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1. Introduction. A recent survey article by J. Fournier and J. Stewart (BULL.AMS 13 (1985), 1–21) explains how amalgams of L^p with l^q (as function spaces over any locally compact abelian group G) can be used as an effective tool for the treatment of various problems in harmonic analysis. The present article may be seen as a complement to this survey, indicating further advantages that arise if one works with generalized amalgams (introduced in 1980 under the name of Wiener-type spaces by the author [**10**]). The main difference between amalgams and these more general spaces is the fact that they allow a more precise description of the local behavior of functions (or distributions) by rather arbitrary norms and that the conditions on the global behavior (of the quantity obtained using that chosen local norm) is described in a way that includes both growth and integrability conditions (not only l^q-summability). In this note we shall work on an intermediate level, which turns out to be a suitable tool for a detailed description of the Fourier transformation (going far beyond the information given by the Hausdorff-Young inequality). For $G = \mathbf{R}^m$ one obtains among others a family of Banach spaces of functions (or even ultradistributions) which are invariant under the Fourier transform. Finally, the problem of pointwise multipliers of Sobolev algebras is discussed from this point of view.

2. Notations and preliminaries. Although our results will be presented in their natural generality, i.e. in the setting of a non-compact, locally compact abelian group G with Haar measure dx the reader will not miss the point by thinking of G as of $\mathbf{R}^m$, with the Lebesgue measure. We shall denote the Lebesgue space by $(L^p, \| \; \|_p)$ for $1 \leqq p < \infty$, keeping in mind that $\mathcal{K}(G)$, the spaces of continuous, compactly supported, complex-valued functions is dense in $L^p(G)$ for $1 \leqq p \leqq \infty$; the closure of $\mathcal{K}(G)$ in $L^\infty(G)$ is identified with $C^0(G)$, the space of continuous functions vanishing at infinity.

A *weight function* w on G is a strictly positive, locally integrable function on G satisfying $w(x+y) \leqq w(x)w(y)$ for all $x, y \in G$ (cf. [**25**], Chap. 3). Two weights w_1 and w_2 are called *equivalent* (we write $w_1 \simeq w_2$) if their quotients are bounded (from zero and infinity) over G. Since any weight function is equivalent to a continuous one (cf. [**6**]) we shall assume throughout that our weights are continuous. The following more general class of functions is very

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useful: A positive, continuous function m is called *moderate*, if there exists an (associated) weight function w such that one has $m(yxz) \leqq w(y)m(x)w(z)$ for all $x, y, z \in G$. Typical examples are the real powers of a given weight function. It is easy to see that equivalent weights (and also moderate functions) define the same weighted spaces, given by $L^p_m(G) = \{f \mid fm \in L^p\}$, with the natural norm $\|f\|_{p,m} := \|fm\|_p$. It is well known [25] that $(L^1_w(G), \| \quad \|_{1,w})$ is a Banach algebra with respect to convolution, called *Beurling algebra*. It is also easy to verify (by vector-valued integration for $1 \leqq p < \infty$ and pointwise for $p = \infty$) that $L^1_w(G)$ acts on $L^p_m(G)$ through convolution, and that one has

(1) $\quad \|g * f\|_{p,m} \leqq \|g\|_{1,w}\|f\|_{p,m}$ for $g \in L^1_w(G)$ and $f \in L^p_m(G)$.

In fact, the spaces $L^p_m(G)$ are invariant under the translation operators L_x, given as $L_x f(y) := f(y - x)$, and the operator norm on L^p_m satisfies $|||L_x|||_{p,m} \leqq w(x)$. Of course, one has the duality $(L^p_m)' = L^{p'}_{1/m}$, with $1/p' = 1 - 1/p$.

The *dual group* $\hat{G}$ is defined as the set of all *characters* (i.e. continuous homomorphisms from G into the Torus $\mathbf{T}$) $\chi : G \to \mathbf{T}$, with pointwise multiplication as group operation. The dual of the group $(\mathbf{R}^m, +)$ is identified with $\mathbf{R}^m$ through the bijection $\chi_t \leftrightarrow t$, where $\chi_t(x) := \exp(2\pi i \langle x, t\rangle)$, for $x, t \in \mathbf{R}^m$. Given $f \in L^1(G)$ the *Fourier transform* is defined as usual by $\mathcal{F}$ or $\mathcal{F}_G$: $\mathcal{F}f(t) := \hat{f}(\chi) := \int_G f(x)\overline{\chi(x)}\, dx$. Using the Pontryagin duality theorem, which allows to identify $(\hat{G})\hat{}$ in a natural way with G (thus allowing us to write $\langle x, \chi\rangle$ or $\langle \chi, x\rangle$ unambiguously for $\chi(x)$ for $x \in G$ and $\chi \in \hat{G}$) one has the Fourier inversion formula

$$f(x) = \mathcal{F}^{-1}h(x) = \int_G \hat{f}(\chi)\langle x, \chi\rangle\, d\chi \quad \text{a.e.},$$

if $h := \hat{f} \in L^1(\hat{G})$. Here the Haar measure $d\chi$ on $\hat{G}$ has to be normalized suitably. In particular $\mathcal{F}^{-1} = \mathcal{F} \circ S = S \circ \mathcal{F}$, with $Sg(x) := g(-x)$ for $x \in G$.

Let us take the convention of using the letter v for (submultiplicative) weight functions on $\hat{G}$, associated to a given moderate function u on $\hat{G}$.

Writing M_χ for the multiplication by $\chi \in \hat{G}$ we note that

(2) $\quad (L_x f)\hat{} = M_{-x}\hat{f}$ and $(M_\chi f)\hat{} = L_\chi \hat{f}$ for $x \in G$, $\chi \in \hat{G}$.

The convolution and uniqueness theorem for the Fourier transform tell us that $\mathcal{F}L^1_w$, endowed with it's natural norm $\|\hat{f} \mid \mathcal{F}L^1_w\| = \|f\|_{1,w}$, is a dense Banach algebra in $(C^0(\hat{G}), \| \quad \|_\infty)$. We restrict ourselves to the consideration of *symmetric* weights w (i.e. weights satisfying $w(x) = w(-x) =: Sw(x)$ or at least $w \simeq Sw$). Note that this implies that $\mathcal{F}L^1_w$ is closed under complex conjugation and coincides with $\mathcal{F}^{-1}L^1_w$, which will be convenient for notational reasons. Moreover, w has to be bounded away from zero, hence L^1_w is a subspace of L^1. We are only interested in *regular* pointwise algebras, i.e., for any point s_0 and a neighborhood $U(s_0)$ it is assumed that there exists $h \in \mathcal{F}L^1_w$ with $h(s_0) = 1$ and supp $h \subseteq U(s_0)$. We assume THROUGHOUT THE PAPER that all weights used (also those in the description of moderateness) should satisfy the so-called *non-quasianalyticity condition* due to Beurling-Domar ([25], Chap. 6, &3):

$$\text{(BD)} \qquad \sum_{n=1}^{\infty} n^{-2} \log(w(nx)) < \infty \quad \text{for all } x \in G.$$

Thus the typical weights on $\mathbf{R}^m$ included in our discussion are of form

$$w : x \longrightarrow \exp(\beta |x|^{\alpha})(1 + |x|)^{\gamma} \log^{\delta}(1 + |x|),$$

with $0 < \alpha < 1$ and $\beta > 0$, $\gamma, \delta \in \mathbf{R}$; or $\alpha = 0$ and $\gamma > 0$, $\delta \in \mathbf{R}$ or $\alpha = \gamma = 0$, $\delta \geqq 0$. The set of moderate functions includes of course any $m := 1/w$, w as above. For $\alpha = 0$ or $\beta = 0$ the corresponding spaces L^p_m may be considered as subspaces of $\mathcal{S}'$ (space of all tempered distributions; here $\mathcal{S}(\mathbf{R}^m)$ is the space of rapidly decreasing C^{∞}-functions on $\mathbf{R}^m$ in the sense of Schwartz). Since $A := \mathcal{F}L^1_w$ satisfies all requirements of A as described in [7] it thus follows that there exist *BUPU*s in A, i.e. *bounded uniform partitions of unity*. These are bounded families $\Psi = (\psi_i)_{i \in I}$ in $(A, \| \; \|_A)$ satisfying (for some compact set $Q = -Q \subseteq G$):

$$\operatorname{supp} \psi_i \subseteq y_i + Q, \sup_{i \in I} \#\{j \mid (y_i + Q) \cap (y_j + Q) \neq \phi\} < \infty$$

and

$$\sum_{i \in I} \psi_i(x) \equiv 1.$$

Given now any Banach space $(B, \| \; \|_B)$ of functions (or distributions) on which A acts by pointwise multiplication, the Wiener type spaces (or generalized amalgams) $W(B, l^q_m)$ can be defined as follows (cf. [10]): $f \in W(B, l^q_m)$ if and only if $f \in B_{loc}$ and

$$\|f \mid W(B, l^q_m)\| := \left(\sum_{i \in I} \|f \psi_i\|^q_B m(y_i)^q \right)^{1/q} < \infty.$$

As shown in [10] and [19] these spaces do not depend on Ψ or $(y_i)_{i \in I}$ and different systems give equivalent norms. The family of spaces of this form behaves also nicely with respect to duality (under the usual restrictions), i.e. one may calculate the dual space in the local and the global component separately; the same is true for (abstract) interpolation or the characterization of pointwise multipliers (cf. [19]). For $m \equiv 1$ and $B \equiv L^p$ one comes back to the ordinary amalgam spaces discussed in [22]. In our discussion only spaces of the form

$B = \mathcal{F}L^p_m$ (note that $\mathcal{F}L^1_w \cdot \mathcal{F}L^p_m = \mathcal{F}(L^1_w * L^p_m) \subseteq \mathcal{F}L^p_m$) with their natural norm will be of interest. Note that these spaces are defined by the usual Fourier transform if $L^p_m \subseteq L^1$, or as spaces of tempered distributions on $G = \mathbf{R}^n$, if m is of at most polynomial growth; however, a very general definition will be given below (including the FT of ultra-distributions).

Finally we observe that we shall have – presupposing a good definition of the spaces $W(\mathcal{F}L^p_u, l^q_m)$ at the moment, that they are invariant under translations and multiplications with characters, with estimates

$$(3) \qquad \|L_x f \mid W(\mathcal{F}L^p_u, l^q_m)\| \leqq C_\Psi w(x) \|f \mid W(\mathcal{F}L^p_u, l^q_m)\|$$

and

$$(4) \qquad \|M_t f \mid W(\mathcal{F}L^p_u, l^q_m)\| \leqq v(t) \|f \mid W(\mathcal{F}L^p_u, l^q_m)\|.$$

Applying now the convolution theorem (Theorem 3 in [**10**]) for Wiener-type spaces we obtain that these spaces are double modules, i.e. Banach modules over $L^1_w(G)$ with respect to convolution and (this follows directly from the definition) pointwise Banach modules over the Banach algebra $\mathcal{F}L^1_v$ (together with the corresponding norm inequalities):

$$(5) \qquad \|g * f \mid W(\mathcal{F}L^p_u, l^q_m)\| \leqq \|g\|_{1,w} C_\Psi \|f \mid W(\mathcal{F}L^p_u, l^q_m)\| \quad \text{for } g \in L^1_w;$$

$$(6) \qquad \|h \cdot f \mid W(\mathcal{F}L^p_u, l^q_m)\| \leqq \|h \mid \mathcal{F}L^1_v\| \, \|f \mid W(\mathcal{F}L^p_u, l^q_m)\| \quad \text{for } h \in \mathcal{F}L^1.$$

The spaces arising for $p = q = 1$ are of particular interest because of the minimality properties of these spaces (cf. below) (cf. [7], [9], [**15**], ...)

Proposition 1. *The space $W(\mathcal{F}L^1_w, l^1_v)(G)$ is a dense Banach ideal in the Beurling algebra $L^1_v(G)$ (under convolution) as well as in $\mathcal{F}L^1_w$ (pointwise). Moreover, it is the minimal, non-trivial Banach space of functions (continuously embedded into $L^1(G)$) satisfying (5) and (6).*

Proof. The decisive argument is the fact that – as a consequence of the BD-condition – the compactly supported elements (which belong of course to $W(\mathcal{F}L^1_w, l^1_v)(G)$) form a dense subspace of $(\mathcal{F}L^1_w)(G)$. The minimality proof follows the line of arguments given in [3] and is left to the reader.

We may mention that it is possible to endow these spaces with an equivalent, "continuous" norm, such as the expression (cf. [**10**]):

$$\left(\int_G \|(L_x k) f \mid \mathcal{F}L^p_u\|^q m(x)^q dx \right)^{1/q},$$

where k can be any non-zero element in $W(\mathcal{F}L^1_v, l^1_w)$. In many situations these norms are more convenient to handle than the discrete description used above.

However, for the results to be discussed here the discrete version seem to more appropriate (mainly because the results concerning duality and pointwise multipliers have been published in the discrete setting, cf. [**19**]). These continuous descriptions, on the other hand, are the reason why we use the symbols $W(B, L^q_m)$ for the spaces $W(B, l^q_m)$ in other papers concerning Wiener type spaces as well.

3. The spaces $W(\mathcal{F}L^p_u, l^q_m)$ and their Fourier transforms. It is our purpose in this chapter to show that the family of Wiener-type spaces $W(\mathcal{F}L^p_u, l^q_m)$ shows a very reasonable behavior under the Fourier transformation. In fact, the general principle that local properties of a function (e.g. smoothness, boundedness ...) are reflected by global (e.g. decay or integrability properties) of its Fourier transform can be given a rather precise meaning in our setting.

In order to prove these results we proceed as follows: First of all the case $p = q = 1$ is treated, then the dual situation. More general cases are then a consequence of results on interpolation theory, applied to these basic results.

Concerning the Wiener-type spaces $W(\mathcal{F}L^1_v, l^1_w)(G)$ it is worth mentioning that the special case $v = w = 1$ gives us the Segal algebra $S_0(G)$, an example which is the prototypical in this field. In fact, as it was shown (cf. [**9**]) this space is the minimal among all Banach ideals in $L^1(G)$ which are also isometrically invariant under multiplications with characters.

THEOREM 2. *Let v and w be weight function on $\hat{G}$ and G respectively, both satisfying the Beurling-Domar condition. Then the Wiener-type spaces $W(\mathcal{F}L^1_v, l^1_w)(G)$ and $W(\mathcal{F}L^1_w, l^1_v)(\hat{G})$ are well defined and the Fourier transform $\mathcal{F}_G : L^1(G) \longmapsto \mathcal{F}L^1(\hat{G})$ establishes an isomorphism between these Banach spaces.*

Proof. That these spaces are well defined has already been discussed above. In order to show that the Fourier transform defines an isomorphism between these spaces it will be sufficient to verify the inclusion $\mathcal{F}_G W(\mathcal{F}L^1_v, l^1_w)(G) \subseteq W(\mathcal{F}L^1_w, l^1_v)(\hat{G})$. In fact, by the closed graph theorem $\mathcal{F}_G$ is automatically a continuous mapping between these spaces in that case, and the converse inclusion is obtained by a symmetry argument (based on Pontrjagin's duality theorem).

We use the discrete (atomic) description of $W(\mathcal{F}L^1_v, l^1_w)$, which implies that (for a suitable constant $C_Q \geqq 1$) for each $f \in W(\mathcal{F}L^1_v, l^1_w)$ a sequence $(f_n)_{n \geqq 1}$ in $\mathcal{F}L^1_v$ with supp $f_n \subseteq Q$, a fixed compact set in G, and $(y_n)_{n \geqq 1}$ in G such that $f = \sum_{n \geqq 1} L_{y_n} f_n$ and $\sum_{n \geqq 1} w(y_n) \| f_n \mid \mathcal{F}L^1_v \| \leqq C \| f \mid W(\mathcal{F}L^1_v, l^1_w) \|$. It is now clear that the individual terms $\mathcal{F}f_n$ belong to L^1_v, but we need more. To this end note that there exists some $g \in L^1_v(\hat{G})$ such that $\hat{g}(x) \equiv 1$ on Q and supp $\hat{g}$ is compact (this follows from condition BD). However, as mentioned the space $W(\mathcal{F}L^1_w, l^1_v)(\hat{G})$ is a (dense) Banach ideal in $L^1_v(\hat{G})$. In view of the localization principle (cf. [**25**], Chap. 2) it is clear that g as described above belongs to this ideal. Making now use of inequalities (4) and (5) we obtain (at least for finite sums):

$$
\begin{aligned}
\|\hat{f} \mid W(\mathcal{F}L^1_w, l^1_v)\| &\leqq \sum_{n\geqq 1} \|M_{y_n}\hat{f}_n \mid W(\mathcal{F}L^1_w, l^1_v)\| \\
&\leqq \sum_{n\geqq 1} w(y_n)\|f_n * g \mid W(\mathcal{F}L^1_w, l^1_v)\| \\
&\leqq \sum_{n\geqq 1} w(y_n)C_\Psi\|f_n\|_{1,v}\|g \mid W(\mathcal{F}L^1_w, l^1_v)\| \\
&\leqq CC_\Psi\|g \mid W(\mathcal{F}L^1_w, l^1_v)\| \, \|f \mid W(\mathcal{F}L^1_v, l^1_w)\| \\
&\leqq C_2\|f \mid W(\mathcal{F}L^1_v, l^1_w)\|.
\end{aligned}
$$

Having now obtained precise information on the image of these spaces (of test functions) under the Fourier transform we can proceed as in the definition of Fourier transforms for tempered distributions, by extending it to the corresponding duals:

THEOREM 3. *Given the weight functions w and v as above, the Fourier transform extends (in a unique w^*-continuous way) to a bounded linear isomorphism between the corresponding dual spaces Banach spaces.*

Proof. Given $\sigma \in W(\mathcal{F}L^1_v, l^1_w)'(G)$ we define $\mathcal{F}\sigma$ by setting:

$$
\mathcal{F}\sigma(h) := \sigma(\mathcal{F} h) \quad \text{for } h \in W(\mathcal{F}L^1_w, l^1_v)(\hat{G}). \tag{7}
$$

In view of Theorem 2 (changing the roles of G and $\hat{G}$ and v, w, respectively) this definition makes sense by Pontrjagin's theorem. It is also clear from this definition that the "new" Fourier transform is a w^*-continuous mapping. Since the ordinary Fourier transform is very well compatible with duality as a consequence of the identity

$$
\int_G \hat{f}(x)\overline{g(x)}\,dx = \int_G \hat{g}(t)\overline{f(t)}\,dt, \quad \text{for } f \in L^1(\hat{G}), g \in L^1(G),
$$

the above definition is actually an extension of the ordinary Fourier transform, and in fact the unique w^*-continuous version (for reasons of density).

Remark 1. It is clear that the above definition is compatible with any other "extended" Fourier transform, in particular with the definition of the Fourier transform of tempered distributions on $\mathbf{R}^m$. For weight functions of at most polynomial growth on $\mathbf{R}^m$ there is a continuous embedding of $L^\infty_{1/w}(G) = (L^1_w(G))'$ into $\mathcal{S}'(\mathbf{R}^m)$. Consequently $\mathcal{F}L^\infty_{1/w}(G)$ is well defined, and the content of Theorem 2 can be reformulated as follows for this case:

COROLLARY 4. *Given two weight functions w, v on $\mathbf{R}^m$ of at most polynomial growth, the Fourier transform extends to an isomorphism between the Banach spaces $W(\mathcal{F}L^\infty_{1/v}, l^\infty_{1/w})(\mathbf{R}^m)$ and $W(\mathcal{F}L^\infty_{1/w}, l^\infty_{1/v})(\mathbf{R}^m)$.*

Proof. Besides the observations already made we only have to note that for Wiener-type spaces the natural duality results hold true, i.e. it is possible to calculate the dual spaces in each "coordinate" (i.e. local and global) separately (cf. [19], Theorem 2.8).

In the general situation described above, which includes already various spaces of ultra-distributions in the sense of Beurling and Björck (cf. [8]) we can take Theorem 2 as a starting point for the definition of a *very general (extended) Fourier transform.* In order to give a good definition of the Fourier transform of L^p_u-spaces we make the following (general) observation, showing that "reasonable" spaces are embedded between the spaces discussed so far and their dual space.

PROPOSITION 5. *Let $(B, \|\ \|_B)$ be a translation invariant Banach space continuously embedded into the space of (Radon) measures $R(G) := \mathcal{K}(G)'$ on G (this is satisfied if B consists of locally integrable functions and convergence in $\|\ \|_B$ implies convergence in measure), which is a pointwise Banach module over the Banach algebra $\mathcal{F}L^1_v(G)$ and a Banach module over $L^1_w(G)$ with respect to convolution, satisfying the estimate $\|L_x f\|_B \leqq w(x)\|f\|_B$ for all $x \in G$, $f \in B$. Then one has the following continuous embeddings*

$$W(\mathcal{F}L^1_v, l^1_w) \hookrightarrow B \hookrightarrow (W(\mathcal{F}L^1_v, l^1_w))'.$$

Proof. The first inclusion is just a consequence of the description of minimal invariant spaces as given in [7]. In order to verify the second one it is sufficient to note first that the assumptions imply that elements of B act in a natural way on the closed subspace

$$(\mathcal{F}L^1_v)_Q := \{f \mid f \in \mathcal{F}L^1_v, \operatorname{supp} f \subseteq Q\}$$

of $\mathcal{F}L^1_v$, i.e. there exists $C > 0$ such that

$$|\langle b, f\rangle| \leqq C\|b\|_B\|f \mid \mathcal{F}L^1_v\| \quad \text{for } b \in B, f \in (\mathcal{F}L^1_v)_Q.$$

That this action can be extended to the whole space $W(\mathcal{F}L^1_v, l^1_w)(G)$ is a consequence of the following estimate (which holds for arbitrary finite sums $f = \sum_{n\geq 1} L_{x_n} f_n$ with $\operatorname{supp} f_n \subseteq Q$):

$$\begin{aligned}|\langle b, f\rangle| &\leqq \sum_{n\geq 1} |\langle L_{-x_n} b, f_n\rangle| \leqq \sum_{n\geq 1} C\|L_{-x_n} b\|_B\|f_n \mid \mathcal{F}L^1_v\| \\ &\leqq C\sum_{n\geq 1} w(-x_n)\|b\|_B\|f_n \mid \mathcal{F}L^1_v\| \leqq C_2\|f \mid W(\mathcal{F}L^1_v, l^1_w)\|\,\|b\|_B.\end{aligned}$$

It is clear that the Banach spaces L^p_u are special cases of the general space $(B, \|\ \|_B)$ discussed in Proposition 5 (as function spaces on $\hat{G}$). In particular, the spaces $\mathcal{F}L^p_u$ (as space of distributions on $(\hat{G})\hat{} = G$), *are now defined for* *the general situation* $1 \leqq p \leqq \infty$ and enjoy of course the same properties that we have stated for the restricted family of L^p_u-spaces discussed so far. Thus, for example it would now be possible to formulate Corollary 4 for the family of spaces $W(\mathcal{F}L^\infty_{1/v}, l^\infty_{1/w})(G)$, where w and v run through the family of weights satisfying estimates $w(x) = O(w_0(x))$ and $v(t) = O(v_0(t))$, with w_0 and v_0 being fixed weight functions (on G and $\hat{G}$ respectively) for which Theorem 2 is valid. It is also possible to define the general Wiener-type spaces of the transform $W(\mathcal{F}L^p_u, l^p_m)$ and to study their behavior under the Fourier transform. For the special case $G = \mathbf{R}^m$ the following results give a whole family of Banach spaces of (tempered) distributions which are invariant with respect to the Fourier transform. It will be an immediate consequence of the fact that our family of spaces is closed in a natural way under complex interpolation, which we need not state explicitly here (cf. the proof below).

THEOREM 6. *Let w, v be weight functions on G and $\hat{G}$ respectively, both satisfying condition (BD). Furthermore, let $\alpha, \beta \in \mathbf{R}$ be given and $p \in (1, \infty)$. Then the Fourier transform $\mathcal{F}_G$ (as considered in Theorem 1) extends to an isomorphism between the spaces $W(\mathcal{F}L^p_u, l^p_m)$ and $W(\mathcal{F}L^p_m, l^p_u)$, with $u := v^\alpha$ and $m := w^\beta$.*

Proof. It is no problem to check that the weights m and u satisfy (BD) themselves. The basic tool, in order to derive Theorem 5 by complex interpolation from Theorems 1 and 2 is the corresponding result for interpolation of Wiener-type spaces [**11**]. It follows as a special case from Corollary 2.4 in [**19**]. Given α, β we find the appropriate end-spaces for interpolation as follows (we treat only one typical special case in detail: $\alpha > 0$, $\beta > 0$: Then

$$W(\mathcal{F}L^p_u, l^p_m) = [W(\mathcal{F}L^1_{u^p}, l^1), W(\mathcal{F}L^\infty, l^\infty_{m^{p'}})]_{1/p]}, \text{ with } 1/p' = 1 - 1/p.$$

The case $p = 2$ of the Theorem 6 is of special interest. It gives us informations concerning potential spaces and their pointwise multipliers spaces, including anisotropic versions and extensions to arbitrary lca. groups (cf. [**4**], p. 10, [**27**], for results in this direction). For simplicity we describe only a typical result:

COROLLARY 7. *Given a weight function w on $\hat{G}$ let us denote the space $\mathcal{F}^{-1}L^2_w(\hat{G})$ by $\mathcal{L}^2_w(G)$. Then we have $\mathcal{L}^2_w(G) = W(\mathcal{L}^2_w(G), l^2)$.*

Proof. It suffices to mention that we have by Plancherel's theorem the following identity: $L^2_w(G) = W(L^2_w, l^2)$. Thus Theorem 6 applies.

This notation has of course been chosen in accordance with the usual symbol for Bessel potential spaces. These correspond to the special case $G = \mathbf{R}^m$ and $w = w_s$, with $w_s(t) := (1 + |t|^2|)^{s/2}$, and Coifman/Meyer speak of "l^2-puzzles" in this connection. The above characterization is of particular interest if $\mathcal{L}^2_w(G)$ happens to be a Banach algebra with respect to pointwise multiplication. A general result in this direction is contained in the following corollary:

COROLLARY 8. *Assume that w is a weakly subadditive weight function on $\hat{G}$ (i.e. that there exists some constant $C_w > 0$ such that $w(x+y) \leqq C_w(w(x)+w(y))$ for $x, y \in G$), and furthermore that $1/w \in L^2(\hat{G})$. Then the algebra of point-wise multipliers coincides with $W(\mathcal{L}_w^2(G), l^\infty)$.*

Proof. It is not difficult to derive from the weak subadditivity of w that the Banach space $L^1 \cap L_w^2$ is a Banach algebra with respect to convolution (cf. [**6**]). However, by the second assumption this space coincides (also as a Banach space) with $L_w^2(\hat{G})$. By consequence the 'potential space' $\mathcal{L}_w^2(G)$ is a pointwise Banach algebra. The assertion thus follows from a general result about multipliers of decomposition spaces (cf. [**19**] , Corollary 2.14).

Our next result will be a fairly general variant of the Hausdorff – Young inequality, now in the setting of generalized amalgams:

THEOREM 9. *Let w, v, α, β and p be as in Theorem 5. Then for $1 < r \leqq p \leqq \infty$,*

$$\mathcal{F}_G(W(\mathcal{F}L_u^p, l_m^r)(G)) \subseteq W(\mathcal{F}L_m^r, l_u^p)(\hat{G}) \text{ with } u := v^\alpha \text{ and } m := w^\beta$$

Proof. Choosing $\theta \in (0, 1)$ such that $1/r = (1-\theta)/p + \theta$ we obtain

$$W(\mathcal{F}L_u^p, l_m^r)(G) = [W(\mathcal{F}L_{v_1}^p, l_{w_1}^p)(G), W(\mathcal{F}L^p, l^1)(G)]_{[\theta]},$$

with $v_1 := v^{\alpha(1-\theta)}$ and $w_1 := w^{\beta/\theta}$, and the result follows from general interpolation principles (cf. e.g. [**1**]).

As already mentioned in [**11**] for the special case of trivial weights the condition $r \leqq p$ are of importance for the validity of Theorem 9. In that note also the "ordinary" Hausdorff-Young theorem for amalgams was derived from such an inequality.

4. Convolution and pointwise algebras, compactness, embeddings. In this section among others those spaces $W(\mathcal{F}L_m^p, L_u^q)$ shall be characterized which are pointwise (Banach) algebras, continuously embedded into $C^0(G)$. Moreover, (compact) embeddings will be described and a new property for Banach spaces of distribution (p-localization) is introduced. Let us start with questions of pointwise multiplication.

THEOREM 10. *Assume that w is a weakly subadditive weight function such that $1/w \in L^{q'}$, then the spaces $W(\mathcal{F}L^p, L_w^q)$ are Banach algebras with respect to convolution. For $1 \leqq p \leqq 2$ these spaces are continuously embedded into $(L^1(G), \| \ \|_1)$.*

Proof. Following the arguments used in the proof of Corollary 8 we realize that the assumptions imply that L_w^q is continuously embedded into L^1 and a

Banach convolution algebra. On the other hand it is clear that $\mathcal{F}L^p(G)$ is a Banach convolution module over $\mathcal{F}L^\infty(G)$ (simply because $L^p(\hat{G})$ is a pointwise module under $L^\infty(\hat{G})$). Applying the convolution theorem for Wiener-type spaces ([**10**] , Theorem 3) we find that $W(\mathcal{F}L^\infty, L^q_w)$ is a Banach convolution algebra, and $W(\mathcal{F}L^p, L^q_w)$ is a Banach convolution module over this algebra. However, locally the spaces $\mathcal{F}L^p(G)$ are ordered by (strict) inclusion for increasing p, which implies that $W(\mathcal{F}L^p, L^q_w)$ is a Banach ideal in $W(\mathcal{F}L^\infty, L^q_w)$. If, in addition, we have $p \in [1,2]$ then the ordinary $\mathcal{F}(L^p(G))$ is contained in $L^{p'}(G)$, hence contained in $L^1_{loc}(\hat{G})$. Since translations act isometrically on $\mathcal{F}L^p(\hat{G})$ it follows that $W(\mathcal{F}L^p, L^q_w) \subseteq W(L^1, L^q_w) \subseteq L^1(\hat{G})$, and the proof is complete.

It is perhaps worth being mentioned next that these spaces are ordered by (strict) inclusions, increasing for p and q going from 1 to infinity. More precisely, we have the following inclusions:

PROPOSITION 11. *The spaces $W(\mathcal{F}L^p_u, l^q_m)$ are ordered by inclusions in the following way:* i) *For $1 \leqq p_1 \leqq p_2 \leqq \infty$ and $1 \leqq q_1 \leqq q_2 \leqq \infty$ one has*

$$W(\mathcal{F}L^{p_1}_{u_1}, l^{q_1}_{m_1}) \subseteq W(\mathcal{F}L^{p_2}_{u_2}, l^{q_2}_{m_2}).$$

ii) *Assume that $p_2 > p_1$, and that the quotient u_1/u_2 belongs to $L^r(\hat{G})$, for some $r \geqq 1/p_1 - 1/p_2$. Then $W(\mathcal{F}L^{p_1}_{u_1}, l^q_m) \subseteq W(\mathcal{F}L^{p_2}_{u_2}, l^q_m)$.*

iii) *Assume that $q_q > q_1$, and that the quotient m_1/m_2 belongs to $L^r(G)$, for some $r \geqq 1/q_1 - 1/q_2$. Then $W(\mathcal{F}L^p_u, l^{q_1}_{m_1}) \subseteq W(\mathcal{F}L^p_u, l^{q_2}_{m_2})$.*

Proof. The monotonicity with respect to the global component, i.e. the weighted sequence spaces follows from the obvious inclusions for weighted sequence spaces by Hölder's inequality. In order to verify monotonicity with respect to the local component observe first that the spaces $\mathcal{F}L^p_u$ are isometrically translation invariant (because multiplications with characters is isometric on L^p_u). Therefore it is sufficient to verify that any compactly supported element $f \in \mathcal{F}L^{p_1}_u$ (with supp $f \subseteq Q$, Q compact in G) belongs to $\mathcal{F}L^{p_2}_u$. Since there exists (cf. above) some compactly supported element $h \in W(\mathcal{F}L^1_v, l^1_w)(G)$ such that $h(z) \equiv 1$ on Q this follows (using once more the convolution relations) from

$$\begin{aligned}\hat{f} = \hat{f} * \hat{h} &\in L^{p_1}_u * W(\mathcal{F}L^1_w, l^1_v) \subseteq W(\mathcal{F}L^1_w, l^{p_1}_u)\\ &\subseteq W(\mathcal{F}L^1_w, l^{p_2}_u) \subseteq L^{p_2}_u \quad \text{for } p_1 \leqq p_2.\end{aligned}$$

This completes the proof of the second part of ii). Assertion iii) is verified in a similar way, using Hölder's inequality.

Combining the above results on inclusions with the natural interpolation results we arrive at other, perhaps more interesting inclusions, yielding sufficient conditions (involving decay conditions on the function and its Fourier transform, i.e. smoothness of the function) for the membership of a given function in such a space under consideration. We do not treat the most general case, but instead a typical one, giving sufficient conditions for the membership of $f \in L^2(\mathbf{R}^m)$ in the Segal algebra $S_0(\mathbf{R}^m) = W(\mathcal{F}L^1, L^1)(\mathbf{R}^m)$.

THEOREM 12. *Let $s, r > 0$ be given, such that* $\min(s, r) > m/2$, *and* $r > ms/(2s-m)$. *Then any $f \in L^2_s(\mathbf{R}^m)$ with $\hat{f} \in L^2_r(\mathbf{R}^m)$ belongs to $S_0(\mathbf{R}^m)$.*

Proof. Note that we have in view of Corollary 8 and the general rules concerning complex interplation that f belongs to

$$W(\mathcal{F}L^2_r, L^2) \cap W(\mathcal{F}L^2, L^2_s)(\mathbf{R}^m) \subseteq W(\mathcal{F}L^2_t, L^2_z) \subseteq W(\mathcal{F}L^1, L^1),$$

whenever we have $t > m/2$, $z > m/2$, and $t = (1-\varphi)r$ and $z = \varphi s$ for some $\varphi \in (0,1)$, which can be found under the given assumptions.

A modification of arguments used in the proof of Corollary 8 gives now a subfamily of spaces under consideration which are pointwise Banach algebras:

THEOREM 13. *Assume that v is a weakly subadditive weight function on $\hat{G}$, with $v^{-1} \in L^{p'}(\hat{G})$, and that $l^q_m \subseteq l^\infty$ (e.g. $m(x) \geqq 1$ for all $x \in G$). Then $W(\mathcal{F}L^p_u, l^q_m)$ is a Banach algebra with respect to pointwise multiplication and the algebra of pointwise multipliers coincides with $W(\mathcal{F}L^p_u, l^\infty)$.*

Proof. Once more we argue (cf. [**6**] for details) that, as a consequence of Hölder's inequality $L^p_u(\hat{G})$ coincides with $L^p_u(\hat{G}) \cap L^1(\hat{G})$, which is a Banach convolution algebra for any weakly subadditive weight v on $\hat{G}$. It follows from the results on pointwise multiplication of Wiener-type spaces (cf. [**19**] or use a direct argument) that $W(\mathcal{F}L^p_u, l^\infty)$ operates on $W(\mathcal{F}L^p_u, l^q_m)$ by pointwise multiplication. On the other hand, $\mathcal{F}L^p_u$ being a pointwise algebra which contains local units (i.e. plateau-like functions, which are uniformly bounded in $\mathcal{F}L^p_u$, together with all their translates) it is clear that a function which defines a pointwise multiplier has to belong locally to $\mathcal{F}L^p_u$, and actually to $W(\mathcal{F}L^p_u, l^\infty)$.

Now it is also possible to formulate the result that the dual spaces of our minimal spaces are maximal spaces under certain invariance conditions. For simplicity let us consider only the case $v = w = 1$:

LEMMA 14. *Given any pair w, v of weight functions on G and $\hat{G}$ respectively. Then the dual space $S_0'(G) = W(\mathcal{F}L^\infty, l^\infty)$ is the maximal subspace of $W(\mathcal{F}L^\infty_{1/v}, l^\infty_{1/w})$ which is isometrically invariant with respect to translations and multiplications with characters.*

Proof. The shortest argument is based on the atomic characterization of the Segal algebra $S_0(G)$ as given in [**15**], [**17**]. Given $\sigma \in W(\mathcal{F}L^\infty_{1/v}; l^\infty_{1/w})$ which is non-zero there exists $f \in W(\mathcal{F}L^1_v, L^1_w)$ such that $\sigma(f) \neq 0$. The isometric invariance (and the appropriate notion of a translate and pointwise product of a functional) imply that $|M_t L_x \sigma(f)| = |\sigma(L_{-x} M_t f| \leqq C$ for all $x \in G$, $t \in \hat{G}$. However, in view of the atomic characterization of $S_0(G)$ (cf. [**15**]) this implies that σ is continuous as a functional with respect to the norm of $S_0(G)$, q.e.d.

In a certain sense one can say that the spaces considered in Theorem 6 are the only spaces which are invariant under Fourier transform and have a certain global behavior. We discuss this in detail for the nonweighted case (corresponding weighted versions are available).

We call a Banach space $(B, \| \ \|_B)$ of functions or distributions (which is assumed to be translation invariant and a pointwise Banach module over some Banach algebra $\mathcal{F}L^1_v$) *p-localizable* if the following condition holds true: Let $(y_i)_{i \in I}$ be any family as in the definition of *BUPUS* (one could call such families *well spread in the group G*). Then one has: $f \in B$ if and only if there is a family $(f_i)_{i \in I}$ in B with common compact support, satisfying $[\sum_{i \in I} \|f_i\|_B^p]^{1/p} < \infty$, such that $f = \sum_{i \in I} L_{y_i} f_i$.

Note that is not difficult to verify that this definition is in fact independent of the family $(y_i)_{i \in I}$ under consideration. Moreover, for any isometrically translation invariant space $(B, \| \ \|_B)$ the above mentioned property is equivalent to the condition that B coincides with the Wiener-type space $W(B, l^p)$. On the other hand it is possible to prove (using either direct arguments or the methods developed in **[19]**) that the spaces $W(B, l^p)$ are p-localizable in the above sense under these circumstances. (Note that our definition has the advantage that a p-localizable space is automatically translation invariant and that the family of translation operators acts uniformly bounded on B; thus B can be endowed with an equivalent, isometrically translation-invariant norm). Of course, the idea is that these spaces are completely known as soon as one knows them just locally. Using this terminology we can show the following result:

THEOREM 15. *The space $W(\mathcal{F}L^p, L^p)(\mathbf{R}^m)$, $1 \leq p < \infty$ is the only p-localizable Banach space of tempered distributions which is invariant under the Fourier transform.*

Proof. It is clear from what has been said so far that this space has the required properties. Assume now, on the other hand, that a p-localizable Banach space $(C, \| \ \|_C)$ is given. Since the uniform boundedness of the family of translation operators, as mentioned above, implies via Fourier transformation that the family of character multiplications acts uniformly bounded on the space as well, it follows (cf. Lemma 14) that $(C, \| \ \|_C)$ as well as $\mathcal{F}C$ is continuously embedded into $W(\mathcal{F}L^\infty, L^\infty)$, in particular, the elements of this space are at least locally pseudomeasures (thus they are so-called quasimeasures), and consequently $\mathcal{F}C \hookrightarrow W(\mathcal{F}L^\infty, L^p)(\mathbf{R}^m)$. Applying the generalized Hausdorff-Young inequality for Wiener-type spaces (for the inverse Fourier transform, cf. Theorem 9) it follows that $C \subseteq W(\mathcal{F}L^p, L^\infty)(\mathbf{R}^m)$, which in turn implies that the local structure of C is at least $\mathcal{F}L^p$, hence $C \hookrightarrow W(\mathcal{F}L^p, L^p)(\mathbf{R}^m)$.

In order to show the converse we may start with the minimality property of $S_0(\mathbf{R}^m)$, which implies (the invariance properties mentioned above) that $S_0(\mathbf{R}^m) \hookrightarrow C = \mathcal{F}C$, hence $\mathcal{F}^{-1}W(\mathcal{F}L^1, L^p) \subseteq \mathcal{F}^{-1}C = C$, due to the p-localization property. However, in view of Theorem 5 of **[10]** this tells us that the Fourier transforms of elements of $L^p = W(\mathcal{F}L^p, L^p)$ are locally the same as those of $W(L^1, L^p)$. We thus obtain that compactly supported elements from $\mathcal{F}L^p$ are contained in C (with a embedding constant depending only on the support). Applying once more the p-localization it follows that one has actually $W(\mathcal{F}L^p, L^p)(\mathbf{R}^m) \hookrightarrow C$, and the proof is complete.

5. Concluding remarks. There are many remarks that could be made. We shall restrict ourselves to the most interesting ones.

A) Most of the results of the present note could also be translated into the setting of the so-called modulation spaces, which are just inverse images under the generalized Fourier transform of the spaces $W(\mathcal{F}L^p_u, L^q_m)$ (cf. the more extensive technical report [**13**], [**14**], or [**31**]). These papers contain also information concerning trace theorems for these spaces, which are quite similar to those for the (inhomogeneous) Besov spaces.

B) As has been seen in the discourse the spaces considered above, especially of the spaces without weights, the two families of operators, namely translations and modulation (multiplication with characters) and the behavior of these spaces under these operators (e.g. p-localizability has to do with a certain behavior under translations) play a significant role in their description. Since these operators may be combined to a group, the so-called Weyl-Heisenberg group, it is possible to look at these spaces from the point of view of non-commutative harmonic analysis (cf. [**20**] for a discussion of the spaces $W(\mathcal{F}L^p, L^p)(\mathbf{R}^m)$ from this point of view).

C) The description of these spaces has been given, starting from a selective description, i.e. the elements under consideration had been selected from the big dual spaces (of certain minimal spaces), thus they are actually spaces of ultradistributions in the sense of Beurling-Björck for the natural pairs of weights w, ν in their definition (cf. [**3**], [**29**], [**30**] for more details). This reminds one of Peetre's K-method of real interpolation (cf. [**1**], [**24**]), where the elements of a new space are those which show a certain behavior of their K-functional (which is a function on $\mathbf{R}^+$). In the present case we had a different class of seminorms, and our selective criterion is another one, namely weighted summability over a lc. group. In a certain sense dual to the selective K-method one has the equivalent J-method, which can be considered a "constructive" method. One has also a constructive description of the spaces, i.e. atomic characterization of these spaces. A more direct approach to such decompositions is given in [**17**]. Another, more abstract, but also more general approach to atomic decomposition (at the same time stressing the analogy between various similar looking families of function spaces, such as the Besov-Triebel spaces and the Wiener-type and modulation spaces) is considered in [**20**] and [**19**].

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