1989/4

NATO - Proc.

427 - 440,

Kluwer

COHERENT FRAMES AND IRREGULAR SAMPLING

Hans G. Feichtinger

Department of Mathematics
University of College Park [1]
MD, 20742 , U S A

Abstract. It is the purpose of this short note to highlight perhaps unexpected connections between the topics described in the papers [G2] *and* [HW1] *in this volume, and at the same time between various papers within a series of joint publications* [FG1-8] *with K.Groechenig on the two topics indicated in the title: Algorithms that allow to recover a function (or tempered distribution)* f *from a suitable family of coefficient, which arise as integrals of* f *against a countable coherent family of functions (such as Heisenberg or affine frames) and the problem of reconstructing a band-limited function from a (sufficiently rich) set of irregularly taken sampling values. As will be pointed out in detail below the basic observation, which may be taken as an explanation of most common results for these two settings, concerns properties of functions which arise as convolution products with nice, integrable functions on a locally compact group.*

INTRODUCTION

The paper is going to describe in which sense the constructions of frames (as described in [DGM],[HW1,2],...) can be seen as a result related to the classical Shannon sampling problem. In both cases the "classical" approach puts its main emphasis on the use of Hilbert spaces methods (such as orthonormal bases, frames, positive operators), using various formulas (such as Poisson's formula, expansion of functions into Fourier series, ..), based on the fact that the sampling points or the parameter sets involved in describing these frames form a regular lattice.

As will be pointed out a *group theoretical approach*, using convolution relations between suitable function spaces on the Euclidean space $\mathbb{R}^m$ or suit-

[1] Permanent address: Dept.Math.,Univ.Vienna,Strudlhofg.4. A-1090 Wien, AUSTRIA

able locally compact groups naturally associated with these frames gives a much more flexible tool, which allows much more freedom in the choice of "mother wavelets" and sampling coefficients than one might expect from the "regular" methods. Also, these methods (which are iterative ones) are not restricted to the Hilbert space setting (thus a number of other function spaces, including many classical ones, such as Besov spaces,...) can be covered in the discussion. Finally, the methods allow error and stability analysis with respect to the "right" norms, i.e. whenever a function belongs to a reasonable normed and invariant function space, that the error with respect to the norm of this space can be estimated .

As giving background information and pointing out connections is the main concern of this note we only state few of the main resultsof the two topics under discussions, in a non-technical (hence not most general) form. Thus we shall restrict the discussion here to spaces related to (unweighted) L^p-spaces, because in this setting the relevant theory is already available in the literature (cf. [FG1,5]) and does not require complicated technical explanations.

We recall that for $1 \leq p < \infty$ the space $L^p(\mathbb{R}^m)$ consists of all measurable functions F on $\mathbb{R}^m$ such that $\|F\|_p := [\int_{\mathbb{R}^m} |F(x)|^p dx]^{1/p} < \infty$. We use the symbol T_y for the translation operator, given by $T_y F(z) := F(z-y)$). We under-stand the Fourier transform $\mathcal{F}F = \hat{F}$ of an integrable functions $F \in L^1(\mathbb{R}^m)$ is given by the usual integral formula. It is well known that the Fourier transform $\mathcal{F}$ may be extended to a mapping from $L^p(\mathbb{R}^m)$ for $1 \leq p \leq 2$, whereas one has use distribution theory (tempered distributions) in order to define the Fourier transform for p>2 (cf. [F4] for an alternative) .

Thus the spectrum of a function is well defined as the support of the Fourier transform: $\mathrm{spec}(F) := \mathrm{supp}(\hat{F})$. A function in L^p is called <u>band-limited</u> (to some bounded set Ω) if $\mathrm{spec}(F) \subseteq \Omega$. Thus $\mathrm{spec}(F) \subseteq \Omega$ if $\hat{F}$ vanishes on the complement of Ω. It is also well known that band-limited functions are continuous (by the Paley Wiener theorem in fact analytic), so pointwise evaluations make perfect sense for these functions.

We start with the description of the most simple version of our problem, the so-called <u>(regular) sampling problem.</u> We first recall it in a slightly more general form than usually presented in textbooks (cf.[G2] for more details, and see [BSS],[H],[J] for surveys of this topic and its relevance).

Theorem : There exists some positive constant C_o (only depending on the definition of the Fourier transform involved) such that for any $C > 0$ the following is true: Given any $\gamma \leq C_o C^{-1}$ (the sampling rate), such that any band-limited function $F \in L^p(\mathbb{R}^m)$, for $1 \leq p < \infty$, with $\hat{F}(t) = 0$ For $|t| \geq C$ can be completely recovered From the sampling values $(F(\gamma \cdot n))_{n \in \mathbb{Z}^m}$. Actually, there is an explicit formula, telling us that

$$F = \sum_{n \in \mathbb{Z}^m} F(\gamma \cdot n)\, T_{\gamma \cdot n} G \quad (*)$$

whenever G is a band-limited and integrable function, satisfying $\hat{G}(t) \equiv 1$ on Ω , with $\hat{G}$ having sufficiently small support. Moreover, the series converges in the pointwise sense (uniformly over bounded sets) as well as in the L^p-sense.

Proof. Although this result can be easily obtained as a minor modification of the usual Shannon sampling theorem we have no simple reference available. Thus we indicate the main arguments (which actually work also in the setting of weighted L^p-spaces) to be used (besides Poisson's formula, of course):

First we observe that for band-limited functions (with bounded spectrum) in L^p the sampling values $(F(\gamma \cdot n))_{n \in \mathbb{Z}^m}$ are p-summable, in fact $(\sum_{n \in \mathbb{Z}^m} |F(\gamma \cdot n)|^p)^{1/p} \leq C \cdot \|F\|_p$ for all $f \in L^p(\mathbb{R}^m)$ with fixed compact spectrum (due to Theorem 3.7.i) and Proposition 3.4. in [FG7], cf. also Remark 2.5 in [FG5]). Using now some notations from [FG5] we may interpret the series as a convolution of the discrete measure $\sum_{n \in \mathbb{Z}^m} F(\gamma \cdot n)\, \delta_{\gamma \cdot n}$ (which is convergent in ML^p) with the function G . Since we have assumed $G \in L^1$ to be band-limited, the local maximal function $G^{\#}$ also belongs to L^1 and thus the convolution product in (*) is convergent in CL^p, in particular in L^p and locally uniform over compact sets (by Theorem 2.1.iii) and ii) of [FG5]).

Finally we mention, that the dependency of the necessary sampling density from the band-width (size of the spectrum of f = support of f) is easily obtained by proving the result first for Ω being the unit cube and by reducing the problem to that case by means of dilations in the general case.

Remark: For irregular (we should call them quasi-regular here) lattices, which can be obtained from a regular one by the application of a regular matrix, such as rotation or (anisotropic) dilation the same arguments go through.

The proof of this so-called Whittaker-Shannon Sampling theorem on regular sampling is usually based on Poisson's formula (in a more or less disguised way). Since there is no substitute for Poisson's formula for irregular sets (this has been shown recently by Cordoba) it is clear that one has to use a completely different approach in the irregular setting. In our papers [FG5-8] we were able to show that the reconstruction is still possible (even if the sampling points are completely irregular)supposing that the sampling density (which is defined in the most natural way) is high enough (in relationship to the size of the spectrum). Several variants of a possible constructive approach are described in more detail in [G2] and [FG5-8]) have been found. One version of the resulting theorems reads as follows:

<u>Theorem :</u> Let Ω be some compact subset of $\mathbb{R}^m$. Then there exists an open neighborhood U (only dependent of Ω), such that for any U-dense, discrete family $\Lambda = (\lambda_i)_{i\in I}$ of points in $\mathbb{R}^m$ (which means that the system $(\lambda_i+U)_{i\in I}$ covers $\mathbb{R}^m$) any band-limited function $F \in L^p(\mathbb{R}^m)$ with $\mathrm{supp}(\hat{F}) \subseteq \Omega$, can be completely reconstructed from the irregular sampling values $(F(x)_i)_{i\in I}$.

As a starting point for our arguments we took the following observation:

A function $F \in L^p(\mathbb{R}^m)$ is band-limited if and only if there exists some other function $G \in L^1(\mathbb{R}^m)$ (which may be assumed to be band-limited itself) such that $G*F = F$. In fact, if we choose $G \in L^1(\mathbb{R}^m)$ such that $\hat{G}$ equals 1 over spec(F) (e.g. some De la Vallée-Poussin kernel), then $\hat{F} = \hat{G}\cdot\hat{F}$. Assume conversely that this holds true for some $G \in L^1(\mathbb{R}^m)$ and some F with unbounded spectrum. Then $\hat{G}$ has to take the value 1 at points arbitrarily far from the origin, which is not possible in view of the Riemann Lebesgue Lemma (stating that the Fourier transform of an integrable functions tends to zero at infinity).

Actually, we were using this reproducing convolution equation for F as our starting point (also not relying on methods from complex analysis). At present there are different methods and algorithms, using this equation in order to recover F from it's sampling values.

The first approach (described in [F3] for weighted L^1-spaces, and in [FG5] for the most general situation, including weighted L^p-spaces) was putting the main emphasis on an iterative approach. The main idea being the following: Use the given information (the sampling values), to approximate the function by some auxiliary function, involving only the sampling values. This can be either (in the most simple case) be a step function (taking the value $F(x_i)$ near x_i (cf. $V_X F$ in [FG6]), or some more sophisticated spline type function ($Sp_\Psi F$, used in [FG5]). Since a band-limited function is quite smooth we may expect that such a function will yield a good approximation to our given function F. But how to recover the remainder? We cannot directly take the remainder term and iterate, because that remainder term does not have any smoothness (at least if we use step functions). In order to circumvent this problem we smooth that first and simple approximation out, in order to get a smooth remainder term, which allows to apply the same procedure over and over and to recapture finally F completely from the sampling values given. In fact, since F itself is not changed by convolution with G we may expect that any approximation to F will still be a good approximation to F after convolving it by G. Altogether, one only has to take care that in this procedure the loss of approximation (due to that additional smoothing) and the win (obtained by having a smoother remainder term for the next step) are in a good balance (so we reduce our approximation quality by investing a small part of it into additional smoothness which in return allows us to keep the machinery running).

The theorem above actually applies to more general situations, based on the same arguments. Thus convergence can be shown to hold true with respect to certain weighted L^p-norms if F belongs to these classes (which simply means that both decay properties and summability properties of f imply better convergence of the series near infinity). In the argument G has to be taken in a suitable weighted L^1-space in that case. The presentation in [FG6] gives the shortest available proof for a (still general) special case. In [FG7] an operator theoretic setting has been choosen to describe the most general version. This is also the basis for the error estimates given in [FG8].

An interesting variant of this result, which might me of great practical use, is the following one (cf. the section on error analysis):

<u>Corollary.</u> In the above situation there exists a bounded family $(E_i)_{i\in I}$ in L^1 (actually uniformly decaying, continuous functions) such that any $F\in L^p(\mathbb{R}^m)$ with $spec(F) \subseteq \Omega$, $1 \leq p < \infty$, can be written as

$$F = \sum_{i\in I} F(x_i)E_i ,$$

with unconditional convergence in $L^p(\mathbb{R}^m)$ and uniformly over compact sets.

Coming to the next topic now we have to speak about <u>coherent</u> <u>frames.</u> Actually, the concept of a frame in a Hilbert space is a classical one (cf. [DS] for a basic account of properties of frames, and [Y] for applications in the theory of non-harmonic Fourier series). It underwent a glorious revival after the discovery of the "wavelet orthonormal system" by Y.Meyer (cf. [M1,2], [LM], [D2].. for details). These are orthonormal systems for $L^2(\mathbb{R}^m)$ having the useful property of being generated from a single function g (called the mother wavelet) by means of elementary operations only, i.e. translations by elements from the lattice $\mathbb{Z}^m \subseteq \mathbb{R}^m$ and dilations (of the argument by powers of 2). Thus the typical basis vector is given by $g_{k,n}(x) := 2^{-m/2}g((x-k)/2^m)$. We consider this family as a <u>coherent</u> one, coherence being understood as the possibility of applying certain elements from a (continuous) group of transformation to one single function in order to obtain the full system. In the wavelet case these transformations are of course unitary transformation induced on $L^2(\mathbb{R}^m)$ induced by the the affine transformations of the argument. We only mention in passing that the notion of "coherent states" is an important concept in quantum mechanics and is the place where this terminology comes from (see [HW1,2] for details).

The observation, that under certain circumstances (e.g.the Heisenberg setting) the requirement of orthonormality turns out to be too strong in companion with coherence, smoothness and decay properties apparently lead to the discussion of frames. First the so-called tight frames came up. These are families $(g_i)_{i\in I}$ in $L^2(\mathbb{R}^m)$ which behave very much like an orthonormal system without being a basis in the usual sense. By definition a <u>tight</u> <u>frame</u> (cf.[DGM]) is identified by the property that $\|f\|_2 = (\sum_{i\in I}|\langle f,g_i\rangle|^2)^{1/2}$. As a simple functional analytic consequence one obtains $f = \sum_{i\in I}\langle f,g_i\rangle g_i$. Thus, in order to expand a given function f one only has to calculate the coefficients $\langle f,g_i\rangle$ and to write the above sum, as if we had an orthonormal bases. As the most simple example of a coherent tight frame one may take as a Hilbert space $\mathbb{R}^2$, and (up to normalization) the unit vectors $(e_1,e_2,-e_1,-e_2)$, which are coherent, because they aris under the action of rotation by multiplies of 90° from e_1. This set is of course no basis (so expansions are not unique), however we are free to require additional properties on the coefficients (in this simple case we could ask for positivity).

An important fact concerning these tight frames is the following. All information about a function is contained in these coefficients, and one can say, if one has an affine tight frame (arising as above by translates and dilates only), that the coefficients $\langle f,g_{k,n}\rangle$ for fixed an (and running k) correspond to the contribution to f at a scaling (or smoothness) level n, also indicating where (depending on the indices k for which large coefficients arise) in the function f relevant contributions at this fixed level prevail. It allows also to find partial sums of the double sum, involving only comparatively few terms and representing nevertheless the function to a variable degree at different places (very precisely on some interval, and only roughly at infinity, for example).

This requirement of storing all information in coefficients (together with the possibility of recovering f "easily" from these coefficients) can also be considerably weakened. Th4e general notation of a <u>frame</u> $(g_i)_{i\in I}$ in the Hilbert space $L^2(\mathbb{R}^m)$ only requires to have constants $A,B > 0$ such that

$$A\cdot\|f\|_2 \leq (\sum_{i\in I}|\langle f,g_i\rangle|^2)^{1/2} \leq B\cdot\|f\|_2 \quad \text{for all} \quad f \in L^2(\mathbb{R}^m) .$$

It is then still possible to recover f by means of an iterative procedure (described in detail in the referred papers).

It was Gröchenig who extended the notion of a frame to a larger class of spaces (so making it a Banach spaces concept as opposed to a Hilbert space notion which is was up to that time). In fact, if we restrict our attention to coherent frames (with few technical restrictions) then it is possible to speak of a <u>frame for a Banach space</u> of functions (actually a coorbit space) by means of an associated Banach space of sequences, if the coefficient mapping allows to embed the Banach space in a complemented way into that sequence space. So, in the case of the Hilbert space the associated Banach space if of course the expected one, namely the space of square summable sequences l^2. Again, there are in this setting norm convergent (iterative) methods to recover the function from its coefficient.

The following turn-around brings us to the heart of the matter, showing that the question of coherent frames (as described in [DGM],[HW],..) is close related to the irregular sampling problem.

As we mentioned, coherent systems arise from a (continuous) group of operators. We formalize the setting by assuming the we have (unitary) operators T_x, the index x being taken from some group $\mathcal{G}$, fulfilling the rules $T_{x^{-1}} = T_x^*$ (the adjoint corresponds to the inverse group element) and

$T_x \circ T_y = T_{x \cdot y}$ (i.e. a group representation). Usually $\mathcal{G}$ can be thought of as a matrix group, so we may use for the wavelets (we describe for simplicity only the case m=1) the group of affine transformation of the real line, given by $x \longmapsto ax+b$, $a > 0$, $b \in \mathbb{R}$, and $T_{(a,b)}$ associated with $x=(a,b) \in \mathcal{G}$ by:

$$f \longmapsto T_{(a,b)}f, \text{ with } T_{(a,b)}f(x) := a^{-m/2}g((x-b)/a).$$

Fixing now some "nice" function g_o we can define the following function on the group $\mathcal{G}$ (depending on your background you may call it either a representation coefficient of the above representation, or simply a <u>continuous wavelet transform</u>, also called <u>generalized wavelet transform</u>, if the domain is not only an Hilbert space, but some space of distributions :

$$f \longrightarrow F, \quad \text{with} \quad f(x) := \langle f, T_x g \rangle .$$

In this situation F is a continuous function on the group $\mathcal{G}$ .Since any locally compact group carries a unique (left) translation invariant measure, the Haar measure on $\mathcal{G}$, the notion p-integrability with respect to that Haar measure ($F \in L^p(\mathcal{G})$) makes sense, and actually F is square integrable for the elements $f \in L^2(\mathbb{R}^m)$. Also, convolution make sense, being defined by $F*G(x) = \int_{\mathcal{G}} G(y^{-1}x)F(y)dy$ for decent functions F,G. Moreover, as in the case of $\mathbb{R}^m$ we have estimates of the form $\|G*F\|_p \le \|G\|_1\|F\|_p$, but one has to careful (if the group is not unimodular, i.e. if left and right invariant Haar measures are not the same for $\mathcal{G}$) that one has in general only $\|F*G\|_p \le \|G\|_{1,w}\|F\|_p$, for some weighted L^1-norm $\| \ \|_{1,w}$. Having a "nice" function means now to assume that the function G (i.e. the one associated with g itself by the mapping to induced via g) is integrable over $\mathcal{G}$. In practice this corresponds to decay and/or smoothness or moment conditions on g (cf. [FG1] for details). Moreover, it is possible to recover f from F by some inversion formula (involving again g).

Speaking now about about having a coherent frame of the form $(\pi(x_i)_{i\in I})$ turns out to be equivalent (if appropriately defined) to say: f is completely determined by the sampling values of F at the points of $X = (x_i)_{i\in I}$ in $\mathcal{G}$, if $F \in L^p(\mathcal{G})$, for example. Looking at the examples of the wavelet theory and other papers on (tight) frames (cf.[DGM],[FJ],...) one might get the impression that the sets X arising there have to be lattices in the group (not in the sense of discrete subgroups with compact quotient, but still lattices in a very strict geometric sense). This is actually useful for labeling the points by double indices and allows to use certain formulas, but at the same time restricts the flexibility of choosing X). So what what about sampling information from irregular sets X? Or what can be said about continuous wavelet transforms based on wavelets g satisfying no special properties. The answer turns out to by quite similar to the above one on band-limited functions: Given g (which somehow corresponds to the information of the size of the spectrum in the band-limited case) we can find some neighborhood U of the neutral element in the group (the size of U depends only on certain smoothness and decay properties of G, not on p) such that any U-dense family X (which now means of course that the family $x_i \cdot U$ covers $\mathcal{G}$) gives rise to a coherent frame. In other words, f can be completely be recovered from the coefficients $\langle f, T_{x_i} g \rangle, i \in I$ in this case if $f \in \mathcal{C}o(L^p) = \{f \mid F \in L^p(\mathcal{G})\}$.

BASIC CONNECTIONS

In order to reveal now the connection between the two topics and to to stress at the same time the importance of coherence the and of the group theoretic background in this theory we add the following informations (cf.[FG1,2] for details, or [F4]). Under the conditions described (i.e. g to be "nice") we actually have the following facts:
G belongs to the weighted L^1-space $L^1_w(\mathcal{G})$, and the p-integrable functions F arising as continuous wavelet transforms via g can be characterized (among all elements in $L^p(\mathcal{G})$ by the reproducing convolution equation

$$F = F * G .$$

In particular, G is a (symmetric) convolution idempotent, hence the mapping $H \longmapsto H * G$ from $L^p(\mathcal{G}))$ into itself is a projection mapping. This it is not completely unlikely (and as has been shown in fact true) that one can recover F from its sampling values by pretty much the same type of arguments that one can use in the irregular reconstruction problem. Of course, the concept of uniform density of the sampling points X has to be understood in the group theoretical sense now (thus with respect to the hyperbolic metric in the case the upper half plane, identified with the group of affine transformation, the so-called ax+b group), smoothness of F is not described in terms of derivatives (since we do not assume to have a Lie group structure, which however would be available in most examples), but using a suitable family of spaces (the so-called Wiener type spaces) or concepts of local maximal functions and oscillations over groups allows to follow similar patters in both settings at this point (they were introduced in [F1] in full generality, and the underlying method is that of splitting functions into equal parts.

IRREGULAR SAMPLING OF THE SHORT TIME FOURIER TRANSFORM

Although the question of recovering a function from irregular sampling of its STFT (Short Time Fourier Transform) looks more like a problem similar to the sampling problem of band-limited functions at a first sight (Fourier transform being involved), it turns out to be related to the sampling problem of generalized wavelet transforms over the Heisenberg group.

Recall that for a function f (assume for simplicity in $L^2(\mathbb{R})$) the STFT with respect to the window function g (which we will assume to be nice, in the sense of compactly supported and continuous, for example) is defined by $STFT_g(f)(x,s) := \mathcal{F}((T_x g)f)(s)$. A priori, it is understood as a (bounded an continuous) function on $\mathbb{R}^2$ (the so-called <u>time-frequency plane</u>). A reconstruction method, allowing to reconstruct f (given g) from the sampling values of $STFT_g(f)$ at a sufficiently small regular lattice in $\mathbb{R}^2$ was given by [B]. In order to obtain the irregular sampling theorem in this situation we indicate only shortly here how to reinterpret the STFT. In fact, the so-called Heisenberg group acts in a very natural way on $L^2(\mathbb{R})$ by means of translation operators plus modulation operators (cf. [FG1],[G1],[HW1,2]) and (in order to get the group law) multiplications with complex numbers of absolute value one (so the <u>Heisenberg group</u> as a set is just $\mathbb{R}^2 \times \mathbb{T}$, where $\mathbb{T}$ denotes the torus group). In an abstract setting this action is defined as the Schrödinger representation of the Heisenberg group. Now looking at the definition of the generalized wavelet transform with respect to this representation (cf. [GF1], Ex.7.1) one can check that for any function f (fixing the analyzing wavelet g) the function $STFT_g(F)$ coincides with the function $F(x,t,1)$. Since $F(x,t,u) = u \cdot F(x,t,1)$ knowledge about the sampling values of $STFT_g(f)$ at sufficiently many points in the time frequency plane gives also sufficient information about F and allows therefore to reconstruct F (and therefrom f) by means of an iterative procedure. Besides the irregularity of the sampling points for $SFTF_g(f)$ (the density depends only on g, not on f !) it is remarkable, that for this reconstruction only minimal requirements on g have to be made.

STABILITY AND ERROR ANALYSIS

The group theoretical approach to coherent frames also allows to cope with questions of stability and to give error estimates in the right norms.

Thus, typically one can handle (using the appropriate function spaces on the acting group $\mathcal{G}$) questions of the following type ("jitter error problem"):

To what extent does the reconstruction deliver a wrong variant of f (the difference being measured with a very wide range of norms), if the sampling values (or the coefficients are not picked up by means of the functions $T_{x_i} g$, as assumed, but instead by some "close by" functions such as $T_{y_i} g$, with elements y_i being close to the elements x_i (in the sense that there is a small neighborhood V t of the neutral element in $\mathcal{G}$ such that $y_i \in x_i V$ for $i \in I$).

The stability considerations are of interest in connection with the description of the reconstruction methods in the form $F = \sum_{i \in I} F(x_i) E_i$ (cf. Corollary above; see [F5],Cor.A', [F6],Cor.4.3, or [F7], Theorem1) or $f = \sum_{i \in I} \langle f, T_{x_i} g \rangle e_i$ in the frame setting (cf.[G1],5.10), with suitable families of functions E_i on $\mathcal{G}$ or e_i on $\mathbb{R}^m$. These functions can be precalculated (given only the information about the sampling geometry, i.e. the set of points X, and the reproducing function G), which is an aspect of possible great practical use (in a situation where one has to recover many functions with equal smoothness (in the sense of having the same reproducing function G)from their respective sampling values, taken over the same set of sampling points. In this situation the a priori calculation of these functions E_i could be done on a main-frame, whereas the stored values of the functions E_i could be used to recover F by simple summation (without going into the iterative procedure for each F individually).

In this setting of course the question comes up, to what extent does the particular form of the functions E_i depend on the parameters involved. So, stability in this setting means, that minor changes of the sampling geometry or starting the procedure with a function G_1 only slightly different from a given function G results in an overall system $(E_i^1)_{i \in I}$ of (reconstructing) functions which are in some uniform sense not far from the original functions $(E_i)_{i \in I}$.

LOOKING BACK

In contrast to the presentation given above our approach to both problems (within the last three years) went the other way round. We first worked on atomic decompositions in functions spaces (one should speak perhaps better of non- orthogonal series representations with respect to coherent systems of atoms), which was based on the convolution equation $F = F * G$ and the transfer method available through the generalized wavelet transform (associated with integrable group representations, as pointed out in [FG1-3]). Observing then the connection to the theory of frames, Groechenig pushed the theory of frames to the same general setting (not anymore restricted to Hilbert spaces or regular lattices) which has turned out to be the most general and most natural one for atomic decompositions along coherent systems. [G1]. During these discussions we found the explained analogy with the situation arising in the context of band-limited functions an inspiring background. Working then on the irregular sampling problem we also found that this point of view allows to develop parallel the reconstruction problem methods of expanding band-limited function into a series of functions, each of them obtained from a single function g using only certain of it's translates (cf. [FG6], and [FG7] for the most general version of these results), which again is intimately related to the general atomic decomposition problem solved in [FG2].

Our first hope, that the technical background would be much easier on $\mathbb{R}^m$ turned out to be misleading in one very important point: Whereas one has plenty of integrable idempotents (for convolution) on any group having an integrable and irreducible representation (such as the affine group, the Heisenberg group, but also the $SL(2,\mathbb{R})$ group, mentioned in [FG1]) there are no such function on $\mathbb{R}^m$ ($g \in L^1$ cannot be an idempotent with respect to convolution due to the Riemann Lebesgue Lemma). On the other hand, particular choices of function G satisfying $F = F*G$, such as Schwartz functions G, allow to invoke more elegant estimates in order to verify convergence of the iterative reconstruction method, not available on groups.

SUMMARY

We have tried to point out that from an harmonic analyst's point of view there is a close relationship between previous constructions of coherent (tight) frames or orthonormal wavelet bases and to the regular sampling problem for band-limited function. The same kind of relationship can be found between the results, saying that any nice function can be used to construct a frame as long as the sampling rate on the group is high enough, and the reconstruction of band-limited functions in L^p-spaces from irregular sampling values. The background is in each case a convolution equality and the fact, that it is possible (under various circumstances) to recover functions F satisfying $F = F*G$, for some nice and integrable G, from any family of sampling values, as long as the sampling density of these points is high enough.

In the case of coherent frames this strategy (which applies to the continuous wavelet transform) has to be complemented with a transfer method (allowing to transfer questions about f to questions about an appropriate continuous wavelet transform and transferring the result back to f), but this can be done using the machinery of coorbit spaces and the appropriate inversion formulas given in [FG2]. Moreover, since the iterative methods involved in this process work for a large variety of translation invariant function spaces on any locally compact group the range of application of this technique, which also allows good error estimates and the verification of a variety of stability results, is not restricted to the Hilbert space setting.

REFERENCES

[B] Bastiaans, M.J.: Signal description by means of local frequency spectrum SPIE 373, Trans.in Opt.Sign.Proc. (1981), 49-62.

[BSS] Butzer,P. W.Splettstößer and R.L.Stens (1988): The sampling theorem and linear prediction in signal analysis. Jber.d.Dt.Math.-Ver. **90**, 1-70.

[D1] Daubechies, I: The wavelet transform, time-frequency localization and signal analysis.

[D2] - " - : Frames of coherent states, phase space localization, and signal analysis, preprint.

[D3] - " - : Orthonormal bases of compactly supported wavelets, Comm.Pure Appl.Math. 41 (1988), 909-996.

[DG] Daubechies, I. and A.Grossmann: Frames in the Bargmann space of entire functions.

[DGM] Daubechies, I., A. Grossmann and Y.Meyer: Painless on-orthogonal expansions. J.Math. Phys. 27 (1986), 1271-1283.

[DS] Duffin,R; and A.Schaeffer : A class of nonharmonic Fourier series. Trans. Amer.Math.Soc. 72 (1952), 341-366.

[F1] Feichtinger H.G.: Banach convolution algebras of Wiener's type. "Functions, Series, Operators", Proc.Conf., Budapest 1980, Coll. Soc.Janos Bolyai, North Holland, 1983, 509-524.

[F2] - " - : Discretization of convolution and reconstruction of band-limited functions from irregular sampling. J.Approx.Theory, to appear.

[F3] - " - : Nonorthogonal expansions using the Heisenberg group, with applications to signal analysis. Techn.Report, prepared for a talk presented at AT&T Bell Labs.. Murray Hill, April 1988.

[F4] - " - : An elementary approach to the generalized Fourier transform, in "Topics in Mathematical Analysis", World Sci.Publ., Ed. Th.Rassias, Athens, 1989; in honor of Cauchy, 200nd anniversary.

[FG1] Feichtinger, H.G. and K.Gröchenig: A unified approach to atomic characterizations via integrable group representations. Proc. Conf. Lund June 1986, Ed. M.Cwikel et al, Springer Lect.Notes 1302, (1988), 52-73.

[FG2] - " - : Banach spaces related to integrable group representations and their atomic decompositions,I. J.Funct.Anal., to appear, 1989.

[FG3] - " -: Banach spaces related to integrable group representations and their atomic decompositions,II. Monatsh. f. Mathematik, to appear. 1990.

[FG4] - " - : Non-orthogonal expansions of signals and some of their applications. Proc. ECMI conference, Strobl, Mai 1989.

[FG5] - " -: Reconstruction of band-limited functions from irregular sampling values (1989). To appear.

[FG6] - " - :Multidimensional irregular sampling of band-limited functions in L^p-spaces, Proc.Conf. Oberwolfach, Feb. 1989, ISNM, Birkhäuser Publ., 1989.

[FG7] - " - : Irregular sampling theorems and series expansions of band-limited functions I (1989), to appear.

[FG8] - " - : Irregular sampling theorems and series expansions of band-limited functions II. (stability results and error analysis), in preparation.

[FJ1] Frazier,M. and B.Jawerth: Decomposition of Besov spaces. Indiana Univ.Math.J. **34** (1985), 777-799.

[FJ2] - " -: The φ-transform and decompositions of distribution spaces, Proc. Conf. "Functions Spaces and Applications", Lund 1986, Lect.Notes Math. 1302, Springer, Heidelberg, 1988.

[G1] Gröchenig, K.H: Describing functions: Atomic decompositions versus frames, to appear.

[G2] - " - : A new approach to irregular sampling; this volume.

[GMP] Grossmann, A. J.Morlet and T.Paul: Transforms associated to square integrable group representations I, J.Math.Phys. 26 (1985), 2473-2479.

[HW1] Heil, C. and D. Walnut: Continuous and Discrete wavelet transform. To appear in SIAM Rev., 1989.

[HM2] - " - : Gabor and wavelet expansions. This volume.

[H] Higgins, J.R: Five short stories about the cardinal series. Bull. Amer. Math.Soc. 12(1985), 45-89.

[J] Jerri, A.J.: The Shannon sampling theorem - its various extensions and applications, a tutorial review. Proc. IEEE 65 (1977), 45-89.

[LM] Lemarié, P. and Y.Meyer: Ondelettes et bases hilbertiennes, Rev.Mat. Iberoamericana 2 (1986), 1-18.

[M1] Meyer, Y.: De la recherche petroliere a la geometrie des espaces de Banach en passant par les paraproduits. Sem.Equ.Der.Part. 1985/86, Ecole Polytechn. Paris.

[M2] Meyer, Y.: Principe d'incertitude, bases hilbertiennes et algèbres d'opérateurs. Sém. Bourbaki 662 (1985-86).

[P] Papoulis, A: Error analysis in sampling theory. Proc. of the IEEE 54/7 (1966), 947-955.

[Y] Young, R: An Introduction to Nonharmonic Fourier Series. Acad.Press, New York, 1980.