Wiener Amalgams Over Euclidean Spaces And Some of Their Applications

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1 Introduction

Wiener amalgams have been introduced by the author in 1980 ([F1,2]). The concept was aimed at the possibility of describing local and global properties of a function or distribution independently (allowing to speak of increasing smoothness at infinity . . .). For a very readable survey of 'ordinary' amalgams, built up on local L^p- and using global l^q-spaces, see the article by J. J. Fournier and J. Stewart [FSt], explaining applications of amalgams to a wide range of problems in analysis. In the present paper further applications of Wiener amalgam spaces are explained, among them various results which use these spaces only for the proof, but not in their statements. Much of the material presented here was influenced by the experiences the author has made using Wiener amalgam spaces while proving results on atomic decompositions and during the work on the irregular sampling problem of functions (joint papers with K. Gröchenig).

2 A Review of Wiener Amalgam Spaces

We start with a description of the general approach to Wiener amalgam spaces (formerly called Wiener type spaces or generalized amalgams). We restrict our attention in this paper to an important subclass, i.e. to those Wiener amalgam spaces on $\mathbb{R}^m$ which are spaces of tempered distributions. In the final section we shall indicate some features of Wiener amalgams over locally compact groups and their applications.

We denote the Schwartz space of rapidly decreasing functions by $\mathcal{S}(I\!\!R^m)$. It is endowed with a natural family of seminorms, and the space $\mathcal{S}'(I\!\!R^m)$ of tempered distributions is its topological dual. $\mathcal{K}(I\!\!R^m)$ denotes the space of all continuous, complex-valued functions with compact support on $I\!\!R^m$, hence $\mathcal{D}(I\!\!R^m) = \mathcal{S}(I\!\!R^m) \cap \mathcal{K}(I\!\!R^m)$, the space of test functions. We write T_x for the translation operator, given by $T_x f(y) := f(y - x)$.

The basic idea is to generate spaces of functions or distributions which show a certain local behavior (local membership in a certain function space) and a certain global behavior (expressed on terms of the local norm). If the local behavior is just L^p-summability, and the global behavior is just ℓ^q-summability these spaces where treated earlier as amalgams of L^p and ℓ^q in the literature. An informative survey has been given by J. J. Fournier and J. Stewart [FSt], cf. also [BDD]. If we want to use more general local norms (including norms of Besov- or Sobolev spaces) or if we would like a more general global notion (such as decay at a certain rate at infinity) the following concept turns out to be very useful.

Definition 1. A Banach space $(B, \|\cdot\|_B)$ is *uniformly localizable*, if

(A1) $\mathcal{S}(\mathbf{R}^m) \hookrightarrow (B, \|\cdot\|_B) \hookrightarrow \mathcal{S}'(\mathbf{R}^m)$ are continuous embeddings, and

(A2) For every $h \in \mathcal{D}(I\!\!R^m)$ one has $h \cdot f \in B$ for any $f \in B$ and there is some $C = C(h) > 0$ such that $\|T_x h \cdot f\|_B \leq C \cdot \|f\|_B$ for all $x \in I\!\!R^m,\ f \in B$.

Note. Make sure to read $T_x hf$ as $(T_x h)f$, and *not* as $T_x(hf)$ in the sequel!

Remark 1. In view of (A2) the space

$$B_{loc} := \{f \in \mathcal{S}'(I\!\!R^m)\ ,\ h \cdot f \in B \text{ for any } h \in D(I\!\!R^m)\}$$

is well defined. Fixing any non-zero function $h \in \mathcal{D}(I\!\!R^m)$ the *control function* (f localized my means of h, and measured in the B-norm) is defined by

$$F_h(x) := \|T_x h \cdot f\|_B\ . \tag{1}$$

Following standard terminology in the theory of short time Fourier transforms h might be called a *window function*, and the most natural choice is actually to take a function h having a plateau type graph, i.e. taking values between zero and one, with $h(m) \equiv 1$ for all $m \in M$, M being some open set around the origin. However, these extra requirements are *not* part of the definition given above. Uniformly localizable spaces will be used as *local components.*

Examples. There is an abundance of uniformly localizable spaces, like the ordinary L^p-spaces (used for 'ordinary amalgams', cf. [FSt]) or the spaces $\mathcal{F}L^q$ (image of L^p under the generalized Fourier transform), which have been used for 'generalized amalgams' in [F3]. In fact $\mathcal{F}L^p := \{\hat{\sigma} \,|\, \sigma \in L^p\}$, the space of all tempered distributions which arise as Fourier transforms of L^p-functions (with the norm being taken from L^p), is uniformly localizable, since the embedding $\mathcal{S}(I\!\!R^m) \hookrightarrow \mathcal{F}L^1$ and the convolution relation $L^1 * L^p \subseteq L^p$ imply that the elements of (the isometrically translation invariant Banach algebra) $\mathcal{F}L^1$ define bounded multipliers on $\mathcal{F}L^p$. One may take also Lipschitz spaces, Besov spaces, potential spaces or, more generally, Triebel-Lizorkin spaces, in order to measure local smoothness (see [F1] or [F4] for more examples).

For a description of the *global behavior* of the control function F_h given by (1) (which already involves the use of the local norm $\| \cdot \|_B$) we need a *global component* $(Y, \| \cdot \|_Y)$. It has to satisfy

(A3) $(Y, \| \cdot \|_Y)$ is a solid BF-space, i.e. a Banach space of locally integrable functions, continuously embedded into $L^1_{loc}(I\!R^m)$, with the extra property that for $f \in Y$ and $g \in L^1_{loc}(I\!R^m)$ the pointwise (a.e.) estimate $|g(x)| \leq |f(x)|$ implies $g \in Y$ and $\|g\|_Y \leq \|f\|_Y$.

(A4) $(Y, \| \cdot \|_Y)$ is translation invariant, satisfying $\|T_x f\|_Y \leq C \cdot w_\alpha(x) \|f\|_Y$, with $w_\alpha(x) := (1 + |x|)^\alpha$, for some $\alpha \geq 0$.

Furthermore we assume that $(Y, \| \cdot \|_Y)$ is a Banach convolution module with respect to the Beurling algebra $L^1_\alpha(I\!R^m) := \{f \mid f w_\alpha \in L^1(I\!R^m)\}$, i.e. $f * g \in Y$ for $f \in Y$ and $g \in L^1_\alpha(I\!R^m)$, together with the corresponding norm estimate $\|f * g\|_Y \leq \|g\|_{1,\alpha} \cdot \|f\|_Y$. In most cases this property follows from the first one, e.g. if $\mathcal{S}(I\!R^m)$ is dense in Y.

Examples. Weighted L^q-spaces $L^q_s := \{f \mid f w_s \in L^q(I\!R^m)\}$ with natural norm $\|f\|_{q,s} := \|f w_s\|_q$, $0 < |s| \leq \alpha$ are the typical choice for Y, showing that the global space Y controls the behavior of F_h by means of decay conditions due to w_s (for $s < 0$ this may as well mean setting bounds on the rate of growth) as well as summability conditions (through the parameter q). Of course, in the case of several dimensions one might take mixed norms spaces and anisotropic weights (many other choices for Y are available then). All weights w in this note are supposed to be continuous, strictly positive, and α-*moderate* in the following sense: $w(x + y) \leq w_\alpha(y) w(x)$ for some $\alpha \geq 0$ and all $x, y \in I\!R^m$. This makes sure that $L^q_s(I\!R^m) \hookrightarrow \mathcal{S}'(I\!R^m)$. The solidity of Y also implies that $Y \hookrightarrow W(L^1, Y)$ for any global component Y, hence $W(B, Y) \hookrightarrow \mathcal{S}'(I\!R^m)$.

Definition 2. Given B, Y as above we define the **Wiener Amalgam Space**

$$W(B, Y) := \{ f \in B_{loc} \, , \; F_h \in Y \} \tag{2}$$

with the natural norm $\|f \,|W(B, Y)\| := \|F_h\|_Y$.

Remark 2. It has been shown that under these circumstances (the conditions stated here are actually more restrictive than those described in [F1]) the space $W(B, Y)$ is a well defined Banach space, continuously embedded into $\mathcal{S}'(I\!R^m)$, and that *different*, non-zero window-functions h define *equivalent* norms. Since we shall make frequent use of this fact let us give a proof.

Proof. Given $h^{(1)}$ and $h^{(2)}$ let us estimate the Y-norm of $F_{h^{(1)}}$ by means of $F_{h^{(2)}}$. First of all we observe that we may assume $h^{(2)}$ to be positive, since otherwise the local multiplier property (A2) allows us to show that for $h := |h^{(2)}|^2$ one has the estimate $\|F_h\|_Y \leq C\overline{h} \cdot \|F_{h^{(2)}}\|_Y$. Assuming thus positivity of $h^{(2)}$ we find a linear combination of shifted versions of $h^{(2)}$, i.e. $h^{(3)} := \sum_{n=1}^k T_{x_k} h^{(2)}$, such that $h^{(3)}(z) \geq \delta_0 > 0$ on $\mathrm{supp}(h^{(1)})$. On the other hand the translation invariance of Y allows us to estimate the norm of $F_{h^{(3)}} \leq \sum_{n=1}^k T_{-x_k} F_{h^{(2)}}$ by $C_4 \cdot \|F_{h^{(2)}}\|_Y$. Finally, we choose some $h^{(4)} \in \mathcal{S}(I\!R^m)$ such that $h^{(4)}(t) = 1/h^{(3)}(t)$ for all $t \in \mathrm{supp}(h^{(1)})$. Then clearly $h^{(1)} = h^{(3)} \cdot (h^{(4)} h^{(1)})$ and therefore we end up with $\|F_{h^{(1)}}\|_Y \leq C_4 \cdot \|F_{h^{(2)}}\|_Y$. For reasons of symmetry this shows the required norm equivalence. $\square$

Remark 3. Observe that ordinary weighted L^p-spaces are special Wiener amalgam spaces, since $L^p_w(I\!R^m) = W(L^p, L^p_w)$. We also mention that the norm on the spaces $W(B,Y)$ can be described equivalently using discrete global norms (cf. [F1]), which helps much in 'guessing' what the correct inclusion between various spaces spaces are. However, we will prefer to stay with the 'continuous' norm based on (1) and (2) in this paper.

Remark 4 (*Duality*). It is also easy to verify that $\mathcal{D}(I\!R^m)$ (and with little extra arguments also $\mathcal{S}(I\!R^m)$) is embedded into $W(B,Y)$, and that each $h \in \mathcal{D}(I\!R^m)$ defines a bounded multiplier on $W(B,Y)$ by (A2). If $\mathcal{D}(I\!R^m)$ is a dense subspace of B as well as of Y, the dual space of $W(B,Y)$ can be identified in the natural way with $W(B',Y')$, and for many other operations 'coordinatwise' arguments may be used, such as for pointwise multipication of Wiener amalgam spaces (see [FGr] for these results in even greater generality) or for interpolation of Banach spaces [F2]. It will be convenient to use the notion $\langle \cdot, \cdot \rangle$ to describe duality between suitable pairs of Banach spaces.

Remark 5. The perhaps most useful single result concerns convolution. We have the following general statement.

Assume that (B^1, B^2, B^3) and (Y^1, Y^2, Y^3) are Banach convolution triples, i.e. that

$$\|f * g\|_{B^3} \leq C_1 \cdot \|g\|_{B^1} \cdot \|f\|_{B^2} \text{ for all } g \in B^1,\ f \in B^2$$

and

$$\|F \cdot G\|_{Y^3} \leq C_2 \cdot \|G\|_{Y^1} \cdot \|F\|_{Y^2} \quad \text{for all } G \in Y^1,\ F \in Y^2\ .$$

Then the spaces $(W(B^1,Y^1), W(B^2,Y^2), W(B^3,Y^3))$ form a Banach convolution triple, i.e. for some constant $C_3 > 0$ one has

$$\|f * g\,|W(B^3,Y^3)\| \leq C_3 \cdot \|g\,|W(B^1,Y^1)\|\, \|f\,|W(B^2,Y^2)\|\ . \tag{3}$$

This may be considered as a far reaching generalization of Young's inequality for convolutions of L^p-functions (see [F1], [BS] for a special case).

In view of this property and the examples, all of which satisfy (A5), we assume for the rest of the paper that the *local component* $(B, \|\cdot\|_B)$ satisfies:

(A5) $(B, \|\cdot\|_B)$ is isometrically translation invariant,
i.e. $\|T_x f\|_B = \|f\|_B$, with $L^1 * B \subseteq B$ and $\|g * f\|_B \leq \|g\|_1 \|f\|_B$ for $g \in L^1,\ f \in B$.

Remark 6. By the convolution relations this implies that $W(B,Y) * L^1_\alpha \subseteq W(B,Y)$, since we may identify L^1_α with $W(L^1, L^1_\alpha)$. It also makes sure that $\mathcal{D}(I\!R^m)$ is dense in $W(B,Y)$, if $\mathcal{K}(I\!R^m)$ is dense in Y. We actually show that any $f \in W(B,Y)$ can be boundedly approximated by a net in $\mathcal{D}(I\!R^m)$.

Proof. Choosing first some $h \in \mathcal{D}(I\!R^m)$ such that $\|f - hf\,|W(B,Y)\| \leq \varepsilon/2$ we find $k \in \mathcal{D}(I\!R^m)$ with $\|hf - k * hf\,|W(B,Y)\| \leq \varepsilon/2$, thus $\|f - k * hf\,|W(B,Y)\| \leq \varepsilon$. Since $k * hf$ is the convolution product of two compactly supported distributions it is compactly supported itself. It also belongs to $\mathcal{D}(I\!\!R^m) * \mathcal{S}'(I\!\!R^m) \subseteq \mathcal{S}(I\!\!R^m)$, which guarantees the required smoothness. Since it is always possible to choose the function $h \in \mathcal{D}(I\!\!R^m)$ within a given a priori bound of the multiplier algebra (cf. below) and the L^1_α-boundedness of k (from some approximate unit) implies, that for some a priori constant $C_0 > 0$ one has $\|k * hf \,|W(B,Y)\| \leq C_0 \|f\,|W(B,Y)\|$, which completes our argumentation. □

Remark 7. There are also two version of the *Hausdorff-Young inequality* (HY in the sequel, see [BD], [F2]). The first one, which works under the expected restrictions $1 \leq p, q \leq 2$, implies that the Fourier transform maps $W(L^p, L^q)$ into $W(L^{q'}, L^{p'})$, with $1/p' + 1/p = 1$, thus interchanging the roles of the local and the global component. The *generalized HY inequality*, which is strictly stronger, is based on the use of generalized amalgams (cf. [F2], [F3]).

$$\mathcal{F}(W(\mathcal{F}L^p, L^r)) \hookrightarrow W(\mathcal{F}L^r, L^p) \ \text{ for } 1 \leq r \leq p \leq \infty \ . \tag{4}$$

3 Spline Approximation And Discretization Operators

In this section we assume throughout that Ψ denotes a positive family of functions ($0 \leq \psi_i(x) \leq 1$ for all x), which is *δ-fine* and *C-bounded*, i.e. satisfies $\mathrm{supp}(\psi_i) \subseteq B_\delta(x_i)$ for some $\delta > 0$ and all $i \in I$, and $\sum_{i\in I} \psi_i(x) \leq C < \infty$ for all $x \in I\!\!R^m$. In most cases we will work with δ-BUPUs (*Bounded Uniform Partitions of Unity*), i.e. families satisfying $\sum_{i\in I} \psi_i(x) \equiv 1$. In that case the family of 'midpoints' has to be δ-dense in $I\!\!R^m$, i.e. $I\!\!R^m = \bigcup_{i\in I} B_\delta(x_i)$.

Examples. In the most simple one-dimensional case, for a given δ-dense family $(x_i)_{i\in \boldsymbol{Z}}$, we could have the rectangular partitions of unity using the indicator functions $\mathbf{1}_{[m_{i-1}, m_i]}$, with $m_i := (x_i + x_{i-1})/2$ for $i \in \boldsymbol{Z}$. The analogue of this in higher dimensions is the indicator functions of the Voronoi region associated with x_i, defined by

$$V_i := \{\, x \mid |x_i - x| \leq |x - x_j| \ \forall j \neq i \,\} \ . \tag{5}$$

Another simple version of a δ-BUPU is a system of triangular functions $(\Delta_i)_{i\in I}$, which are piecewise linear on the interval $[x_i, x_{i+1}]$ and satisfy $\Delta_i(x_i) = 1$, and $\Delta_i(x_j) = 0$ for $i \neq j$.

Contrary to earlier papers we do *not* assume that the family of points is *relatively separated* (cf. section 5), i.e. no assumption is made on how dense the family of points can be, and there may be even accumulation points to X. On the other hand it is clear from the assumptions, that in this case the functions ψ_i have to be very small (some of them might be zero). Given such a family Ψ we define the following two operators.

Definition 3. The *spline quasi-interpolant* for the family Ψ is defined for any continuous function f by

$$Sp_\Psi f := \sum_{i\in I} f(x_i)\, \psi_i \quad . \tag{6}$$

Along with this operator we also define a 'discretization operator', mapping locally integrable functions into discrete measures

$$f \in L^1_{loc}, \quad D_\Psi f := \sum_{i\in I} \langle f, \psi_i \rangle \cdot \delta_{x_i} = \sum_{i\in I} \left(\int f(x) \psi_i(x)\, dx \right) \cdot \delta_{x_i} \quad . \tag{7}$$

Remark 8. Actually, D_Ψ is well defined for arbitrary Radon measures, if Ψ is a family of continuous functions and if we interpret $\langle \mu, \psi_i \rangle$ as the natural duality between $\mathcal{K}(I\!\!R^m)$ and $\mathcal{R}(I\!\!R^m) = \mathcal{K}(I\!\!R^m)'$. In fact, for continuous Ψ the operator Sp_Ψ is continuous on $\mathcal{K}(I\!\!R^m)$ and D_Ψ is the dual operator, since

$$\langle D_\Psi \mu, f \rangle = \langle \mu, Sp_\Psi f \rangle \quad \text{for } f \in \mathcal{K}(I\!\!R^m) \text{ and } \mu \in \mathcal{R}(I\!\!R^m) \quad . \tag{8}$$

Clearly Sp_Ψ maps $C^b(I\!\!R^m)$, the bounded continuous complex-valued functions on $I\!\!R^m$, endowed with the sup-norm $\|f\|_\infty := \sup_{x \in I\!\!R^m} |f(x)|$, into itself, with $\|Sp_\Psi f\|_\infty \leq \|f\|_\infty$ for all $f \in C^b(I\!\!R^m)$ for continuous Ψ. On the other hand both of these operators are not well defined or at least not bounded on any of the L^p-spaces. On the other hand for 'smooth' functions in L^p one may expect that $Sp_\Psi f$ is a good approximation to f in the L^p-sense for small δ. It turns out the that space $W(C^0, Y)$ gives the appropriate setting for this statement.

Lemma 1. For any $\delta > 0$ and $C > 0$ there exists some $C_\delta > 0$ such that for all C-bounded families Ψ which are δ-fine one has

$$\|Sp_\Psi f \,|W(C^0, Y)\| \leq C_\delta \cdot \|f \,|W(C^0, Y)\| \ . \tag{9}$$

Proof. Since we are free to choose our 'window' function defining the norms on both sides of the inequality, we choose any $h \in D(I\!\!R^m)$ for the right side, and use a $k \in \mathcal{D}(I\!\!R^m)$ for the left norm, with $k(t) \equiv 1$ on $\mathrm{supp}(h) + B_\delta(0)$. In this case we have the pointwise inequality $F_h(Sp_\Psi f) \leq C \cdot F_k(g)$ (a local version of the sup-norm estimate $\|Sp_\Psi f\|_\infty \leq \|f\|_\infty$), from which the required estimate follows. □

The easiest way to verify that these spline-type functions approximate f in the appropriate sense is by means of the concept of a local oscillation. For $\delta > 0$ we define the *δ-oscillation* of a continuous function by

$$\mathrm{osc}_\delta f(x) := \sup_{|y| \leq \delta} |f(x) - f(x+y)| \ . \tag{10}$$

Using it we can formulate:

Lemma 2. For any $f \in W(C^0, Y)$ with $\mathrm{osc}_\delta f \in W(C^0, Y)$. If $\mathcal{K}(I\!\!R^m)$ is a dense subspace of Y then $\|\mathrm{osc}_\delta f \,|W(C^0, Y)\| \to 0$, hence $\|Sp_\Psi f - f \,|W(C^0, Y)\| \to 0$ for $\delta \to 0$.

Proof. Observe that one has the pointwise estimate $\mathrm{osc}_\delta f \leq 2 \cdot F_h$, for any $h \in \mathcal{D}$ which satisfies $h(x) \equiv 1$ on $B_\delta(0)$. Since Y is solid and $\mathcal{K}(I\!\!R^m)$ is dense in Y there exists $k \in \mathcal{D}$ such that $\|F_h \cdot (1-k)\|_Y \leq \varepsilon/4$. Uniform continuity of f on compact sets implies that we have uniform convergence of $\mathrm{osc}_\delta f$ on compact subsets, therefore one has for sufficiently small $\delta > 0$

$$\|\mathrm{osc}_\delta f \cdot k \,|W(C^o, Y)\| \ \leq \ \sup_{u \in \mathrm{supp}(k)} |\mathrm{osc}_\delta f(u)| \cdot \|k \,|W(C^0, Y)\| \ < \ \frac{\varepsilon}{2} \quad .$$

Altogether this implies for sufficiently small δ

$$\|\mathrm{osc}_\delta f \,|W(C^0, Y)\| \leq \|\mathrm{osc}_\delta f \cdot (1-k) \,|W(C^0, Y)\| + \frac{\varepsilon}{2} \leq 2 \,\|F_h \cdot (1-k) \,|W(C^0, Y)\| + \frac{\varepsilon}{2} < \varepsilon$$

In order to verify the second statement we use the pointwise estimate

$$|Sp_{\Psi} f - f| \leq \operatorname{osc}_\delta(f) \ \text{ for any } \delta\text{-BUPU } \Psi \ , \tag{11}$$

which follows easily by summation over the pointwise estimates for the terms

$$|f(x) \cdot \psi_i(x) - f(x_i) \cdot \psi_i(x)| \ \leq \operatorname{osc}_\delta f(x) \cdot \psi_i(x) \ \ \forall x \in I\!\!R^m \ \ .$$

□

In connection with these operators (as well as with convolution operators) and with the characterization of compact subsets in function spaces the following two notions turn out to be useful (cf. [F4]).

Definition 4. Let M be a bounded set in a Banach space $(B, \| \cdot \|_B)$ of tempered distributions.

(i) M is called *(uniformly) tight* if there exists some bounded net of multipliers $(h_\gamma)_{\gamma \in \Gamma}$ in $\mathcal{D}(I\!\!R^m)$, such that $\lim_\gamma h_\gamma \cdot f = f$ uniformly over M, in B-norm sense, i.e. for every $\varepsilon > 0$ there exists $h \in \mathcal{D}(I\!\!R^m)$ such that

$$\|h \cdot f\|_B \leq C \cdot \|f\|_B \ \ \forall f \in B \ ,$$

and

$$\|h \cdot f - f\|_B \leq \varepsilon \cdot \|f\|_B \ \ \forall f \in M \ . \tag{12}$$

(ii) M is *equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|T_x f - f\|_B \leq \varepsilon \ \ \forall f \in M, \ x \in I\!\!R^m \text{ with } |x| \leq \delta \ . \tag{13}$$

Remark 9 (*Stability of these concepts*). Note that both properties (tightness and equicontinuity) descends on the closure, i.e. a set M is tight in Y if and only if $\overline{M}$ is tight in Y. Actually, the following holds. If a set can be approximated by tight (or equicontinuous) sets, then it is tight (equicontinuous) itself. More precisely, if, for a set M and any $\varepsilon > 0$, there exists a tight M_ε such that $M \subseteq \bigcup_{x \in M_\varepsilon} B_\varepsilon(x)$, then M is tight itself.

Remark 10 (*Stability under convolution and multiplication*). It is an easy exercise to check the following. Let M_1, M_2 be tight subsets of B^1, B^2. Then $M_1 * M_2$ is tight in B^3, if (B^1, B^2, B^3) is a convolution triple. In a similar way equicontinuity is preserved under pointwise multiplication, if one has a pointwise convolution triple. Moreover, a convolution product is equicontinuous if only one factor is equicontinuous, and a pointwise product $M_1 \cdot M_2$ is tight if only one factor is tight (cf. [F3,4] for details).

The concept of oscillation is actually very useful in order to describe equicontinuity in the spaces $W(C^0, Y)$. The following result also sheds some light on the usefulness of the concept of oscillation and indicates the importance of equicontinuity for the approximation behavior of spline approximations (cf. [F4] for results on equicontinuity in general spaces).

Theorem 3 (*characterization of equicontinuity in $W(C^0, Y)$*). For a bounded subset M in $W(C^0, Y)$ the following conditions are equivalent.

(a) M is equicontinuous in $W(C^0, Y)$.

(b) For every $\varepsilon > 0$ there exists $k \in \mathcal{D}(I\!\!R^m)$ such that

$$\|k * f - f \,|W(C^0, Y)\| \leq \varepsilon \ \ \forall f \in M \ .$$

(c) The oscillation functions tend to zero uniformly over M, i.e. for every $\varepsilon > 0$ there exists $\delta_0 > 0$ such that $\|\mathrm{osc}_\delta f\|_Y \leq \varepsilon$ for all $f \in M$, $\delta \leq \delta_0$.

(d) The same as (c), with $\|\mathrm{osc}_\delta f \,|W(C^0, Y)\| \leq \varepsilon$.

(e) The family of spline quasi-interpolants $Sp_\Psi f$ is uniformly convergent to f in the norm of $W(C^0, Y)$, i.e. for any $\varepsilon > 0$ there exists $\delta_0 > 0$ such that $\|f - Sp_\Psi f \,|W(C^0, Y)\| \leq \varepsilon$ for all $f \in M$, and all δ-BUPU's, with $\delta \leq \delta_0$.

Proof.
(a) $\Rightarrow$ (b): Assume that M is equicontinuous. Choose $\delta > 0$ such that $\|T_x f - f \,|W(C^0, Y)\| \leq \varepsilon$ for all $f \in M$. Then any function $k \in \mathcal{K}^+(I\!\!R^m)$ with $\int k(x)\,dx = 1$ and support in $B_\delta(0)$ satisfies $\|k * f - f\|_{W(C^0,Y)} \leq \varepsilon$ for all $f \in M$. In fact, writing the convolution as a vector-valued integral of the form

$$\|k * f - f \cdot \int k(y)\,dy \,|W(C^0, Y)\| \leq \int \|T_y f - f \,|W(C^0, Y)\| \cdot k(y)\,dy \leq \varepsilon \ \ \forall f \in M \quad .$$

(b) $\Rightarrow$ (d), hence (c): We start with the pointwise estimate

$$\mathrm{osc}_\delta(k * f) \leq \mathrm{osc}_\delta(k) * |f| \ . \tag{14}$$

This gives us $\mathrm{osc}_\delta(f) \leq \mathrm{osc}_\delta(k * f) + \mathrm{osc}_\delta(f - k * f)$, hence

$$\|\mathrm{osc}_\delta(f) \,|W(C^0, L^p)\| \leq C \cdot \|\mathrm{osc}_\delta k \,|W(C^0, L^1)\| \cdot \|f \,|W(L^1, L^p)\| + \|\mathrm{osc}_\delta(f - k * f) \,|W(C^0, L^p)\| \ .$$

The second term is small, since $\mathrm{osc}_\delta(f - k * f)$ is dominated by the control function of $2\,(f - k * f)$, which is not only small in the Y-norm but also in the $W(C^0, Y)$-norm (as a consequence of the independence of the norm on the specific window). The uniform continuity of $k \in \mathcal{K}(I\!\!R^m)$ allows us to choose δ in a way that the first expression on the right side is small as well.
(d) $\Rightarrow$ (a): This implication is obvious, since the definition of $\mathrm{osc}_\delta f$ implies $|T_x f - f| \leq \mathrm{osc}_\delta f$ for $|x| \leq \delta$, given the estimate in $W(C^0, Y)$.
(c) $\Rightarrow$ (e): Follows from (11) (cf. the proof of Lemma 1).
(e) $\Rightarrow$ (a): Actually we suppose only, that the convergence of $Sp_\Psi f$ to f is uniform for some subnet, satisfying a few extra conditions. We consider in this paragraph only δ-BUPUs which are equicontinuous (each for itself) and of a finite degree of overlap.

Explicitly we suppose that we have: Ψ is a δ-BUPU such that $(B_{3\delta}(x_i))_{i \in I}$ defines a covering of finite (maximal) height of $I\!\!R^m$, and such that the functions $\{\psi_i \,|\, i \in I\}$ form an equicontinuous subfamily in $C^0(I\!\!R^m)$. This set of conditions is of course satisfied by any family, derrived by tensor products of triangular functions (with basic width δ) over a regular lattice. In this case the amount of overlap between of their supports depends only on dimension of the space.

Under these circumstances we claim: For any such Ψ the set

$$\{Sp_\Psi f \,|\, f \in W(C^0,Y),\ \|f\,|W(C^0,Y)\| \leq 1\} \tag{15}$$

is equicontinuous in $W(C^0,Y)$.

The proof of this is based on the following. We assume that $|y| \leq \delta$. Choosing first some non-negative function ϕ such that $\mathrm{supp}(\phi) \subseteq B_{3\delta}(0)$ and $\phi(t) \equiv 1$ on $B_{2\delta}(0)$ we have of course $\psi_i(x-y) = \psi_i(x-y)T_{x_i}\phi(x)$, since $\mathrm{supp}(T_y\psi_i) \subseteq B_{2\delta}(x_i)$. This gives the following pointwise estimate.

$$\mathrm{osc}_\delta(Sp_\Psi f)(x) = \sup_{|y|\leq\delta} |Sp_\Psi f(x-y) - Sp_\Psi f(x)| \leq \sup_{|y|\leq\delta} \sum_{i\in I} |f(x_i)||\psi_i(x-y) - \psi_i(x)| \leq$$

$$\leq \sum_{i\in I} |f(x_i)| \cdot \sup_{|y|\leq\delta} |\psi_i(x-y) - \psi_i(x)| \cdot T_{x_i}\phi(x) \leq \sum_{i\in I} |f(x_i)| \cdot \|\mathrm{osc}_\delta \psi_i\|_\infty \cdot T_{x_i}\phi\ .$$

On the basis of the equicontinuity of Ψ we can find $\delta_0 > 0$ such that $\|\mathrm{osc}_\delta\psi_i\|_\infty \leq \eta$ for any given $\eta > 0$. Using also the fact that the finite height condition implies (and is actually equivalent to the statement that the discrete measure $\sum_{i\in I}\delta_{x_i}$ belongs to $W(M,L^\infty)$) we obtain the required norm estimate.

$$\|\mathrm{osc}_\delta(Sp_\Psi f)\,|W(C^0,Y)\| \leq \sup_{i\in I} \|\mathrm{osc}_\delta\psi_i\|_\infty \cdot \|\sum_{i\in I} |f(x_i)| \cdot \delta_{x_i} * \phi\,|W(C^0,Y)\| \leq$$

$$\leq \eta \cdot C_1 \cdot \|\sum_{i\in I} |f(x_i)| \cdot \delta_{x_i}\,|W(M,Y)\| \cdot \|\phi\,|W(C^0,L^1)\| \leq$$

$$\leq \eta \cdot C_2 \cdot \|\sum_{i\in I} \delta_{x_i}\,|W(M,L^\infty)\| \cdot \|f\,|W(C^0,Y)\| \cdot \|\phi\,|W(C^0,L^1)\|\ .$$

Since M is bounded in $W(C^0,Y)$ it is possible to chose η (hence δ_0) in a way such that

$$\|\mathrm{osc}_\delta(Sp_\Psi f)\,|W(C^0,Y)\| \leq \varepsilon \quad \forall f \in M \quad .$$

We have thus verified that if M satisfies (e) it can be approximated uniformly in the $W(C^0,Y)$-norm by equicontinuous set $\{Sp_\Psi f | f \in M\}$, and the proof is complete in view of Remark 9. □

Remark 11. Although $\mathrm{osc}_\delta f$ has many features of a modulus of continuity (with respect to $W(C^0,L^p)$) it does not seem to be equivalent thereto (in terms of rates of convergence etc., at least we have not been able to verify this).

Next we discuss the discretization operators.

Lemma 4.

(i) For any $\gamma > 0$ the family of operators $\mu \mapsto D_\Psi\mu$, where Ψ runs through the system of bounded and γ-supported families (without necessarily giving a BUPU), is uniformly bounded on $W(M,Y)$.

(ii) As Ψ runs through the family of δ-BUPUs one has convergence of the family $D_\Psi \mu \to \mu$ in the vague topology, i.e. $D_\Psi \mu(k) \to \mu(k)$ for any $k \in \mathcal{K}(\mathbb{R}^m)$. In particular, $D_\Psi \mu \to \mu$ in the w*-sense, if Y is the dual of a space X, i.e. in the $\sigma(W(M,Y), W(C^0,X))$ sense.

Using this terminology we next describe the general compactness criterion for distribution spaces (cf. [F4]) in the setting of Wiener amalgams, known as the Riesz-Weil criterion in the case of L^p-spaces.

Proposition 5. Assume that $(B, \|\cdot\|_B)$ is a Banach space satisfying (A1) to (A5), and that $\mathcal{K}(\mathbb{R}^m)$ is dense in Y. Then a closed bounded subset $M \subseteq W(B,Y)$ is *compact* if and only if it is uniformly tight and equicontinuous with respect to the $W(B,Y)$-norm. This holds if and only if it is possible to determine (for any $\varepsilon > 0$) two functions $h, k \in \mathcal{D}(\mathbb{R}^m)$ such that

$$\|h(k * f)\,|W(B,Y)\| < \varepsilon \quad \text{(and} \quad \|k * (hf)\,|W(B,Y)\| < \varepsilon) \quad \forall f \in M \quad . \tag{16}$$

Proof. We have to check a few properties as described in [F4]. First of all it is already clear that $W(B,Y)$ has continuous shift, i.e. $\|T_x f - f\|_B \to 0$ for $x \to 0$, for all $f \in W(B,Y)$. The second requirement is the existence of an approximate unit in $\mathcal{D}(\mathbb{R}^m)$ for B, bounded in the operator norm, i.e. some family $(h_\gamma)_{\gamma \in \Gamma}$ in $\mathcal{D}(\mathbb{R}^m)$ satisfying

$$\|h_\gamma \cdot f\|_B \le C \cdot \|f\|_B \quad \text{and} \quad \|h_\gamma \cdot f - f\|_B \to 0 \quad \forall f \in W(B,Y) \; .$$

This property, however can be derived from assumption (A2) as $\mathcal{D}(\mathbb{R}^m)$ is contained in the pointwise multiplier A of B, and the multiplier space of $W(B,Y)$ contains $W(A, L^\infty)$ (cf. [F3]). □

Since we want to discuss operators preserving (or improving) these properties we also use a related concept for (families of) operators.

Definition 5. We call a bounded family of operators $(T_\gamma)_{\gamma \in \Gamma}$ *uniformly tight* (a single operator just tight) if the set $T_\Gamma M := \{T_\gamma x \mid \; x \in M, \; \gamma \in \Gamma\}$ is uniformly tight for every tight set M. We call it *strictly tight* if there exists some compact set $K_0 \subseteq \mathbb{R}^m$ such that $\mathrm{supp}(f) \subseteq S \subseteq \mathbb{R}^m$ implies $\mathrm{supp}(T_\gamma f) \subseteq S + K_0$. Finally, we call such a family *equicontinuous* if the set $T_\Gamma M$ is equicontinuous for every bounded set M.

Remark 12 (*Stability of these concepts*). As a consequence of the stability of the notions of tightness and equicontinuity (cf. remarks 9/10) the (norm) limit of a sequence of tight (or equicontinuous) operators is tight (equicontinuous) itself. It follows that these operators form closed subspaces of the space of operators between two Banach spaces of distributions, and closed subalgebras in the operator algebra of any given space.

Remark 13. Due to the boundedness assumption on the family $(T_\gamma)_{\gamma \in \Gamma}$ the tightness is already satisfied if the tightness is satisfied for bounded sets M which have common compact support. Using the concept of the translate of an operator between translation invariant function spaces, given by $T_x(T) : f \mapsto T_{-x}(T_x f)$ one may describe the above notion of equicontinuity equivalently as equicontinuity in the Banach space of operators (in the norm sense).

By means of these concepts we can show, for example, that for any $k \in \mathcal{K}(I\!R^m)$ we have uniform convergence of $D_\Psi f * k$ to $f * k$, on the unit ball of L^p, i.e. the operators $T_\Psi : f \mapsto D_\Psi f * k$ converge to the convolution operator $f \mapsto f * k$, for $|\Psi| \to 0$.

To prove this we note that the family $\{D_\Psi f,\ |\Psi| \leq 1\}$ is a tight family in $W(M, L^p)$. Since convolution by k is a tight and equicontinuous operator from $W(M, L^p)$ into $W(C^0, L^p)$ (this is an immediate consequence of the convolution relations and the fact that convolution commutes with translation) we find that $\{D_\Psi f * k,\ |\Psi| \leq 1\}$ is actually relatively compact in $W(C^0, L^p)$. Since the w*-convergence shows that we have at least pointwise convergence $D_\Psi f * k(x) \to f * k(x)$ for every $x \in I\!R^m$ the proof is complete. In fact, if we would *not* have norm convergence of this net there would be a subnet staying away from $f * k$. Applying the above argument to that subnet would produce a contradiction. What we have actually shown can be summarized as follows (in a typical but not the most general form).

Lemma 6. For every $f \in W(M, L^p)$ and $k \in W(C^0, L^1)$ we have

$$\|(D_\Psi f - f) * k \,|W(C^0, L^p)\| \to 0 \text{ for } |\Psi| \to 0\ ; \tag{17}$$

in particular, we have uniform as well as L^p-convergence.

Equipped with these arguments it is now easy to check that most product-convolution or convolution-product operators (for short PC or CP operators, cf. [BS]) act compact on L^p-spaces (or other spaces).

Lemma 7. Let $g \in W(L^{p'}, L^1)$ and $h \in W(L^p, L^\infty)$ be given. Then the CP-operator $Tf := (f * g) \cdot h$ is a bounded operator L^p. If moreover $h \in W(L^p, C^0)$, then T is a compact operator on L^p for $1 \leq p < \infty$.

Proof. Boundedness follows from the multiplier and convolution theorems

$$Tf = (f*g)\cdot h \in (W(L^p,L^p) * W(L^{p'},L^1)) \cdot W(L^p,L^\infty) \subseteq W(C^0,L^p)\cdot W(L^p,L^\infty) \subseteq L^p \quad .$$

If h belongs to $W(L^p, C^0)$ (this is the closure of $\mathcal{K}(I\!R^m)$ in $W(L^p, L^\infty)$), then pointwise multiplication by h is a tight operator (as norm limit of tight operators). Thus convolution by g maps the unit ball of L^p into an equicontinuous subset of $W(C^0, L^p)$. Pointwise multiplication by h transforms this equicontinuity (because h is also equicontinuous in the $W(L^\infty, L^p)$ sense) into equicontinuity in the L^p-sense and furthermore produces tightness. Thus, altogether, we obtain relative compactness of $T(M)$ in L^p, for any bounded subset $M \subseteq L^p$ by the compactness criterion. □

Remark 14. The last result should be only a prototype of general results of this form. Many similar statements can be derived, using especially the convolution theorems for weighted Wiener amalgam spaces, for example. Results on the boundedness of $PC - CP$ operators have been motivating for Busby and Smith [BS].

Almost all of the above results hold for locally compact Abelian groups. The following result concerns a special problem on $I\!R^m$, where dilations play a role, i.e. a special form of spline approximations for L^p-spaces over $I\!R^m$.

The question is the following: Since the operators Sp_Ψ are not bounded on L^p-spaces it is natural to use regularization operators (convolution by decent functions) before applying the spline operators. A simple cascade argument yields the following (approximation by splines).

Given $f \in L^p(I\!R^m)$ and $\varepsilon > 0$ there exists a $k \in \mathcal{K}(I\!R^m)$ with $\|k * f - f\|_p \le \varepsilon/2$. Fixing k we find some $\delta > 0$ such that $\|Sp_\Psi(k * f) - k * f\|_p \le \varepsilon/2$ for any δ-BUPU Ψ, hence $\|Sp_\Psi(k * f) - f\|_p \le \varepsilon$.

We want to conclude this section with a discussion of the following question, arising from this description. If we obtain the function k by suitable contraction (isometrically in L^1) of a fixed function k_0, can we choose the sampling rate (used to obtain the values $f(x_i)$ needed to built the function $Sp_\Psi f$) at the minimal expected rate, which would be given by the contraction factor ? Since the answer is positive in case of L^2 (see [Au]) we can be optimistic.

We need some additional notations for dilation operators

$$St_\rho f(x) := \rho^{-m} f(x/\rho)\ . \tag{18}$$

These satisfy the following rules with respect to convolution, writing q for the dual index, given by $1/q + 1/p = 1$

$$\|St_\rho f\|_p = \rho^{-m/q}\|f\|_p \quad \forall f \in L^p, \rho > 0\ ; \tag{19}$$

$$St_\rho(f * g) = St_\rho f * St_\rho g \tag{20}$$

The following convention keeps notations short: $\rho\Psi := (\rho^m St_\rho \psi_i)_{i\in I}$ (note that with this notation it is clear that one has $|\rho\Psi| = |\rho||\Psi|$, where $|\Psi|$ is the size of Ψ given as $\sup_{i\in I} \mathrm{diam}(\mathrm{supp}(\psi_i))$, and $\rho\Psi$ is a $\rho\delta$-BUPU if Ψ is a δ-BUPU.

Our p-version of the spline-approximation theorem now reads as follows.

Theorem 8. For $h \in W(C^0, L^1)$ with $\int_{I\!R^m} h(x)\,dx = 1$ for any $f \in L^p(I\!R^m)$, $1 \le p < \infty$

$$\|f - Sp_{\rho\Psi}(St_\rho h * f)\|_p \to 0\ ,\ \text{for } \rho \to 0\ . \tag{21}$$

Proof. We are going to derive the result from the following two assertions:
(i) The operators $f \to Sp_{\rho\Psi}(St_\rho h * f)$ are uniformly bounded on $L^p(I\!R^m)$;
(ii) For $f \in \mathcal{K}(I\!R^m)$ and $h \in \mathcal{K}(I\!R^m)$ one has common compact supports of the functions $Sp_{\rho\Psi}(St_\rho h * f)$ and $\|f - Sp_{\rho\Psi}(St_\rho h * f)\|_\infty \to 0$ for $\rho \to 0$.
The key to our proof is the identity

$$Sp_{\rho\Psi}(St_\rho h * f) = St_\rho(Sp_\Psi(h * St_{\rho^{-1}} f)) \text{ for } \rho > 0\ . \tag{22}$$

It implies (together with the above rules) that (i) holds for $h \in W(C^0, L^1)$:

$$\begin{aligned}
&\|St_\rho(Sp_\Psi(h * St_{\rho^{-1}} f))\|_p = \rho^{-m/q}\|Sp_\Psi(h * St_{\rho^{-1}} f)\|_p \le \\
&\le\ C_0 \cdot \rho^{-m/q}\|Sp_\Psi(h * St_{\rho^{-1}} f)\,|W(C^0, L^p)\| = \\
&=\ C_0 \cdot \rho^{-m/q}\|Sp_\Psi\|\,\|h * St_{\rho^{-1}} f)\,|W(C^0, L^p)\| = \\
&=\ C_1 \cdot \rho^{-m/q}\|Sp_\Psi\|\,\|h\,|W(L^q, L^1)\|\,\|St_{\rho^{-1}} f)\,|W(L^p, L^q)\| = \\
&=\ C_2 \cdot \rho^{-m/q}\|Sp_\Psi\|\,\|h\,|W(L^q, L^1)\|\rho^{mq}\|f\|_p = C_3\|h\,|W(L^q, L^1)\| \cdot \|f\|_p\ .
\end{aligned}$$

In order to verify (ii) we note that for $f,\ h \in \mathcal{K}(\mathbb{R}^m)$ the family $St_\rho h * f$ is not only bounded, with common compact support (for $\rho \le 1$), but also (uniformly) equicontinuous, since

$$\|T_x(St_\rho h * f) - St_\rho h * f\|_\infty = \|St_\rho h * (T_x f - f)\|_\infty \le$$
$$\le \|St_\rho h\|_1 \|T_x f - f)\|_\infty = \|h\|_1 \|T_x f - f\|_\infty \longrightarrow 0 \ ,$$

hence

$$\sup_{\rho \in (0,1)} \|Sp_\Phi(St_\rho h * f) - St_\rho h * f\|_\infty \le \|h\|_1 \sup_{|x| \le |\Phi|} \|T_x f - f\|_\infty \longrightarrow 0$$

for $|\Phi| \to 0$, and

$$\|Sp_\Psi(St_\rho h * f) - St_\rho h * f\|_\infty \ \le \ \|h\|_1 \sup_{|x| \le \rho|\Psi|} \|T_x f - f\|_\infty \longrightarrow 0 \quad \text{for } \rho \to 0 \ .$$

Given this estimate it is easy to see that it extends to $f \in L^p$ by continuity. Having verified (i) for $f \in L^p$ it is also possible to extend (i) to the case $h \in W(L^q, L^1)$. In fact, it is possible to approximate the expression $St_\rho(Sp_\Psi(h * St_{\rho^{-1}} f))$ by an equivalent expression involving only $\tilde{h} \in \mathcal{K}(\mathbb{R}^m)$, due to the estimate (resulting again from (i))

$$\|St_\rho(Sp_\Psi(h * St_{\rho^{-1}} f) - Sp_\Psi(\tilde{h} * St_{\rho^{-1}} f))\|_p \le$$

$$\le C_0 \cdot \rho^{-m/q} \|Sp_\Psi | W(C^0, L^p)\| \ \|(h - \tilde{h}) * St_{\rho^{-1}} f \,| W(C^0, L^p)\| = C_3 \|h - \tilde{h} \,| W(L^q, L^1)\| \ \|f\|_p.$$

This shows that the result holds (for given p) for any $h \in W(L^q, L^1)$. □

Remark 15. Using the last argument of the proof of Theorem 3 it is possible to verify that the stated convergence takes place uniformly on a bounded set $M \subseteq B$ if and only if M is equicontinuous in $L^p(\mathbb{R}^m)$.

4 Variations on the Sampling Theorem, Error Analysis

One of the corner-stones of digital signal analysis is the so-called sampling theorem, according to which a band-limited signal can be completely reconstructed from the sampling values taken at any sufficient fine lattice. In fact, the critical rate, also known as the Nyquist rate, is inversely proportional to the bandwidth of the signal ((2 · (maximal frequency)$^{-1}$ in the usual engineering terminology). Usually this result is presented as a Hilbert space result. In fact, using Plancherel's theorem and Poisson's formula it can be verified that the classical Shannon sampling theorem is indeed equivalent to the Fourier series expansion of periodic functions, a result which is the prototype of the concept of general orthogonal expansion in a Hilbert space (cf. [Br], [Pa2], [LO], and [Bu], [Je] for surveys).

We will describe now the setting in a function space terminology and show how statements about the sampling series (and later results on irregular sampling) can be derived by arguments based on the use of Wiener amalgam spaces. Most of the results in this section are new and cover limiting cases or variants of results given in a series of papers on the irregular sampling problem for band-limited functions [FG1-3].

Definition 6. A tempered distribution $\sigma \in \mathcal{S}(I\!R^m)$ is called *band-limited* with *spectrum* Ω if the (generalized) Fourier transform $\hat{\sigma}$ vanishes on the complement of the closed, bounded set $\Omega \subseteq I\!R^m$. For integrable functions f the statement $\mathrm{spec}(f) \subseteq \Omega$ simply means that $\hat{f}(s) = 0$ for $s \notin \Omega$.

It is a standard result due to Paley and Wiener, and, in its most general version, to Schwartz, that band-limited tempered distributions are actually represented by analytic, hence continuous and differentiable (ordinary) functions. Therefore single function values make sense (are well defined) for band-limited functions in L^p-spaces, for $1 \leq p \leq \infty$. However, we shall not base our arguments on this fact, but use the following result instead (this is actually a special case of Thm.5 of [F1] and holds for lca. groups).

Lemma 9. Let Ω be a compact subset of $I\!R^m$. Then there exists some constant C_Ω (only depending on Ω and h) such that for all $p \geq 1$

$$\|f \,|W(C^0, L^p_w)\| \leq C_\Omega \cdot \|f\|_{L^p_w} \tag{23}$$

for $f \in L^p_w(I\!R^m)$ with $\mathrm{spec}(f) \subseteq \Omega$, and all weights with $L^1_s * L^p_w \subseteq L^p_w$.

Proof. We just choose an arbitrary $h \in \mathcal{S}(I\!R^m) \subseteq W(C^0, L^1_s)$ satisfying $\hat{h}(\omega) \equiv 1$ on Ω. Then $f = h * f$, and therefore

$$\|f \,|W(C^0, L^p_w)\| \leq C_1 \|h \,|W(C^0, L^1_s)\| \cdot \|f \,|W(L^1, L^p_w)\| \leq C_\Omega \cdot \|f\|_{L^p_w} \ .$$

□

Although the following result is true (by almost the same argument) for lca. groups, we present it for simplicity in the setting of $I\!R^m$. The product of multi-indices such as $\boldsymbol{a}n$ is to be understood as $(a_1 n_1, \ldots, a_m n_m)$.

Theorem 10 (*(Weighted L^p-version of the Classical Sampling Theorem*). Given a compact set $\Omega \subseteq I\!R^m$ and a band-limited function $h \in L^1_s(I\!R)$ satisfying $\hat{h}(t) = 1$ on Ω there exists $\boldsymbol{c} = (c_1, \ldots c_m),\ c_i > 0$ (depending only on h) such that for any $\boldsymbol{a} \leq \boldsymbol{c}$ (coordinatewise) one has:
Any band-limited function $f \in L^p_w(I\!R^m)$ with $\mathrm{spec}(f) = \mathrm{supp}(\hat{f}) \subseteq \Omega$ can be reconstructed from the sampling values over the lattice $(\boldsymbol{a}n)_{n \in \boldsymbol{Z}^m}$ by means of the *cardinal series*

$$f = \sum_{n \in \boldsymbol{Z}^m} \boldsymbol{a}^{-1} f(\boldsymbol{a}n) T_{\boldsymbol{a}n} h \tag{24}$$

Unconditional convergence of the series takes place in the $W(C^0, L^p_w)$-norm, hence in L^p_w as well as uniformly over compact subsets of $I\!R^m$ for $1 \leq p < \infty$.

Proof. We shall use the symbol III for the so-called 'shah-distribution' $\mathrm{III} = \mathrm{III}_1$, given by

$$\mathrm{III}_{\boldsymbol{a}} := \sum_{n \in \boldsymbol{Z}^m} \delta_{\boldsymbol{a}n} \quad .$$

It is clear that $\mathrm{III}_{\boldsymbol{a}} \in W(M, L^\infty)$ for each $\boldsymbol{a}$, actually $\boldsymbol{a} \cdot \mathrm{III}_{\boldsymbol{a}} := \prod_{j=1}^m a_j \cdot \mathrm{III}_{\boldsymbol{a}}$ is uniformly bounded in $W(M, L^\infty)$. Since f is continuous by Lemma 9, $f \cdot \mathrm{III}_{\boldsymbol{a}}$ is well defined as a discrete Radon measure, but the pointwise multiplier result for amalgams gives more: $f \cdot \boldsymbol{a}\mathrm{III}_{\boldsymbol{a}} \in W(C^0, L^p_w) \cdot W(M, L^\infty) \subseteq W(M, L^p_w)$, and even uniform bounded in $W(M, L^p_w)$.

Interpretation of (24) in the distribution theoretic sense shows that we have to verify

$$(\boldsymbol{c}f \cdot \mathrm{III}_{\boldsymbol{c}}) * h = f \ . \tag{25}$$

Given Poisson's formula in the form $(\boldsymbol{a}\mathrm{III}_{\boldsymbol{a}})^{\wedge} = \mathrm{III}_{\boldsymbol{b}}$ for $\boldsymbol{b} = \boldsymbol{a}^{-1}$ we may rewrite (applying the usual rules for the Fourier transform on $\mathcal{S}'(\mathbb{R}^m)$) that this conditions is equivalent to $(\hat{f} * \mathrm{III}_{\boldsymbol{b}}) \cdot \hat{h} = \hat{f}$. Drawing a picture of the compactly supported function $\hat{f}$ and its β-periodic extension $\hat{f} * \mathrm{III}_{\boldsymbol{b}}$ the reader will immediately verify that the given plateau-condition allows to find $\boldsymbol{c}$ such that $\boldsymbol{b} = \boldsymbol{c}^{-1}$ is large enough for the formula to hold, if $\boldsymbol{a} \leq \boldsymbol{c}$ (Ω has to fit into a rectangle of the form $[c_1^{-1}, \dots, c_m^{-1}]$).

In order to check convergence let us observe that Lemma 9 implies convergence of $\boldsymbol{a} \cdot \sum_{|n| \leq k} f(\boldsymbol{a}n)\delta_{\boldsymbol{a}n}$ to $\boldsymbol{a}f \cdot \mathrm{III}_{\boldsymbol{a}}$ for $k \to \infty$ in the norm of $W(M, L^p_w)$ for any $p < \infty$. Applying $W(M, L^p_w) * W(C^0, L^1_s) \subseteq W(C^0, L^p_w)$ we derive that

$$\boldsymbol{a} \cdot \sum_{|n| \leq k} f(\boldsymbol{a}n)\, \delta_{\boldsymbol{a}n} * h = \boldsymbol{a} \cdot \sum_{n=-k}^{k} f(\boldsymbol{a}n)\, T_{\boldsymbol{a}n} h$$

is convergent in $W(C^0, L^p_w)$, and therefore in L^p_w as well as locally uniform. □

Remark 16. Using invertible linear transformations of $\mathbb{R}^m$ the above results can be easily transformed into a result on more irregular sampling schemes without significant change in the arguments. That this is the most general approach to the sampling problem based on a Poisson-type formula can be seen from a recent result of Cordoba [Co].

We want to show next how amalgams can be used to describe the aliasing error in L^p-norms, i.e. the consequences of applying the above formula to $f \in L^p$ which are not band-limited. Obviously the part of $\hat{f}$ exceeding Ω will be responsible for the error, but a simple L^p-estimate of that part is certainly not sufficient. After all, the sampling points are just a set of measure zero in $\mathbb{R}^m$. A sufficient extra condition on $\hat{f}$ is integrability, which allows to obtain uniform estimates for the aliasing error (cf. [BSS], Theorem 3.8). We show that $W(C^0, L^{p'})$-estimates can be obtained under a slightly stronger $W(L^p, L^1)$ assumption on $\hat{f}$.

Lemma 11 (*aliasing error estimate using amalgams*) Assume $\hat{f} \in W(L^p, L^1)$ for some $p \in [1, 2]$. Then $f \in W(C^0, L^{p'})$, and the aliasing error can be estimated as follows. For for any $\hat{h}(t) \equiv 1$ on Ω and $0 \leq \hat{h}(t) \leq 1$ on $\mathbb{R}^m$ there exists some $C_2 > 0$ such that

$$\| f - (\boldsymbol{a}f \cdot \mathrm{III}_{\boldsymbol{a}}) * h \,|W(C^0, L^{p'})\| \leq C_2 \, \| \hat{f} - \hat{f} \cdot \mathbf{1}_\Omega \,|W(L^p, L^1)\| \ . \tag{26}$$

In particular, the aliasing tends to zero for $\boldsymbol{a} \to (0, \dots, 0)$.

Proof. The first statement is a simple consequence of the generalized HY-theorem, by which $\mathcal{F}$ maps $W(L^p, L^1) \subseteq W(\mathcal{F}L^{p'}, L^1)$ to $W(\mathcal{F}L^1, L^{p'}) \subseteq W(C^0, L^{p'})$ (cf. [F2], see [F3] for weighted versions). In order to estimate the aliasing error we split f into a good and a bad part by setting $f_\Omega := \mathcal{F}^{-1}(f \cdot \mathbf{1}_\Omega)$ and $f_r := f - f_\Omega$. Then of course $f_\Omega = (\boldsymbol{a}f_\Omega \mathrm{III}_{\boldsymbol{a}}) * h$, and therefore the aliasing error can be estimated by

$$\begin{aligned}
&\| f - (\boldsymbol{a}f \cdot \mathrm{III}_{\boldsymbol{a}}) * h \,|W(C^0, L^{p'})\| = \| f_r - (\boldsymbol{a}f_r \mathrm{III}_{\boldsymbol{a}}) * h \,|W(C^0, L^{p'})\| \leq \\
&\quad \leq \ \| f_r \,|W(C^0, L^{p'})\| + \| (\boldsymbol{a}f_r \mathrm{III}_{\boldsymbol{a}}) * h \,|W(C^0, L^{p'})\| \leq \\
&\quad \leq \ \| f_r |W(C^0, L^{p'})\| + C \cdot \| \boldsymbol{a}\mathrm{III}_{\boldsymbol{a}} \,|W(M, L^\infty)\| \, \| h \,|W(C^0, L^1)\| \cdot \| f_r \,|W(C^0, L^{p'})\| \ .
\end{aligned}$$

Since the family $\boldsymbol{a}\text{III}_{\boldsymbol{a}}$ is uniformly bounded in $W(M, L^\infty)$ we obtain

$$\|f - (\boldsymbol{a} f \cdot \text{III}_{\boldsymbol{a}}) * h \,|W(C^0, L^{p'})\| \leq C_1 \cdot \|f_r \,|W(C^0, L^{p'})\| \leq C_2 \|\hat{f} - \hat{f} \cdot \mathbf{1}_\Omega \,|W(L^p, L^1)\| \quad .$$

From this it is clear that the aliasing error tends to zero as Ω increases to $\mathbb{R}^m$, and even the speed of convergence can be controlled by the decay of the norm of the high frequency tails, measured in the norm of $W(L^p, L^1)$. □

In Theorem 10 we have restricted ourselves to the use of band-limited functions $h \in L^1_s$ in the reconstruction process, because we wanted to have the result for the full range of values $p \geq 1$ and weights. In fact, the use of the traditional sinc-function, the inverse Fourier transform of the rectangular function, which is *not* in L^1, has to be excluded from the discussion for this reason. On the other hand, the sinc-function (or its multidimensional analog, obtained by pointwise products), belongs to L^p, for any $p > 1$. Thus there is some hope that estimates involving the sinc-function can be obtained in this case. It turns out, however, that the convolution estimate based on the fact that $\sum_{n \in \boldsymbol{Z}^m} f(\boldsymbol{a}n)\, \delta_{\boldsymbol{a}n} \in W(M, L^p_w)$, and therefore $\in W(C^0, L^r)$ for any $r > 1$, is too weak to ensure L^p_w (or even $W(C^0, L^p_w)$ convergence) of the sampling series, even for the trivial weight w. The problem even gets worse if one wants to study jitter errors, because then the usual argument for the L^2-case (it is based on orthogonal series expansions) breaks down. We will show how the generalized HY-theorem can be used to establish appropriate estimates. We write $\mathbf{sinc}(\boldsymbol{x}) := \text{sinc}(x_1) \cdots \text{sinc}(x_m),\ \boldsymbol{x} \in \mathbb{R}^m$.

But first a very useful corollary to the generalized HY inequality.

Lemma 12. The *ideal low pass filter*, i.e. convolution by **sinc**, defines a bounded multiplier from $W(M, L^p)$ into $W(C^0, L^p)$. In particular, for any convergent sequence $(\mu_n)_{n \geq 1}$ in $W(M, L^p)$, with limit μ_0, the sequence $\mu_n * \mathbf{sinc}$ is convergent in $W(C^0, L^p)$.

Proof. We will apply the generalized HY inequality twice. First we observe that $\mu \in W(M, L^p) \subseteq W(\mathcal{F}L^\infty, L^p)$ implies that $\hat{\mu} \in W(\mathcal{F}L^p, L^\infty)$. Since rect $= \mathcal{F}(\mathbf{sinc})$ is known to be a bounded pointwise multiplier for $\mathcal{F}L^p$ for $1 < p < \infty$ (cf. [Pe], Chap.7 or [St], Chap.4), hence $\hat{\mu} \cdot \text{rect} \subseteq \mathrm{W}(\mathcal{F}\mathrm{L}^p, \mathrm{L}^1)$, thus $\mu * \mathbf{sinc} \in \mathcal{F}^{-1}(W(\mathcal{F}L^p, L^1)) \subseteq W(\mathcal{F}L^1, L^p) \subseteq W(C^0, L^p)$, and the required norm estimates hold as well. □

The following result is a partial extensions of Theorem 5 in [Go].

Corollary 13. Let $X = ((x_i)_{i \in I})$ be a well-spread family in $\mathbb{R}^m$ and $\Omega \subseteq \mathbb{R}^m$ be compact. Then

$$S_X f := \sum_{i \in I} f(x_i) T_{x_i} \mathbf{sinc}$$

is unconditionally convergent in $W(C^0, L^p)$, and there exists $C_2 = C(X, \Omega) > 0$ such that

$$\|S_X f \,|W(C^0, L^p)\| \leq C_2 \cdot \|f\|_p \tag{27}$$

for any $p \in (1, \infty)$ and any $f \in L^p(\mathbb{R}^m)$ with spec$(f) \subseteq \Omega$.

Proof. If X is well-spread then $\delta_X := \sum_{i \in I} \delta_{x_i} \in W(M, L^\infty)$, and by Lemma 9

$$\sum_{i \in I} f(x_i) \delta_{x_i} = f \cdot \delta_X \in W(C^0, L^p) \cdot W(M, L^\infty) \subseteq W(M, L^p), \tag{28}$$

for any band-limited $f \in L^p(I\!R^m)$ and the previous Lemma applies. $\square$

In various situations, especially in the discussion of the irregular sampling problem with highly irregular sampling sets (which might have arbitrary high density at some places), the following modification is of interest.

Lemma 14. Let $\Psi = (\psi_i)_{i \in I}$ be a family of measurable functions which satisfy $0 \leq \psi_i(x) \leq 1$, $\mathrm{supp}(\psi_i) \subseteq B_\delta(x_i)$ for all $i \in I$ and $\sum_{i\in I} \psi_i(x) \leq C_\Psi < \infty$ for $x \in I\!R^m$. If $c_i := \|\psi_i\|_1$ then the discrete measure $\mu_\Psi := \sum_{i \in I} c_i \delta_{x_i}$ belongs to $W(M, L^\infty)$, and $\|\mu \,|W(M, L^\infty)\| \leq C_\delta \cdot C_\Psi$ for some $C_\delta > 0$.

Proof. We shall make use of the duality $W(M, L^\infty) = W(C^0, L^1)'$. Thus we only have to obtain an estimate for $f \in \mathcal{K}(I\!R^m)$ (using the positivity of c_i)

$$|\mu_\Psi(f)| \leq \sum_{i \in I} c_i |f(x_i)| = \| \sum_{i \in I} f(x_i) \psi_i \|_1 \tag{29}$$

Fixing $h \in \mathcal{K}(I\!R^m)$ with $h(x) \equiv 1$ on $B_\delta(0)$ we check that $\sum_{i \in I} f(x_i)\psi_i(x) \leq \|T_x h \cdot f\|_\infty$, and consequently we complete the proof by

$$|\mu_\Psi(f)| \leq \|T_x h \cdot f\|_1 = \|F_h\|_1 = \|f \,|W(C^0, L^1)\| \ \ .$$

$\square$

Corollary 15. For Ψ, δ as above and any compact set $\Omega \subseteq I\!R^m$ there exists $C = C(X, \delta, \Omega) > 0$ such that $S_X f := \sum_{i \in I} f(x_i) c_i \cdot T_{x_i}\mathbf{sinc}$ is unconditionally convergent in $W(C^0, L^p)$, and satisfies

$$\|S_X f \,|W(C^0, L^p)\| \leq C \cdot \|f\|_p \tag{30}$$

for any $p \in (1, \infty)$ and any $f \in L^p(I\!R^m)$ with $\mathrm{spec}(f) \subseteq \Omega$.

Proof. This corollary follows from Lemma 14 by means of Lemma 12.

In the discussion of the sampling theorem various kinds of error analysis have to be done. In some of the classical papers (cf. [Pa1,2]) uniform error estimates for L^2-data were considered as sufficient for practical purposes). As we have seen, one may expect $W(C^0, L^p)$ estimates in many cases. That these can actually be obtained has been shown for a variety of situations in [FG3]. Some limiting cases and situations mostly not covered in [FG3] are discussed in the sequel.

Theorem 16 (*Jitter error estimate for p-norms*). Let X be well spread, Ω bounded in $I\!R^m$ and $p, 1 < p < \infty$ be given. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any family $\tilde{X} = (\tilde{x}_i)_{i \in I}$ satisfying $|x_i - \tilde{x}_i| \leq \delta$ the jitter error is small in the $W(C^0, L^p)$-sense, i.e.

$$\| \sum_{i \in I} (f(x_i) - f(\tilde{x}_i))\, T_{x_i}\mathbf{sinc} \,|W(C^0, L^p)\| \leq \varepsilon \cdot \|f\|_p \tag{31}$$

Proof. Without loss of generality we may assume that $f \in \{f \in L^p(I\!R^m),\ \|f\|_p \leq 1 \text{ and } \mathrm{spec}(f) \subseteq \Omega\}$, a set which is known to be equicontinuous in $L^p(I\!R^m)$. Next we observe that

$$|f(x_i) - f(\tilde{x}_i)| \leq \mathrm{osc}_\delta f(x_i) \text{ for } i \in I\ . \tag{32}$$

By Theorem 3 $\mathrm{osc}_\delta f \in W(C^0, L^p)$ (with arbitrary small norm for sufficiently small δ), thus it is possible to find to any given $\eta > 0$ some $\delta > 0$ such that

$$\| \sum_{i \in I} (f(x_i) - f(\tilde{x}_i)) \delta_{x_i} \, | W(M, L^p_w) \| \leq \eta \quad .$$

An application of Lemma 12 concludes the proof. □

Let et us look at a delicate point in the above argument: We did *not* go to absolute values of the sinc-function in the proof. However, we used the fact that smaller (by the absolute value) complex coefficients in the series allow better norm estimates of the corresponding discrete measures in $W(M, L^p)$.

The above result has an important corollary.

Corollary 17. Let Ω be a bounded subset of $I\!\!R^m$ and $1 < p < \infty$ be given. Then for any sufficiently small $\boldsymbol{a} > 0$ there exists some $\delta = \delta(\boldsymbol{a}, \Omega, p)$ such that the operator $f \mapsto S_X f$ is invertible over $L^{p,\Omega}(I\!\!R^m)$ if $|x_n - \boldsymbol{a} n| \leq \delta$ for all $n \in \boldsymbol{Z}^m$. In particular, it is possible to recover f from the irregular sampling values $(f(x_n))^m_{n \in \boldsymbol{Z}}$ by applying the inverse operator S_X^{-1} to $S_X f$ (it can be obtained by Neumann's series).

Remark 17. The above result can also be described alternatively as an iterative algorithm (involving iterative application of $Id - S_X$), which is convergent in the sense of the $W(C^0, L^p)$-norm. Actually, it has been shown that the family $\{T_{x_i} \mathbf{sinc}\}$ is a *Banach frame* in the Banach space

$$L^{p,\Omega}(I\!\!R^m) := \{\, f \in L^p(I\!\!R^m), \ \mathrm{spec}(f) \subseteq \Omega \,\} \quad .$$

This means that the mapping $f \mapsto \langle f, T_{x_i} \mathbf{sinc} \rangle = f(x_i)$ is a mapping from $L^{p,\Omega}$ to the sequence space ℓ^p, $(\sum_{i \in I} |f(x_i)|^p)^{1/p}$ defines an equivalent norm on $L^{p,\Omega}$, and there is a bounded operator $U : \ell^p \to L^{p,\Omega}$, with $U \circ S_X = Id$ on $L^{p,\Omega}$.

The jitter error discussed above is the traditional one. Plotkin (cf. [PRS1,2]) also mentioned a *jitter error of second kind*, arising from the considerations of sums $\sum_{i \in I} \lambda_i T \tilde{x}_i h$ instead of sums $\sum_{i \in I} \lambda_i T_{x_i} h$. It is an open question whether it is possible to give a satisfactory estimate for this jitter error in the L^p-norm for the case of the **sinc**-function (another argument for the use of better decaying kernels), but we can at least prove the following (which guarantees at least uniform estimates).

Proposition 18 (*Jitter error of the second kind*). Let $h \in W(C^0, L^1)$, $p \in [1, \infty)$, and a well-spread family X be given. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\| \sum_{i \in I} \lambda_i T \tilde{x}_i h - \sum_{i \in I} \lambda_i T_{x_i} h \, | W(C^0, L^p) \| \leq \varepsilon \cdot \| \sum_{i \in I} \lambda_i \delta_{x_i} \, | W(M, L^p) \| \tag{33}$$

$$\text{whenever } \ |x_i - \tilde{x}_i| \leq \delta \text{ for } i \in I \ .$$

In the limiting case of $h = \mathbf{sinc}$ we can obtain an estimate in $W(C^0, L^r)$ for any r, $p < r < \infty$, and in particular the uniform jitter error will be small.

Note. The typical application of this result involves $h \in L^1(I\!\!R^m)$, which is band-limited, e.g. a classical de la Vallée Poussin kernel.

Proof. In this case we are forced to use absolute values involving h.
The pointwise estimate $|T\tilde{x}_i h - T_{x_i} h| \leq T_{x_i}(\mathrm{osc}_\delta h)$, gives via Lemma 12

$$\| \sum_{i \in I} \lambda_i T\tilde{x}_i h - \sum_{i \in I} \lambda_i T_{x_i} h \,|W(C^0, L^p)\| \leq C \cdot \| \sum_{i \in I} \lambda_i \delta_{x_i} \,|W(M, L^p)\| \cdot \|\mathrm{osc}_\delta h \,|W(C^0, L^1)\| \quad .$$

By Theorem 3 $\mathrm{osc}_\delta h$ will be small in $W(C^0, L^1)$ for sufficiently small δ. The choice $h = \mathbf{sinc}$ is not covered by this statement, but since the **sinc**-function belongs only to $L^s(I\!\!R^m)$ for any $s > 1$, hence to $W(C^0, L^s)$ by Lemma 12 (actually the norms tend to infinity for $s \to 1$). Therefore the following estimate is possible (but requires smaller and smaller δ for $\varepsilon > 0$, as s goes to 1):

$$\| \sum_{i \in I} \lambda_i T_{\tilde{x}_i} \mathbf{sinc} - \sum_{i \in I} \lambda_i T_{x_i} \mathbf{sinc} \,|W(C^0, L^r)\| \leq$$

$$C \cdot \| \sum_{i \in I} \lambda_i \delta_{x_i} \,|W(M, L^p)\| \cdot \|\mathrm{osc}_\delta \mathbf{sinc} \,|W(C^0, L^s)\| \quad \text{for } 1/r = 1 - (1/p + 1/s) \quad .$$

Uniform convergence follows in each case, or by the choice $s = p'$. □

Corollary 19 (*uniform jitter estimate for sinc-functions*). For any compact Ω, well-spread X, and p, with $1 < p < \infty$, the uniform total jitter is small if $\tilde{X}$ is close to X, i.e. for any $f \in L^p(I\!\!R^m)$ with $\mathrm{spec}(f) \subseteq \Omega$

$$\| \sum_{i \in I} f(x_i) T_{x_i} \mathbf{sinc} - f(\tilde{x}_i) T_{y_i} \mathbf{sinc} \|_\infty \leq \varepsilon \cdot \|f\|_p \tag{34}$$

as long as $|x_i - y_i| \leq \gamma$ and $|\tilde{x}_i - x_i| \leq \gamma$ for some $\gamma \leq \gamma_0 = \gamma_0(\varepsilon)$. If **sinc** is replaced above by some function $h \in W(C^0, L^1)$ the estimate even holds true in the sense of $W(C^0, L^p)$ and for $p \geq 1$.

There is, however, no hope to extend the jitter error estimate for the sinc-function to the case $p = \infty$, as can be shown by the following counterexample. It also gives an answer to the following problem: Given an irregular sampling family $X = (x_n)_{n=1}^\infty$, is it possible that the series $\sum_{n=1}^\infty T_{x_n}$ sinc can have a zero, in other words, is it possible that for some choice of x_n one has $\sum_{n=1}^\infty \mathrm{sinc}(x_n - x_0) = 0$ for some x_0 ? This series may be considered as a low pass filtered version of the discrete measure $\delta_X := \sum_{n=1}^\infty \delta_{x_n}$ (using the rectangular filter), and is proposed as a correction term in the reconstruction procedure suggested by Plotkin.

Proposition 20. Given any lattice constant $\boldsymbol{a} > 0$ and any $\delta > 0$ (the allowed jitter constant). Then for any $\gamma \in I\!\!R$ (or $\gamma = \infty$) it is possible to find a jitter-sequence j_n with $|j_n| \leq \delta$, such that

$$\sum_{n=1}^{\infty} \mathrm{sinc}(\boldsymbol{a} n - j_n) = \gamma \quad ,$$

i.e. the 'jittered series' can take arbitrary values at zero, or may be divergent, even if the jitter error is uniformly small (it would be easy to use summation over $\boldsymbol{Z}$ as well).

Proof. The argument is based on the fact that the sinc-function, defined by $\mathrm{sinc}(x) := \sin(\pi x)/(\pi x)$ for $x \neq 0$, decays only like $1/x$, and that the harmonic series $\sum_{n=1}^\infty 1/n$ is known to be divergent. Without loss of generality we assume that $\boldsymbol{a} = (1, \ldots, 1)$, and that $\gamma > 0$. By means of dilations and choosing some j_n's negative, our argument covers the remaining cases.

The idea is to set those jitter errors, which would give negative contributions, zero (we also could jitter in the other direction) and to have a jitter error in the same direction for every other lattice point. Setting $M := \{1 + 2k,\ k \in I\!N\}$ we plan to keep $j_n = 0$ for $k \notin M$, and to choose j_n in a suitable way for $n \in M$. We will assume for simplicity that $\delta \leq 1/8$.

The main estimate concerns an estimate for the derivative of the sinc-function over the intervals I_k, defined by $I_k := [2k + 3/8, 2k + 5/8]$, $k \geq 2$. One obtains (since $y > 4$, hence $\pi y > 6$ for $y \in I_k, k \geq 2$):

$$\mathrm{sinc}'(y) = \frac{\cos(\pi y)}{y} - \frac{\sin(\pi y)}{\pi y^2} \leq \frac{(\cos(\pi y) + 1/6)}{y} \quad . \tag{35}$$

Noting furthermore that $\cos(z) \leq -\frac{1}{2y} < -\frac{2}{3}$ if $|(2k+1)\pi - z| < \frac{\pi}{4}$ we conclude that

$$\mathrm{sinc}'(y) < \frac{-2/3 + 1/6}{y} = -\frac{1}{2y} \leq -\frac{1}{4k} \quad \text{for } y \in I_k \quad .$$

The mean value theorem implies that for any $u \in I\!R$

$$\mathrm{sinc}(u - j_n) = \mathrm{sinc}(u) - j_n \mathrm{sinc}'(\xi_n) \quad \text{for some } \xi_n \in (n - j_n, n) \quad , \tag{36}$$

hence

$$\mathrm{sinc}(2k + 1 - \delta) \geq \frac{\delta}{4k} \quad \text{for } k \in I\!N \quad . \tag{37}$$

Since the partial sums of the harmonic series are unbounded there exists $k_1 \in I\!N$ such that

$$s := \sum_{k=0}^{k_1 - 1} \mathrm{sinc}(2k + 1 - \delta) \leq \gamma \ , \quad \text{but} \quad s + \mathrm{sinc}(2k_1 + 1 - \delta) > \gamma \ .$$

Using the fact that sinc is strictly decreasing and continuous on $[2k_1 + 1 - \delta, 2k_1 + 1]$ we can find some δ_1 with $0 \leq \delta_1 \leq \delta$ such that $\mathrm{sinc}(2k_1 + 1 - \delta_1) = \gamma - s$. Setting $j_n = \delta$ for $n = 2k+1$, $2 \leq k \leq k_1$, $j_{k_1} := \delta_1$ and $j_n = 0$ otherwise, we find that $\sum_{n=1}^{\infty} \mathrm{sinc}(n - j_n) = \gamma$, i.e. the jittered sampling series can take any prescribed value. $\square$

As an immediate corollary we obtain the counterexample, showing that the low-pass filtered version of an almost regular lattice may have zeros:

Since $\mathrm{sinc}(0) = 1$ it is sufficient to choose j_n in such a way that $\sum_{n=1}^{\infty} \mathrm{sinc}(n - j_n) = -1$ (note that $\mathrm{sinc}(k) = 0$ for any $k \in \boldsymbol{Z}$, $k \neq 0$). Note that it would be possible to ask for *no* jitter error for any term with $n \leq n_0$ (some given number), and still haveing this disastrous phenomenon. It is also possible to show that there is a uniform error estimate if the sequence of jitter errors is itself square summable, i.e. if $\sum_{n=1}^{\infty} j_n^2 < \infty$. Results in this direction will be discussed elsewhere.

In a discussion of band-limited functions on $I\!R$ ([Cl],[CC]) the following question came up: Given a band-limited, square integrable function and a smooth deformation of the real line $\varphi : \mathbb{R} \to \mathbb{R}$, what can be said about $f \circ \varphi$? Since band-limited functions are differentiable it makes sense to discuss only differentiable functions φ which are strictly increasing, i.e. satisfying $0 < \varphi'(t) < \infty$ for all $t \in \mathbb{R}$. According to the conjecture stated in [Cl1] the composition mapping $f \mapsto f \circ \varphi$ should always produce some non-band-limited functions from band-limited ones, except the special case that φ is an affine mapping of the form $\varphi(z) = \alpha z + \beta$ for $\alpha > 0$ and $\beta \in \mathbb{R}$. Along with this question, however, it has to be checked under which circumstances the composition mapping preserves square integrability. Using Wiener amalgam spaces we can give a partial answer.

Proposition 21 (*Preservation of p-integrability under composition*). Assume that for some positive $a > 0$ and $K \geq 0$ we have $\gamma(t) \in [a(t-K), a(t+K)]$ for all $t \in \mathbb{R}$ (i.e. that the graph of γ is contained in a strip in $\mathbb{R}^2$), then for some $C = C(\Omega, K, a) > 0$

$$\|f \circ \varphi\|_p \leq C \cdot \|f\|_p \quad \forall f \in L^p \text{ with } \operatorname{spec}(f) \subseteq \Omega \quad . \tag{38}$$

Note that this statement is *not* valid without the hypothesis of band-limitedness on f. In fact, if for some point $t_n \in \mathbb{R}$ with $|\varphi'(t_n)| \leq \frac{1}{2n^2}$, then $|\varphi'(t)| \leq 1/n^2$ for some interval $[a_n, b_n]$. Setting $\alpha_n := \varphi^{-1}(a_n)$ and $\beta_n := \varphi^{-1}(n_n)$ it follows from the mean value theorem, that $|a_n - b_n| \geq n^2 |\alpha_n - \beta_n|$, and that the indicator function $f := \mathbf{1}_{[\alpha_n, \beta_n]}$ satisfies $\|f \circ \varphi\|_2 \geq n \cdot \|f\|_2$.

Proof. Choosing some function $k \in \mathcal{K}(\mathbb{R}^m)$ such that $k(x) \equiv 1$ on $[-aK, aK]$ we have

$$|f \circ \varphi(t)| \leq \sup\nolimits_{z \in [a(t-K), a(t+K)]} |f(z)| \leq \|T_{at}k \cdot f\|_\infty = F_k(at) \ .$$

Since we know that band-limited functions belong to $W(C^0, L^p)$, i.e. that $\|F_k\|_p \leq C_\Omega \|f\|_p$, we end up with the estimate (using the dilation invariance of L^p)

$$\|f \circ \varphi\|_p \leq a^{m(1-1/p)} \cdot \|F_k\|_p \leq C \cdot \|f\|_p \quad . \tag{39}$$

□

The above proposition gives a sufficient condition to preserve square integrability of band-limited L^p-functions. Improving on the necessary conditions on φ (in order to preserve band-limitedness) as given in [Cl] we mention the following result: Any function γ, which has an unbounded derivative will produce functions which are *not* band-limited!

In fact, band-limited functions satisfy (what is usually called Bernstein's inequality, cf. [FG2], Prop.3.4 for an L^p-version)

$$\sup_{t \in \mathbb{R}} |f'(t)| \leq C \sup_{t \in \mathbb{R}} |f(t)| \quad .$$

Functions γ with unbounded $\gamma'(t)$ (arbitrary steep parts in the graph of γ) apparently destroy this property.

5 Geometric Conditions on the Sampling Sets

One of the basic estimates in irregular sampling theory is the following (we state the L^p-version here), taking a discrete sampling family $X = (x_i)_{i \in I}$.

For any bounded set Ω and p, $1 \leq p < \infty$, there is some $C_\Omega > 0$ such that

$$\left(\sum_{i\in I} |f(t_i)|^p\right)^{1/p} \leq C_\Omega \|f\|_p \ \forall f \in L^p(I\!R^m) \text{ with } \operatorname{spec}(f) \subseteq \Omega \ . \tag{40}$$

This estimate has been first shown for $f \in L^2(I\!R^m)$ and regular lattices by S. Nikolskij, and is known as Nikolskij's inequality. In a more classical setting such estimates have been proved by Plancherel and Polya (cf. [PP]). Recently an irregular version (for sets in the plane which arise as products of irregular sets in each coordinate) has been proposed by Butzer and Hinsen (cf. [BH1,2]). For certain irregular sets these results were proved by Duffin and Shaeffer in [DS] in the one-dimensional case (cf. [DS], or [Y]).

It is clear, that X must not be too dense that such an inequality holds, even if we have very smooth and nice functions f. Certainly there must be no accumulation points to the family X. Since the L^p-spaces are translation invariant the right necessary condition is the following one.

Definition 7. A discrete set $X = (x_i)_{i\in I}$ is called *relatively separated* if there exists an upper bound on the local density of X in $I\!R^m$ in the following sense: For some $r_0 > 0$ there is a uniform bound on

$$d(y) := \#\{i \,|\, x_i \in B_{r_0}(y)\}, \text{ i.e. } d_r(X) := \sup_{y\in I\!R^m} d(y) < \infty \ . \tag{41}$$

It is then clear (any ball can be covered by a finite family of balls of any given size) that for any ball $B \subseteq I\!R^m$ the following is true.

The number of elements in any translate $x + B$ is uniformly bounded.

(I) For some $C_B > 0$ we have $\#\{i \,|\, t_i \in (x + B)\} \leq C_B \ \ \forall x \in I\!R^m$.

(II) For any $r > 0$ the family $(B_r(x_i)_{i\in I})$ of balls of radius r is of bounded height $h(r)$, i.e. there is a maximal number $h(r)$ of balls $B_r(x_i)$ covering any given point. Actually, somewhat more holds.

(II′) Given any compact subset $K \subseteq I\!R^m$ there is a uniform bound on the balls intersecting $x + K$, independently of x , i.e.

$$\sup_{y\in I\!R^m} \#\{i \,|\, (y + K) \cap B_r(x_i) \neq \emptyset\} < \infty \ . \tag{42}$$

Theorem 22. A discrete family $X = ((x_i)_{i\in I})$ in $I\!R^m$ is relatively separated if one (hence all) of the following conditions are satisfied: (I), (II) or

(III) The measure $\sum_{i\in I} \delta_{x_i}$ belongs to $W(M, L^\infty)$, i.e. is *translation bounded* in the sense of [AL].

(IV) For some (any) $1 \leq p < \infty$ there is a constant $C > 0$ such that

$$(\sum_{i\in I} |f(x_i)|^p)^{1/p} \leq C \cdot \|f \,|W(C^0, L^p)\| \ \ \forall f \in W(C^0, L^p) \tag{43}$$

(or only for all functions $f = T_x k$, for some non-zero $k \in \mathcal{K}(I\!R^m)$).

(V) For any $\delta > 0$ the family is a finite union of *separated* families $X^k = (x_i^k)_{i\in I}$, each satisfying $|x_i^k - x_j^k| \geq \delta > 0$ for $i \neq j$.

(VI) X is a finite union of sets which are subsets of sequences each of which is *uniformly dense* in the sense of Duffin and Shaeffer:

$$|x_n - \alpha n| \leq L \ \ \forall n \in \boldsymbol{Z}^m (\text{ for some } \alpha > 0 \text{ and } L > 0) \ . \tag{44}$$

Proof. Taking for granted that the concept of relative separation does not depend on the choice of the radius r_0 we obtain the stronger version of (II) by choosing some r_1 such that $K \subseteq B_{r_1}(x_0)$, observing then that $(y+K)\cap B_r(x_i) \neq \emptyset$ only if $x_i \in B_{r_2}(y)$, for $r_2 = r_1+r$, showing thus that (42) is equivalent with (41). In order to show the equivalence with (III) choose some $k \in \mathcal{K}(I\!R^m)$ such that $0 \leq k(y) \leq 1$, $k(x) \equiv 1$ on $B_s(0)$ and $\mathrm{supp}(k) \subseteq B_r(0)$. Then we have for $\mu := \sum_{i\in I}\delta_{x_i}$

$$M_k(x) := \|(T_x k)\cdot\mu\|_M = \sum_{i\in I}|k(x_i - x)| \leq \#\{i|x_i \in B_r(x)\} \ \ , \tag{45}$$

but on the other hand $M_k(x) \geq \#\{i|x_i \in B_s(x)\}$.

To check that a relatively separated family X satisfies (IV) we use

$$\mu\cdot f \in W(M, L^\infty)\cdot W(C^0, L^p) \subseteq W(M, L^p) \ .$$

Assume conversely, that X is *not* well spread. Then actually for each $r > 0$ there are points x_n in $I\!R^m$ such that $\#\{i|x_i \in B_r(x)\} \geq n$. Let now $k \in \mathcal{K}(I\!R^m)$ be any function with $\min_{y\in B_r(0)}(k(y)) \geq \eta > 0$. Then clearly $T_{x_n}k$ is bounded in $W(C^0, L^p)$ for any $p \geq 1$, but $(\sum_{i\in I}|T_{x_n}k(x_i)|^p)^{1/p} \geq \eta\cdot n^{1/p}$ for each $n \geq 1$, in contradiction to (IV).

(V): It is left as an exercise to the reader to verify that a relatively separated set satisfies (V). Conversely, let X be relatively separated. We cover $I\!R^m$ by (almost) disjoint cubes of side length $\geq \delta > 0$. Then in each of these cubes together with all its 2^m neighbors there are a maximal number n_1 of points available. This can be easily used to split X into at most $n_1\delta$-separated subsequences (cf. [FGr]). Conversely any separated set is obviously relatively separated, and the same is true for finite unions.

(VI): We observe first that property (40) has the following features: If this property holds for any subset Y of set X satisfying (40), and it also holds for finite unions of sets X_i, each of them satisfying (40). Since a set satisfying (44) is relatively separated, the sets described in (VI) are relatively separated by the above argument. The converse requires only slight modifications of arguments used for (V). $\square$

The above result also sheds light on the so-called Parseval relationship for non-uniform sampling appearing in a note by Marvasti and Chuande [MC] which has just appeared. Their proof makes implicit use of the definition of a 'sampling set' in the sense of Duffin and Schaeffer [DS], which implies that we have an estimate of the form (43) for $p = 2$, which means that X is relatively separated by Theorem 22. Actually, the argument used in [MC] is not valid without relative separation. In fact, for general frequencies $(\lambda_n)_{n=1}^\infty$ the convergence of $\sum_{n=1}^\infty a_n e^{-i\pi\lambda_n}$ for sequences $(a_n)_{n=1}^\infty$ in ℓ^2 is only guaranteed in the sense of some mean, and *not* locally in L^2 (it is not difficult to set up simple counterexamples which are divergent over some interval). Given this restriction we can give a proof of Parseval's relationship for non-uniform sampling in several variables.

Theorem 23. Let $(x_n)_{n=1}^{\infty}$ be a relatively separated sampling sequence in $I\!R^m$, and M_{lp} be the Fourier transform of $S_X f := \sum_{n=1}^{\infty} f(x_n) \cdot T_{x_n} g$, where $g \in L^2$ satisfies $\hat{g}(t) \equiv 1$ on Ω. Then M_{lp} belongs locally to L^2, and the following relation holds for any $f \in L^2$ with $\operatorname{spec}(f) \subseteq \Omega$

$$\sum_{n=1}^{\infty} |f(x_n)|^2 = \int_{I\!R^m} \hat{f}(s) \cdot \overline{M}_{lp}(s)\, ds \quad . \tag{46}$$

Proof. We have already discussed convergence of the series on the left side. On the other hand Corollary 13 shows that $S_X f \in W(M, L^2) \subseteq W(\mathcal{F}L^\infty, L^2)$ and thus by the generalized HY inequality $M_{lp} \in W(L^2, L^\infty) \subseteq L^2_{loc}(I\!R^m)$. Since $\hat{f}$ is a compactly supported L^2-function converence of the right hand integral follows. These observations allows us to use the duality pairing $\langle \cdot, \cdot \rangle$ (this time considered as the natural extension of the Hilbert space duality in the argument below) in varying pairs in order to obtain.

$$\sum_{n=1}^{\infty} |f(x_n)|^2 = \langle |f|^2, \delta_X \rangle = \langle f, \delta_X \cdot f \rangle = \langle f * g^*, \delta_X \cdot f \rangle = \tag{47}$$

$$= \langle f, (\delta_X \cdot f) * g \rangle = \langle f, S_X f \rangle = \langle \hat{f}, M_{lp} \rangle = \int_{I\!R^m} \hat{f}(s) \overline{M}_{lp}(s)\, ds$$

the third step following from the fact that f has spectrum in Ω and that convolution by $g^* := \mathcal{F}^{-1}(\hat{g}^-)$ acts therefore trivial. The last step being a simple application of Plancherel's theorem. □

Remark 18: Observe that this result is not only true in several dimensions but is *not* restricted to particular sampling schemes arising as product sets of one-dimensional irregular sampling sets, as in the two-dimensional result given in [MC].

6 Product Convolution Operators And Signal Recovery

Estimates for certain product-convolution operators are also at the heart of a reconstruction method suggested by Donoho and Stark [DS1,2]. The situation under discussion is the following: A function f is given with several parts being missing. This missing information is complemented by some a priori information on its Fourier transform. The well known Papoulis–Gerchberg algorithm covers the case where the spectrum is known to be contained in some bounded spectral set, i.e. covers the case of band-limited functions. Given only a small part of the function it is then possible to recover the full function by an iterative procedure. However, the method may be very instable and sensitive to noise. It also does not cover the case of a possibly unbounded spectrum. The results of Donoho and Stark [DS1,2] show that it is possible to solve the problem in certain cases, e.g. if the spectrum is unbounded, but has finite measure. Of course, there has to be a trade-off between the size of the set T of missing values, and the set Ω on which the Fourier transform $\hat{f}$ of f is concentrated (or unknown). It turns out that one has stable reconstruction by means of an iterative algorithm if $|T|\,|\Omega| < 1$, i.e. if the product of the (Lebesgue) measures of these sets is small enough. Although only the one-dimensional case is treated explicitly in [DS1,2] their arguments extend to m dimensions, and actually to locally compact abelian groups, as pointed out by Smith [Sm].

The key estimate for the recovery (cf. [DS1]), concerns the operator

$$f \mapsto PQf \ \text{ with } \ Q : f \mapsto \mathcal{F}^{-1}(\mathbf{1}_\Omega \hat{f}), \quad P : f \mapsto \mathbf{1}_T f \ \ , \tag{48}$$

where $\mathbf{1}_\Omega$ denotes the indicator function of the set $\Omega \subseteq \mathbb{R}^m$ and $T \subseteq \mathbb{R}^m$ is some subset of $\mathbb{R}^m$. Clearly, another way of considering this operator is to see it as a convolution product, with convolution by $\mathcal{F}^{-1}(\mathbf{1}_\Omega)$, followed by pointwise multiplication with $\mathbf{1}_T$. It is shown that $\|P \circ Q\| \le |W|\,|T|$, the operator norm being for L^p, $1 \le p \le 2$. It is easy to verify this result (due to K. Smith [Sm]). By Hausdorff-Young $\|h_\Omega\|_{p'} \le \|\mathbf{1}_\Omega\|_p = |\Omega|^{1/p}$, and by Hölder's inequality $L^p * L^{p'} \subseteq C^0$, thus

$$\|PQf\|_p \le \|P(Qf)\|_p \le \|\mathbf{1}_T\|_p \|Qf\|_\infty \le (|T|\,|\Omega|)^{1/p}\, \|f\|_p \ \ \forall f \in L^p(\mathbb{R}^m) \ \ . \tag{49}$$

It is clear that under the given circumstances $Q \circ (Id - P) \circ Q$ is invertible as an operator on $L^{p,\Omega}$. However, if $f \in L^{p,\Omega}$ is given over $\mathbb{R}^m \backslash T$, we exactly know $(Id - P) \circ Q$, and therefore, applying the inverse operator, we are able to recover f. Of course, the inversion is carried out by means of Neumann's series and can thus be formulated as an iterative procedure (cf. [DS1], section 4).

The other way of looking at their result was to decompose the mapping $f \mapsto PQf$ into 4 different mappings, which in principle could go through arbitrary Banach spaces of functions or distributions (not only through L^p-spaces).

With $B = L^p$ (actually on any lca. group) the following is natural (and gives the same result as mentioned above). Consider the sequence of mappings

$$f \mapsto \mathcal{F}f \mapsto \mathbf{1}_\Omega \circ \mathcal{F}f \mapsto \mathcal{F}^{-1} \circ (\mathbf{1}_\Omega \mathcal{F}f) \mapsto \mathbf{1}_T \circ \mathcal{F}^{-1}(\mathbf{1}_\Omega \mathcal{F}f) \tag{50}$$

The composed operator is treated as a composition of operators each of which is either a (inverse) Fourier transform or a pointwise multiplier of some indicator function. Using the fact that L^p is (contractively) embedded into the pointwise multiplier algebra from $L^{p'}$ into L^1, we see that the optimal way of looking at the above sequence as operators between the spaces L^p, $L^{p'}$, L^1, L^∞ and L^p (in this order), and in each case the norm of the relevant multiplication operator is just $|\cdot|^{1/p}$ of the underlying set.

It is now evident, that the above chain (50) can be run through various other spaces. The general idea behind such an approach is of course to describe situations, which are not covered by the above estimates, but still allow (maybe under some extra conditions on f) to apply the signal recovery algorithm. As a typical result in this direction obtained by using amalgam spaces we discuss a theorem concerning L^2-functions. Furthermore we fix some r such that $r \ge 2$. Then for $1/p := 1/2 + 1/r$

$$\|f\mathbf{1}_W \,|W(L^2, \ell^p)\| \le \|\hat{f}\,|W(L^2, \ell^2)\| \cdot \|\mathbf{1}_W \,|W(L^\infty, \ell^r)\| \ \ , \tag{51}$$

and by the Hausdorff-Young theorem for Wiener amalgams an estimate for Qf:

$$\|Qf \,|W(L^\infty, \ell^{p'})\| \le C \cdot \|\hat{f}\mathbf{1}_W \,|W(L^2, \ell^p)\| \le C \cdot \|f\|_2 \,\|\mathbf{1}_W \,|W(L^\infty, \ell^r)\| \ \ . \tag{52}$$

Applying now the pointwise multiplier rule for amalgams one has

$$\|PQf\|_2 = \|PQf\,|W(L^2,\ell^2)\| \le \|Qf\,|W(L^\infty,\ell^{p'})\| \cdot \|\mathbf{1}_T\,|W(L^2,\ell^{r'})\| \le$$

$$\le C \cdot \|f\|_2\, \|\mathbf{1}_W\,|W(L^\infty,\ell^r)\| \cdot \|\mathbf{1}_T\,|W(L^2,\ell^{r'})\| \quad .$$

We have shown that at the expense of a more sensitive measurement of W ($\|\mathbf{1}_W\,|W(L^\infty,\ell^2)\|$ instead of $\|\mathbf{1}_W\|_2 = |W|^{1/2}$) we can replace $\|\mathbf{1}_T\|_2 = |T|^{1/2}$ by the much less sensitive measure $\|\mathbf{1}_T\,|W(L^2,\ell^\infty)\| = \sup_{x\in\mathbb{R}^m} |T\cap(x+Q)|$, where Q is the unit cube in $\mathbb{R}^m$ (which may be considered as a *local density measure*). This result can be used as follows.

Assume we know that the set W consists of few disjoint intervals (or cubes), far apart from each other (so that the band-width or even the diameter of the spectrum is large). Then, roughly speaking, the norm of $\mathbf{1}_W \in W(L^\infty,\ell^r)$ corresponds to $k^{1/r}$ if k is the number of intervals of unit length needed to cover W.

Theorem 24. For $r > 0$ and Q open in $\mathbb{R}^m$, with compact closure, there is some $\gamma > 0$ such that any $f \in L^2(\mathbb{R}^m)$, with spec(f) contained in at most n balls of radius r, can be completely recovered from $f\mathbf{1}_M$, if only $\inf_{x\in\mathbb{R}^m} |M\cap(x+Q)| \ge |Q| - \gamma$, i.e. if the local density of the set of missing values is not too large.

7 Wiener Amalgams on Locally Compact Groups and Wavelet Theory

Throughout this paper we have treated Wiener amalgam spaces over $\mathbb{R}^m$, which is of course just a typical special case of a locally compact Abelian group with respect to addition. The natural setting for Wiener amalgam spaces are, however, (at least) general locally compact groups, since only the availabilty of a local norm (such as a p-norm based on the existence of a left invariant Haar measure) and some global norm (again typically some L^q-space with some submultiplicative weight) are required for the definition of the spaces $W(B,Y)$. Since left and right translations don't commute in general, a number of technical problems comes up, and 'left' and 'right' Wiener amalgam spaces have to be considered, the space Y has to be assumed to be both, left- and right translation invariant.

However, amalgam spaces turn out to behave quite satisfactory, and it is not hard to verify duality and pointwise multiplier theorems. For the convolution theorem there is no problem for so-called [IN]-groups, i.e. groups which have the property that some neighborhood Q of the identity is invariant under inner automorphisms, i.e. satisfies $y^{-1}Qy = Q$ (see [BS],[F1]). A typical example is the reduced Heisenberg group $\boldsymbol{H}_m := \mathbb{R}^m \times \mathbb{R}^m \times \boldsymbol{T}$ (cf. [FG4]). In this case the result about convolution triples is still valid (cf. [BS],[F1]), and for a while it seemed to be the natural setting. Later it turned out, that in the proof of certain results about atomic decompositions related to integrable and irreducible representations (presented in [FG4-6] and [Gr]) results of this type were required for groups such as the '$ax+b$-group' of affine transformations of the real line. Actually, the use of the oscillation concept was triggered by the successful use of this concept for questions in general wavelet theory in [Gr]. It turned out, that it is possible to prove the required convolution relations by introducing extra weights (e.g. the weight steming from the asymptotic behavior of the norm of right translation operators over Wiener amalgam spaces defined by means of left translations).

These modified convolution relations, together with the use of operators of the form Sp_Ψ and D_Ψ (as described in section 3), turned out to be extremely useful in proving general results about atomic decompositions (cf. [FG4-6]) or about recovery of a distribution from a family of coefficients, taken with respect to a coherent system of test functions (see [Gr]). In fact, these problems can be shown to be equivalent to problems concerning functions on these locally compact groups which satisfy a certain convolution relation, and are thus closely related to questions about band-limited functions over $\mathbb{R}^m$ (a summary of these connections is given in [F5]). A summary of this theory of coherent non-orthogonal series expansions is given in [CFG], a more comprehensive presentations has been given by Heil and Walnut in [HW]. In section 4 of [HW] it is also pointed out how Wiener amalgam spaces can be used within wavelet theory (see also [He] and [W]).

Amalgam spaces are also useful in many other areas of analysis, e.g. in the study of almost periodic functions [AL], [F6].

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