{W(C_0,\ell^1)}. \tag{8.69}$$

In view of the norm equivalence between the ℓ^p-norm of c and $\|f\|_{W(C_0,\ell^p)}$ for $f \in V^p_\varphi$, and the fact that $\|\text{osc}_\delta(\varphi)\|_{W(C_0,\ell^1)} \to 0$ for $\delta \to 0$, the proof of the Prop. 8.1 is complete. □

We are now ready to formulate the following uniform variant of Theorem 8.1.

Theorem 8.2. *Let $\varphi \in W(C_0, \ell^1)$ and $p \in [1,\infty)$ be given. Then, for every $\varepsilon > 0$ there exists some $\delta > 0$ such that $\|\eta\|_\infty \leq \delta$ implies*

$$\left\|Qf - Q_\eta f\right\|_{W(C_0,\ell^p)} \;\leq\; \varepsilon \cdot \|f\|_{W(C_0,\ell^p)} \quad \forall f \in V^p_\varphi. \tag{8.70}$$

Corollary 8.3. *Assume that $s > d/2, a_0 > 0$, and $p \in [1,\infty)$. Then, for every $\varepsilon > 0$ there exists some $\delta > 0$ such that $\|\eta\|_\infty \leq \delta$ implies for any $a \geq a_0$*

$$\left\|Q_{s,a}f - Q_{s,a,\eta}f\right\|_{W(C_0,\ell^p)} \;\leq\; \varepsilon \cdot \|f\|_{W(C_0,\ell^p)} \quad \forall f \in V^p_\varphi. \tag{8.71}$$

Although our jitter error estimates rely essentially on the equicontinuity of those sets of functions within $W(C_0, \ell^p)$ to which the minimal norm interpolation procedures are applied we need a slightly stronger control on the decay of $\text{osc}_\delta f$.

Definition 8.2. A bounded and continuous function on $\mathbb{R}^d$ belongs to $Lip\,(\alpha)$ for some $\alpha \in (0,1)$ if and only if there exists $C = C_f > 0$ such that

$$|f(x) - f(y)| \leq C|x-y|^\alpha \qquad \forall\; x, y \in \mathbb{R}^d. \tag{8.72}$$

The $Lip\,(\alpha)$-norm of f in such a class is given by

$$\|f\|_{Lip\,(\alpha)} := \|f\|_\infty + C_f, \tag{8.73}$$

where C_f is the minimal constant for which the estimate 8.72 is valid.

A crucial estimate allowing us to give a uniform control of jitter errors for functions in fractional L^p-Sobolev spaces is the one given below. Let us start with a trivial pointwise estimate for $\mathrm{osc}_\delta(f)$ for functions $f \in Lip\,(\alpha)$.

Lemma 8.2. *For $f \in Lip\,(\alpha), 0 < \alpha < 1$, one has for $\delta > 0$*

$$\mathrm{osc}_\delta f(x) \leq C_f \cdot \delta^\alpha \leq \delta^\alpha \cdot \|f\|_{Lip\,(\alpha)}. \tag{8.74}$$

In the following proposition it is shown that we have uniform decay of $\mathrm{osc}_\delta f$ in $W(C_0, \ell^p)$ over the unit ball of a fractional Sobolev space.

Proposition 8.2. *Assume $s > d/p$. Then for any $\alpha \in (0, \min(1, s - d/p))$ there exists $C = C(\alpha, s, p) > 0$ such that*

$$\|\mathrm{osc}_\delta f\|_{W(C_0,\ell^p)} \leq C \cdot \delta^\alpha \|f\|_{\mathcal{H}_s^p} \quad \forall f \in \mathcal{H}_s^p(\mathbb{R}^d). \tag{8.75}$$

Proof. According to the localization property expressed in Lemma 5.1 for $\mathcal{H}_s^p(\mathbb{R}^d)$ we may assume from the beginning that $f = \sum_{k\in\mathbb{Z}^d} f \cdot T_k\psi$, for some test function $\psi \in \mathcal{D}(\mathbb{R}^d)$ with $\sum_{k\in\mathbb{Z}^d} \psi_k(x) = 1$, where we write ψ_k for $T_k\psi$, such that the following estimate is valid:

$$(\sum_{k\in\mathbb{Z}^d} \|f\psi_k\|_{\mathcal{H}_s^p}^p)^{1/p} \leq C_1 \|f\|_{\mathcal{H}_s^p}$$

Next we observe that for any $\alpha \in (0, \min(1, s - d/p))$ there exists $C(\alpha) > 0$ such that for all $k \in \mathbb{Z}^d$

$$\|\mathrm{osc}_\delta(f\psi_k)\|_\infty \leq C(\alpha) \cdot \|f\psi_k\|_{\mathcal{H}_s^p} \cdot \delta^\alpha.$$

Since, however, we have to estimate $(\mathrm{osc}_\delta f) \cdot (\psi_k)$ instead of $\mathrm{osc}_\delta(f\psi_k)$, we need one more observation: For $k \in \mathbb{Z}^d$ we have the pointwise inequality

$$(\mathrm{osc}_\delta f \cdot \psi_k)(x) \;\leq\; C_0 \cdot \mathrm{osc}_\delta(f \cdot T_k\phi)(x) \quad \forall x \in \mathbb{R}^d, \tag{8.76}$$

whenever $\phi(y) = 1$ in some sufficiently large neighborhood of the support of ψ. By adding over $k \in \mathbb{Z}^d$, we obtain

$$\big(\sum_{k\in\mathbb{Z}^d} (\|\mathrm{osc}_\delta f \cdot \psi_k\|_\infty)^p\big)^{1/p} \leq C'(\alpha)\,\delta^\alpha \big(\sum_{k\in\mathbb{Z}^d} \|f\psi_k\|_{\mathcal{H}_s^p}^p\big)^{1/p} \leq C''(\alpha)\,\delta^\alpha \|f\|_{\mathcal{H}_s^p}$$

and the proof is complete. □

Theorem 8.3. *Let $\psi \in W(C_0, \ell^1), a > 0, s_1 > d/p$ be given. Then, for every $\varepsilon > 0$ there exists some $\delta > 0$ such that $\|\eta\|_\infty \leq \delta$ implies*

$$\big\|Qf - Q_\eta f\big\|_{W(C_0,\ell^p)} \;\leq\; \varepsilon \cdot \|f\|_{\mathcal{H}_{s_1}^p} \quad \forall f \in \mathcal{H}_{s_1}^p(\mathbb{R}^d). \tag{8.77}$$

Proof. The arguments for the proof of Theorem 8.3 are completely analogue to that of Theorem 8.2. □

9 References

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