AN ELEMENTARY APPROACH TO THE GENERALIZED FOURIER TRANSFORM

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HOMMAGE A CAUCHY:

He has taught us the importance of the notion of completeness. The text given below heavily relies on the basic principles of functional analysis connected with this notion: Given a normed space there is an essentially unique completion. Since any bounded linear operator (especially each continuous linear functional) on a normed space extends in a unique way to an bounded operator (functional) on this completion one may always work with Banach spaces (even if their elements are nor given "concretely"). Another way to obtain complete normed, i.e. Banach spaces is to consider a dual space (i.e. the space of all linear functionals of a normed space). It is always a Banach space. In the presentation below we shall take advantage of this fact in the treatment of various Banach spaces of functions and distributions. As we hope to show a good understanding of these principles is perhaps more helpful for an understanding of the generalized Fourier transform on $\mathbb{R}^m$ (see Ref. Wi or a first systematic attempt in this direction) than detailed knowledge of Lebesgue integration (sometimes stressed too much in this connection), for example.

Usual introductions to Fourier analysis start either with Fourier series (e.g.Ref. K) or the Fourier transfrom for Lebesgue integrable functions. As a unifying theory sometimes the general harmonic analysis over locally compact abelian groups or the theory of tempered distributions in the sense of L. Schwartz (see Ref. Schw) is offered. It is the purpose of this note to present a distribution theoretic approach to the Fourier transform, which allows a unified approach to the above questions, including problems arising in signal analysis and linear system theory. Although in this presentation we shall make use of special features of $\mathbb{R}^m$ (e.g. the existence of the Gauss function, which is invariant under the FT) the theory, with somewhat modified arguments, the same theory can be developed in the general setting of lca. groups (as described as natural setting in Ref. W), but even avoiding the use of structure theory of these groups and leading to some kind of distribution theory suitable for the purposes of harmonic analysis (cf. Ref. F8) . Thus our approach is also of theoretical interest.

This note contains in a compressed form parts of lectures by the author, held at the University of Vienna during the last years .

§1 THE BANACH ALGEBRA $L^1(\mathbb{R}^m)$ AND BASIC PROPERTIES OF THE FOURIER TRANSFORM

In this section we shortly summarize essential properties of the Banach convolution algebra $L^1(\mathbb{R}^m)$ and properties of the Fourier transformation defined on this algebra. We start with the space

$\mathcal{K}(\mathbb{R}^m) := \{ k \in C(\mathbb{R}^m)$ i.e. continuous, $\mathbb{C}$-valued on $\mathbb{R}^m$, supp k is compact$\}$ considered as a normed vector space with respect to the following two norms:

$$\|f\|_\infty := \sup_{x\in\mathbb{R}^m}|f(x)| \quad \text{and} \quad \|f\|_1 := \int_{\mathbb{R}^m}|f(x)|dx.$$

Both norms translation invariant and inversion invariant, i.e. the translation operators L_x , $x \in \mathbb{R}^m$, $L_x f(y) := f(y-x)$, and the involutions $\check{}: f\check{}(x) = f(x^{-1})$ and $^*: f^*(x) := \bar{f}(-x)$ are isometric with respect to both norms. It is useful to observe that for any $f \in \mathcal{K}(\mathbb{R}^m)$

$$\|L_y f - f\|_\infty \to 0 \text{ and } \|L_y f - f\|_1 \to 0 \text{ for } y \to 0 . \qquad \text{(CT)}$$

$\mathcal{K}(\mathbb{R}^m)$ is a normed algebra under pointwise multiplication with respect to the first norm, with involution $f \to \bar{f}$ (complex conjugation). Its completion is the Banach algebra $(C^0(\mathbb{R}^m), \|\ \|_\infty)$, where $C^0(\mathbb{R}^m) :=$

$$\{ f:\mathbb{R}^m \to \mathbb{C} \text{ , continuous, complex-valued, with } \lim_{x\to\infty} f(x) = 0 \} .$$

For our purposes the following multiplication will be even more relevant: Given $f,g \in \mathcal{K}(\mathbb{R}^m)$ we define their convolution product $g * f$ by

$$g * f(x) := \int_{\mathbb{R}^m} f(x-y)g(y)\ dy.$$

For this convolution product one has the following norm estimates:
$\|f * g\|_\infty \le \|g\|_1 \|f\|_\infty$ and $\|f * g\|_1 \le \|g\|_1 \|f\|_1$ for $f, g \in \mathcal{K}(\mathbb{R}^m)$. (NE)

Theorem 1.1: $(\mathcal{K}(\mathbb{R}^m), \| \ \|_1)$ is a commutative, normed algebra with respect to convolution as multiplication and isometric involutions * and $\check{}$, i.e. with $(f * g)\check{} = g\check{} * f\check{}$, and $(f * g)^* = g^* * f^*$, satisfying (translation invariance)

$$L_x(f * g) = (L_x f) * g \quad \text{for } f, g \in \mathcal{K}(\mathbb{R}^m) \qquad \text{(TI)}.$$

Proof. Since all functions involved are compactly supported and continuous no problem about existence of integrals can arise (one may treat them in the sense of Riemann), and the most simple version of Fubinis theorem about interchanging order of integration can be applied.

There is one more (functional analytic connection) between the two norms. Defining the space of bounded measures on $\mathbb{R}^m$ as $M(\mathbb{R}^m) := (C^0(\mathbb{R}^m), \| \ \|_\infty)'$ (this notation is justified due to the Riesz representation theorem) we have:

Lemma 1.2. There is an isometric embedding of $(\mathcal{K}(\mathbb{R}^m), \| \ \|_1)$ into $(M(\mathbb{R}^m), \| \ \|)$ through the linear mapping (FTM = function to measure), which maps $k \in \mathbb{R}^m$ to the linear functional

$$\mu_k : f \longrightarrow \int_{\mathbb{R}^m} f(x) k(x) dx . \qquad \text{(FTM)}$$

(thus mapping k to the measure on $\mathbb{R}^m$ with density $k(x)dx$) .

In view of this observation one may identify the (abstract) completion of $(\mathcal{K}(\mathbb{R}^m), \| \ \|_1)$ with the closure (of its image) in $M(\mathbb{R}^m)$. Since one can show that this is just the space of absolutely continuous measures we may (by the Radon - Nikodym theorem) identify this space with $(L^1(\mathbb{R}^m), \| \ \|_1)$ of all Lebesgue-integrable functions (modulo functions vanishing almost everywhere).

It is also easy to see that $(C^0(\mathbb{R}^m), \| \ \|_\infty)$ has bounded approximate units (of norm 1), actually, for any $h \in C^0(\mathbb{R}^m)$ with $h(0)=1$ and $\|h\|_\infty=1$ the sequence $D_{1/n}h$ of dilates of h, defined through $D_\alpha h(x) := h(\alpha x)$ defines such a net, since $\|(D_\alpha h)k - k\|_\infty \longrightarrow 0$ for $\alpha \longrightarrow 0$ for each $k \in C^0(\mathbb{R}^m)$. Note that the operators D_α, $\alpha > 0$, are isometric for $\| \ \|_\infty$.

Corresponding assertions are true with respect to convolution, for $g \in \mathcal{K}(\mathbb{R}^m)$ (or $L^1(\mathbb{R}^m)$) with $\int_{\mathbb{R}^m} g(x)dx = 1$ and positive (thus $\|g\|_1=1$), if we use the normalized contraction, isometric for $\| \|_1$, given by $St_\alpha g(x) := \alpha^{-m} g(x/\alpha)$. Actually, as a consequence of (CT) one has $\|St_\alpha g * f - f\|_1 \longrightarrow 0$ for $\alpha \longrightarrow 0$ for any $f \in \mathcal{K}(\mathbb{R}^m)$ (or $L^1(\mathbb{R}^m)$).
We note further that the operators $(St_\alpha)_{\alpha>0}$ form a group of automorphism of the Banach algebra $L^1(\mathbb{R}^m)$, i.e. satisfy $St_\alpha(f * g) = St_\alpha f * St_\alpha g$ for $f, g \in L^1(\mathbb{R}^m)$.
On the other hand the ordinary dilations $(D_\alpha)_{\alpha>0}$ are automorphisms for pointwise multiplication, since $D_\alpha(f \cdot g) = D_\alpha f \cdot D_\alpha g$ for $f, g \in C^0(\mathbb{R}^m)$. We also mention another group of isometric automorphism for convolution, namely the operators M_t, $t \in \mathbb{R}^m$, where $M_t f$ is the function f modulated with frequency t, i.e. $\big(M_t f\big)(x) := \chi_t(x) \cdot f(x)$, where χ_t is the character (= continuous homomorphism from $\mathbb{R}^m$ into the torus $\Pi := \{z \mid |z| = 1\}$, given by $\chi_t(x) = \exp\big(2\pi i\langle x, t\rangle\big)$, $t \in \mathbb{R}^m$. Actually one has $M_t(f * g) = M_t f * M_t g$.

Summarizing we may state that we have two Banach algebras with bounded approximate units (of norm one), namely $(L^1(\mathbb{R}^m), \| \|_1)$ with respect to convolution, and $(C^0(\mathbb{R}^m), \| \|_\infty)$ with pointwise multiplication. Both of them are isometrically invariant under suitable automorphism groups of normalized dilation operators and are involutive Banach algebras for the two involutions.

FOURIER TRANSFORM

Our next aim is to recall the definition and basic properties of the (ordinary) Fourier transform:

<u>Definition</u> : Let $\langle x, t\rangle = \sum_{i=1}^m x_i t_i$ be the usual scalar product on $\mathbb{R}^m$. For $f \in \mathcal{K}(\mathbb{R}^m)$ (or $L^1(\mathbb{R}^m)$) the Fourier transform $\hat{f}$ is given by

$$\hat{f}(t) := \int_{\mathbb{R}^m} f(x)\, e^{-2\pi i\langle x, t\rangle} dx$$

We denote the mapping $f \longmapsto \hat{f}$ by $\mathcal{F}$ and call it the Fourier transform.

<u>Lemma 1.3</u>: 1) $\mathcal{F}$ is a linear mapping;

2) $\hat{f}$ is uniformly continuous and bounded, with $\|\hat{f}\|_\infty \leq \|f\|_1$;

3) $\mathcal{F}\check{f} = \big(\mathcal{F} f\big)\check{}$ where $\check{f}(x) := f(-x)$;

4) $\mathcal{F}\bar{f} = \left(\mathcal{F}\bar{f}\right)^{\vee}$, $\mathcal{F} f^{*} = \overline{\mathcal{F} f}$ where $f^{*} := \left(\bar{f}\right)^{\vee}$;

5) $(f * g)^{\wedge} = \hat{f}\cdot\hat{g}$ (Convolution theorem).

Theorem 1.4 (Riemann-Lebesgue Lemma): The Fourier transform is a *-Banach algebra homomorphism of norm one, i.e. satisfying $\|\hat{f}\|_{\infty} \le \|f\|_{1}$ for all $f \in L^{1}(\mathbb{R}^{m})$ from the Banach algebra $L^{1}(\mathbb{R}^{m})$ into the Banach algebra $(C^{0}(\mathbb{R}^{m}), \|\ \|_{\infty})$. Furthermore one has the following rules showing that the Fourier transform is an intertwining operator between the various operators:

$$(M_t f)^{\wedge} = L_t \hat{f} \qquad (L_y f)^{\wedge} = M_{-y}\hat{f}$$

$$(St_{\rho} f)^{\wedge} = D_{\rho}\hat{f} \qquad (D_{\rho} f)^{\wedge} = St_{\rho}\hat{f}$$

Next we need an inversion formula. Using Fubini's theorem one obtains

Lemma 1.5: (Fundamental relation) For $f, g \in L^{1}(\mathbb{R}^{m})$ one has

$$\int_{\mathbb{R}^m} \hat{f}(x)\, g(x)\, dx = \int_{\mathbb{R}^m} f(t)\, \hat{g}(t)\, dt$$

Using this and the Stone-Weierstraß it follows that the image $A(\mathbb{R}^{m}) := \mathcal{F}L^{1}(\mathbb{R}^{m})$ is dense in $(C^{0}(\mathbb{R}^{m}), \|\ \|_{\infty})$. This can be used to verify

Corollary 1.6.

The Fourier transform is an injective mapping from L^{1} into C^{0}.

Theorem 1.7 (Inversion theorem):

For $f \in L^{1} \cap C^{b}$ with $\hat{f} \in L^{1}(\mathbb{R}^{m})$ one has the following inversion formula:

$$f(x) = \int_{\mathbb{R}^m} \hat{f}(\xi)\, \exp\left(2\pi i\langle \xi, x\rangle\right)\, d\xi \text{ for all } x \in \mathbb{R}^{m}.$$

Proof. This results arises as a limit (for $\rho \longrightarrow 0$) of the fundamental relation, where one may choose g as $M_x(D_{\rho} g_o)$, having as Fourier transform $L_x(St_{\rho} g_o)$, a function which is highly concentrated at x for ρ close to zero.

Remark 1.1 : It is easy to see that the inversion formula is equivalent to

$$\mathcal{F}\,\mathcal{F}\, f = \check{f}.$$

Consequently one also has in symmetry to the convolution theorem

$$\mathcal{F}(f \cdot g) = \mathcal{F}(f) * \mathcal{F}(g) \qquad \text{(PC)}$$

Remark 1.2: Putting $h := \bar{\hat{g}}$ (for $g \in L^1$ with $\hat{g} \in L^1$) in the above relation one obtains immediately the so-called Parsevals formula:

$$\int_{\mathbb{R}^m} f(x)\ \bar{g}(x)\ dx = \int_{\mathbb{R}^m} \hat{f}(x)\ \bar{\hat{g}}(x)\ dx \qquad \text{(PF1)}$$

which is of special interest for $g = \bar{f}$, reading as follows (L^2-isometry):

$$\int_{\mathbb{R}^m} |f(x)|^2\ dx = \int_{\mathbb{R}^m} |\hat{f}(x)|^2\ dx. \qquad \text{(PF2)}$$

Applying now the general principle (extension of operators to completions) one obtains therefrom:

Theorem 1.8. (Plancherels Theorem) $\mathcal{F}$ "extends" in a unique way to an isometric operator on $L^2(\mathbb{R}^m)$.

We conclude this section with an important consequence: A continuous function $h \in C^0(\mathbb{R}^m)$ belongs to the Fourier algebra $A := \mathcal{F}L^1$ if (and only if) it can be written as a convolution product of two L^2-functions. Thus there are plenty of compactly supported functions in A. We call $g \in L^1(\mathbb{R}^m)$ band-limited if $\hat{g}$ has compact support, i.e. if $\hat{g} \in \mathcal{K}(\mathbb{R}^m) \cap A$. In view of the injectivity of the Fourier transform the natural norm on A is $\|\hat{f}\|_A := \|f\|_1$. We thus have (using also the theorem of Wiener-Levy, cf. Ref. REI , Chap. 1, §3):

Theorem 1.9.. $(A, \|\ \|_A)$ is a regular Banach algebra, called Fourier algebra, of continuous functions, stable under complex conjugation, with local inversion, and such that $A \cap \mathcal{K}(\mathbb{R}^m)$ is a dense subspace of A . In particular, for any open covering by relatively compact sets there is a subordinate partition of unity in A .

Proof. By convolution of characteristic functions one can produce plateau-like functions in A satisfying $h(x) = 1$ on a given compact set $K \subseteq \mathbb{R}^m$ and $\operatorname{supp} h \subseteq U$, any open set with $U \supseteq K$. For more details cf. Ref. REI , §2. It is clear that any $f \in L^1(\mathbb{R}^m)$ can be approximated by band-limited functions, since one has for any $g \in L^1(\mathbb{R}^m)$ with $\hat{g}(0) = 1$ $\|St_\alpha g * f - f\|_1 \longrightarrow 0$ for $\alpha \longrightarrow 0$, and $(St_\alpha g * f)\hat{} = D_\alpha \hat{g} \cdot \hat{f}$ has compact support.

Finally we mention that the m-dimensional Gauss function g_0, with $g_0(x) := \exp[-\pi\langle x, x\rangle]$ is invariant under the Fourier transform (cf. Ref. StW (p.6)):

$$[g_0]\hat{}(t) = g_0(t)$$

§2 DEFINTION, CHARACTERIZATIONS, BASIC PROPERTIES OF A SPACE OF TEST FUNCTIONS

It is our purpose to extend the ordinary Fourier transform as treated in §1 to a larger space of distributions. This can be done, using the fundamental relation above. The recipe to obtain a suitable Banach space of distributions is to look for an appropriate Banach space of test functions invariant under FT. In this section we shall give various equivalent descriptions of such a space and describe its basic properties. A very useful tool for this purpose is the following, usually arising in connection with local frequency analysis of non-periodic functions (cf. e.g. Refs. BA, PA1).

Definition. Given a function $p \in \mathcal{K}(\mathbb{R}^m)$ the Short-Time Fourier Transform of a locally integrable function $f \in L^1_{loc}(\mathbb{R}^m)$ with respect to the "window function" p is given by

$$STFT_p(f)(x,t) := \int_{\mathbb{R}^m} p(x-y)f(y)e^{-2\pi i \langle y,t \rangle} dy \text{ for } x,t \in \mathbb{R}^m .$$

It can be shown that $STFT_p(f)$ is square integrable, i.e. in $L^2(\mathbb{R}^m \times \mathbb{R}^m)$, if (and only if) $f \in L^2(\mathbb{R}^m)$ (with equivalent norms, by "MOYAL's formula"), and is a bounded function for any $f \in L^1+L^\infty(\mathbb{R}^m)$. It turns out that better resolution in the time AND frequency direction is obtained if one uses instead of p as above the Gauss function g_o, having several other advantages. (Actually this function is, up to dilations, characterized through the property that for it one obtains equality in Heisenberg's uncertainty relation)

Our test space is now defined as follows:

Definition. Denote by g_o the Gauss function and set: $S_o(\mathbb{R}^m) :=$ $\{ f \mid f \text{ continuous and integrable, with } \iint |STFT_{g_o}(f)(x,t)| dx\, dt < \infty \}$.
As a natural seminorm (which will turn out to be a nice norm) on $S_o(\mathbb{R}^m)$ we have $\|f\|_{S_o} := \|STFT_{g_o}\|_1$.

The basic properties concerning this space and its invariance properties are given in the following theorem:

Theorem 2.1. (BASIC PROPERTIES)

i) $(S_o(\mathbb{R}^m), \| \ \|_{S_o})$ is a Banach space, continuously embedded into $L^1(\mathbb{R}^m)$ and $C^o(\mathbb{R}^m)$.

ii) It is isometrically invariant under the translation operators L_x, $x\in\mathbb{R}^m$, and multiplications with characters (i.e. the operators M_t) as well as under the Fourier transformation.

iii) $S_o(\mathbb{R}^m)$ is a Banach ideal in the Banach convolution algebra $L^1(\mathbb{R}^m)$, i.e. $S_o(\mathbb{R}^m) * L^1(\mathbb{R}^m) \subseteq S_o(\mathbb{R}^m)$, and one has the corresponding norm estimate: $\| g * f \|_{S_o} \le \| g \|_1 \| f \|_{S_o}$ for $g \in L^1, f \in S_o$.

iv) $S_o(\mathbb{R}^m)$ is a pointwise Banach ideal in the Banach algebra $\mathcal{F}L^1$ (again with appropriate norm estimates).

v) Translation (and also character multiplication) is continuous from $\mathbb{R}^m$ into $S_o(\mathbb{R}^m)$, i.e. given $f \in S_o(\mathbb{R}^m)$ the mapping $x \mapsto T_x f$ (resp. $t \mapsto M_t f$) from $\mathbb{R}^m$ into $S_o(\mathbb{R}^m)$ is continuous (with respect to $\| \ \|_{S_o}$).

vi) Consequently, one has for any $k \in L^1(\mathbb{R}^m)$, with $\int_{\mathbb{R}^m} k(x)dx = 1$: $\lim_{\rho \to 0} \|(St_\rho k) * f - f\|_{S_o} \to 0$ for $\rho \to 0$; via FT this implies

Remark 2.1. Actually, the fact that $S_o(\mathbb{R}^m)$ is minimal within the family of spaces satisfying i), ii) and v). In Ref. F1 it is shown that $S_o(\mathbb{R}^m)$ (defined in a different way there), endowed with the natural norm, is the smallest strongly character invariant Segal algebra on $\mathbb{R}^m$ (cf. Ref. REI , Chap. 6, §2). It has been the basis for the proof of many useful properties of $S_o(G)$, for G lca. as discussed in Refs. F1-F8. In the present approach for $\mathbb{R}^m$ we use special properties of g_o .

One of the advantages of taking g_o as window function is it's invariance under the Fourier transform which implies the relation

$$|STFT_{g_o}(f)(x,t)| = |STFT_{g_o}(\hat{f})(-t,x)| \quad \text{for } t,x \in \mathbb{R}^m.$$

It follows that we have the following invariance property:

Theorem 2.2. The Fourier transform is an isometric mapping from $S_o(\mathbb{R}^m)$ onto itself. It interchanges the role of pointwise and convolutive structure. Furthermore, $S_o(\mathbb{R}^m)$ is invariant under autormorphisms of $\mathbb{R}^m$, especially the dilation operators D_α and St_α leave $S_o(\mathbb{R}^m)$ invariant for $\alpha>0$. Moreover one has for $h \in \mathcal{F}L^1$ with $h(0)=1$: $\lim_{\rho \to 0} \|(D_\rho h)\cdot f - f\|_{S_o} \to 0$ for $\rho \to 0$.

Thus our space $S_o(\mathbb{R}^m)$ enjoys many properties common with the Schwartz space $S(\mathbb{R}^m)$, such as invariance with respect to the Fourier transformation, but has the advantage of being a Banach space (isometrically invariant under the operators L_x and M_t, i.e. translation and multiplication with pure frequencies).

In order to give some information about the indepence of this space on the particular choice of the window (i.e. g_o in our case) and sufficient conditions for membership in $S_o(\mathbb{R}^m)$ we mention the following:

<u>Theorem 2.3.</u> The following conditions imply that $f \in S_o(\mathbb{R}^m)$:

i) f integrable and band-limited (i.e. if supp $\hat{f}$ is compact).

ii) $f \in \mathcal{K}(\mathbb{R}^m)$ with $\hat{f} \in L^1(\mathbb{R}^m)$.

iii) fw_s and $\hat{f}w_s \in L^2(\mathbb{R}^m)$, with $w_s(x) := (1+|x|)^s$, for some $s > 2m$. In particular $S(\mathbb{R}^m)$ is continuously and densely embedded into $S_o(\mathbb{R}^m)$. Consequently $g_o \in S_o(\mathbb{R}^m)$.

iv) Assume that $f = g * h$, where g, h are two integrable, continuous, radial symmetric and decreasing functions on $\mathbb{R}^m$. Then $f \in S_o(\mathbb{R}^m)$.

<u>Proof.</u> i) and ii) will follow from observations given below, while iii) is a consequence of the fact that

$|STFT_{g_o}(f)(x,t)| = |M_t g_o * f(x)| \le |g_o| * |f|(x)$, and on the other hand

$|STFT_{g_o}(\hat{f})(t,-x)| = |M_x g_o * \hat{f}(t)| \le |g_o| * |\hat{f}|(t)$ for all $x, t \in \mathbb{R}^m$,

and the observation that one has $(h * f)w_s \in C^o(\mathbb{R}^m)$ if hw_s, $fw_s \in L^2(\mathbb{R}^m)$, because the weight function w_s is submultiplicative for $s\ge 0$ (cf.Ref. REI), i.e. satisfies $w_s(x+y) \le w_s(x)w_s(y)$ for all $x, y \in \mathbb{R}^m$ (for s=1 this is obvious, and follow for s>0 therefrom). For arguments concerning iv) cf. Ref. F1 .

We mention that conditions i-iv) are useful in order to show that most classical kernels (such as the De la Vallé-Poussin kernel or the Fejer kernel, but of course not the Dirichlet kernel, belong to $S_o(\mathbb{R})$).

As a consequence the following space is certainly contained in $S_o(\mathbb{R}^m)$ for each $g \in S_o(\mathbb{R}^m)$:

$$S_g := \{ f \,|\, f = \sum_{n=1}^{\infty} a_n M_{t_n} L_{x_n} g \text{ , with } t_n, x_n \in \mathbb{R}^m \text{ and } \sum_{n=1}^{\infty} |a_n| < \infty \}.$$

Actually, it can be shown that for $g \neq 0$ the above space, with its natural norm (the infimum over the expressions $\sum_{n=1}^{\infty} |a_n|$), coincides with $S_o(\mathbb{R}^m)$ a Banach space (cf. Refs. F4,F5).

Another elegant characterization available is the following one: Having verified these properties our next aim is to find alternative descriptions of $S_o(\mathbb{R}^m)$, which finally should end up with the characterization through Gabor-sums indicated in the introduction. Actually, our aim will be to show that we have the following result (in the terminology of Ref. PER this can be called an coherent expansion, because our building blocks are obtained from a basic building block under the action of the Heisenberg group) :

Theorem 2.4. (GABOR-REPRESENTATION)

There exist $\alpha_o > 0$ and $\beta_o > 0$ such that for $\alpha \le \alpha_o$ and $\beta \le \beta_o$ there exists $C = C(\alpha,\beta) > 0$ with the following property:

$f \in S_o(\mathbb{R}^m)$ if and only if $f = \sum_{n,k} a_{n,k} M_{\beta n} L_{\alpha k} g_o$, for some double sequence of coefficients satisfying

$$\sum_{n,k} |a_{n,k}| \le C \, \| f \|_{S_1} .$$

The last result can be conveniently used to verify the restriction properties of $S_o(\mathbb{R}^m)$ (i.e. restrictions to k-dimensional plane in $\mathbb{R}^m$ leaves the corresponding space $S_o(\mathbb{R}^n)$ as space of traces. Using it is possible to consider distributions on $\mathbb{R}^k$ as distributions on $\mathbb{R}^m$, via $f \longmapsto \sigma(f|_{\mathbb{R}^n})$ (cf. Thm. 3.5 below and Refs. F6,F7 or Tr2 for more general results).

One more characterization of $S_o(\mathbb{R}^m)$ as subspace of A is the following one:

Theorem 2.5: A function $f \in A$ belongs to $S_o(\mathbb{R}^m)$ if and only if for some (any) fixed $g \in S_o(\mathbb{R}^m)$, $g \neq 0$, the function $y \longmapsto \| (L_y g)\cdot f \|_A$ belongs to $L^1(\mathbb{R}^m)$. The expression $\int_{\mathbb{R}^m} \| (L_y g)\cdot f \|_A \, dy$ defines an equivalent norm on $S_o(\mathbb{R}^m)$.

Another characteristic feature of $S_o(\mathbb{R}^m)$ is its minimality property:

Definition : Fixing any open, relatively compact subset $Q \subseteq \mathbb{R}^m$ $S_o(\mathbb{R}^m)$ coincides with the set of all (continuous) functions $f \in L^1(\mathbb{R}^m)$, having an (L^1-convergent) representation as a sum $f = \sum_{n=1}^{\infty} L_{y_n} f_n$ with $\sum_{n=1}^{\infty} \| f_n \|_A < \infty$ and $f_n \in A_Q(\mathbb{R}^m) := \{f | f \in A(\mathbb{R}^m), \text{supp } f \subseteq Q\}$, for $y_n \in \mathbb{R}^m$ arbitrary. Every representation of a function $f \in S_o(\mathbb{R}^m)$ as $\sum_{n=1}^{\infty} L_{y_n} f_n$, satisfying the above conditions is called *admissible* representation of f.

More informations are collected in Refs. F1 - F8. We also mention that the elements of $S_o(\mathbb{R}^m)$ can be characterized through integrability of their ambiguity function (cf.Refs. Sch1,Sch2 or PA1 for these notions) or their associated Wigner distribution.

§3 AN INVARIANT BANACH SPACE OF DISTRIBUTIONS AS A TOOL IN HARMONIC ANALYSIS.

In this section we shall introduce the dual space $S_o'(\mathbb{R}^m)$, show how "ordinary" functions can be identified with elements (we call them distributions) of $S_o'(\mathbb{R}^m)$ in a natural way, and how most natural notions and operations (e.g. support, translation, Fourier transform etc.) extend to these "generalized functions" in an essentially unique way.

We shall not only be interested in $S_o'(\mathbb{R}^m)$ as a Banach space with its natural norm $\|\sigma\| := \sup_{\|f\|_{S_o}=1} |\sigma(f)|$, but also as a topological vector space with respect to the w*-topology (i.e. the topology of pointwise convergence), having as a basis for the neighbourhood of $\sigma_o \in S_o(\mathbb{R}^m)'$ the system

$$U(F,c) := \{ \sigma \in S_o'(\mathbb{R}^m),\ |\sigma_o(f) - \sigma(f)| < c \text{ for all } f \in F \},$$

where F runs through the finite subsets of $S_o(\mathbb{R}^m)$ and $c>0$ is arbitrary.

We first identify several important subclasses of elements of $S_o'(\mathbb{R}^m)$:

Theorem 3.1. i) Assume that $h \in L^1 + L^\infty(\mathbb{R}^m)$. Then $\mu_h: f \rightarrow \int_{\mathbb{R}^m} f(x)h(x)dx$ defines an element of $S_o'(\mathbb{R}^m)$ (and $\mu_h = \mu_g$ if and only if

$h(x) = g(x)$ a.e.). Distributions arising in this way are called <u>regular</u> distributions. In this sense any L^p-space (with respect to Lebesgue measure) may be considered as subspace of $S_0'(\mathbb{R}^m)$.

ii) Any bounded measure μ defines a unique distribution in $S_0'(\mathbb{R}^m)$. Especially, the point measures δ_x: $f \longrightarrow f(x)$ are in $S_0'(\mathbb{R}^m)$.

iii) A (possible) unbounded discrete measure of the form $\sum_{n=1}^{\infty} a_n \delta_{x_n}$ belongs to $S_0'(\mathbb{R}^m)$ if one of the following two conditions is satisfied: a) $\sum_{n=1}^{\infty} |a_n| < \infty$ or
b) There is some $\gamma > 0$ such that $|x_n - x_m| > \gamma$ for $n \neq m$. In particular, the Shah-distribution (of impulse train) Ш belongs to $S_0'(\mathbb{R}^m)$.

Next we introduce the notion of a support for our distributions.

<u>Definition.</u> Given $\sigma \in S_0(\mathbb{R}^m)'$ the <u>support</u> supp σ is defined as the complement of the open set of all points x such that the "action of σ near x is trivial", i.e. such that for some neighbourhood U of x one has $\sigma(f) = 0$ for all $f \in A_U \subseteq S_0$, where $A_U := \{ f \in A,\ \text{supp } f \subseteq U \}$. It is worth mentioning that this notion is compatible with the notion of a support of an ordinary function (which may be considerd as distribution also), given by $\{ x |\ f(x) \neq 0 \}^-$. Moreover there is the following expected result:

<u>Lemma 3.2.</u> Assume that $f \in S_0(\mathbb{R}^m)$ and $\sigma \in S_0'(\mathbb{R}^m)$ are given such that $\text{supp } f \cap \text{supp } \sigma = \emptyset$. Then $\sigma(f) = 0$.

Now we are ready to extend various operations to setting of distributions. As we shall see below there is in most cases precisely one definition of such an operation on distributions which coincides with a given operation on ordinary functions and has the property of being continuous with respect to the w*-convergence. However, for simplicity (it is also more efficient) we define these operations by duality. In this connection it is convenient to describe the duality by $\langle \sigma, f \rangle$ instead of $\sigma(f)$, thus stressing the bilinear character of the mapping $(\sigma, f) \longrightarrow \sigma(f)$.

Thus we define dilation, translation, convolution by functions in L^1:

$$\langle St_\rho \sigma, f \rangle := \langle \sigma, D_\rho f \rangle.$$

$$\langle L_y\sigma, f\rangle := \langle \sigma, L_{-y}f\rangle$$

$$\langle M_t\sigma, f\rangle := \langle \sigma, M_t f\rangle$$

$$\langle h\cdot\sigma, f\rangle := \langle \sigma, h\cdot f\rangle$$

$$\langle h * \sigma, f\rangle := \langle \sigma, \check{h} * f\rangle$$

$$\langle \check{\sigma}, f\rangle := \langle \sigma, \check{f}\rangle$$

Finally the fundamental relation suggests to define the Fourier transform of $\sigma \in S_0'(\mathbb{R}^m)$ through $\langle \hat{\sigma}, f\rangle := \langle \sigma, \hat{f}\rangle$

The inversion formula for the ordinary Fourier transform now just gives $\hat{\hat{f}} = \check{f}$ for all for all $f \in S_0(\mathbb{R}^m)$. Thus for $\sigma \in S_0'(\mathbb{R}^m)$

$$\langle \hat{\hat{\sigma}}, f\rangle = \langle \sigma, \hat{\hat{f}}\rangle = \langle \sigma, \check{f}\rangle = \langle \check{\sigma}, f\rangle.$$

<u>Remark 3.1.</u> It is perhaps useful to mention that one can show that $h * \sigma$ is a regular distribution, arising from the function (in $C^b(\mathbb{R}^m)$) $x \longmapsto \sigma(L_x(f^{\check{}}))$. Using the existence of L^1-bounded approximate units in $S_0(\mathbb{R}^m)$ this allows an easy proof that $C^b(\mathbb{R}^m)$ is a w^*-dense subspace of $S_0'(\mathbb{R}^m)$. Applying then pointwise approximate units (e.g. $D_\alpha h$, with $h(0)=1$, $h \in S_0(\mathbb{R}^m)$) it can be shown that actually S_0 is a dense subspace of S_0'. In a similar way one can show that the finite discrete measures of the form $\Sigma_{i\in F}\, a_i \delta_{x_i}$ constitute another w^*-dense subspace of $S_0'(\mathbb{R}^m)$. This implies among others:

<u>Theorem 3.3.</u> The Fourier transform is continuous automorphism of the Banach space $S_0'(\mathbb{R}^m)$, which is furthermore continuous with respect to the w^*-topology. It is therefore the unique w^*-continuous extension of the classical Fourier transform, considered as a mapping from $S_0(\mathbb{R}^m)$ into itself. The inverse mapping is of the form $\sigma \longmapsto \hat{\sigma}^{\check{}}$. At the same time it is the unique $S_0'(\mathbb{R}^m)$ w^*-continuously into itself, and δ_x to χ_{-x} (or χ_t to δ_t).

Now, in conjunction with the notions of the (generalized) FT and of a support we introduce the <u>spectrum</u> of $\sigma \in S_0'(\mathbb{R}^m)$ by $\operatorname{spec}\sigma := \operatorname{supp}\hat{\sigma}$. Although we don't discuss details here we mention that for $h \in L^\infty(\mathbb{R}^m) \subseteq S_0'(\mathbb{R}^m)$ this notion coincides with the traditional one (as described in Ref. REI , Chap 7 or Ref. Ar). In all situations it is a good idea to think of spec σ as the set of "relevant" frequencies for σ , which can be "filtered" out of σ by convolution.

The following result may be considered as a characterization of "plane waves" in $\mathbb{R}^m, m\geq 2$, telling us that for plane waves (constant in one direction) only "plane frequencies" are relevant. More precisely, we have:

Theorem 3.4. Let H be a closed subgroup of $\mathbb{R}^m$, e.g. a lower-dimensional subspace (isomorphic to $\mathbb{R}^n$) or a lattice like $\mathbb{Z}^m$, or a combination of such spaces. Denote by $H^\perp$ the (group theoretical) orthogonal complement of H in $\mathbb{R}^m$, given by

$\{ t \in \mathbb{R}^m | \ \chi_x(t) = \exp(2\pi i<x,t>=1 \text{ for all } x \in H \}$. Then one has

i) supp $\sigma \subseteq H$ if and only if $L_z\hat{\sigma} = \hat{\sigma}$ for all $z \in H^\perp$.

ii) spec $\sigma \subseteq H$ if and only if $L_h\sigma = \sigma$ for all $h \in H$.

The most important special case is that of an ordinary periodic function (thus satisfying ii) for $H = \alpha_1\mathbb{Z}x\alpha_2\mathbb{Z}x\ldots\alpha_m\mathbb{Z}$, hence $H^\perp = \alpha_1^{-1}\mathbb{Z} \times \alpha_2^{-1}\mathbb{Z} \times\ldots\times\alpha_m^{-1}\mathbb{Z})$. The spectrum of σ thus will be concentrated on a lattice (which is wide for small periods and small for large periods) and we may expect (cf. below) $\hat{\sigma}$ to be a discrete measure "living" on that lattice (with coefficients equal to the classical Fourier coefficients).

The most obvious examples of distributions concentrated on subgroups are those arising from distributions "living" on that subgroup. Thus, e.g. $\tau \in S_0'(\mathbb{R}^n)$ for some n < m, defining a distribution on $\mathbb{R}^m$ by $\sigma_\tau(f) := \tau(f|_{\mathbb{R}^n x\{0\}})$, for $f \in S_0(\mathbb{R}^m)$.

That these are all possible distributions with supp $\sigma \subseteq \mathbb{R}^n x\{0\}$ is the content of the following result (special case $H = \mathbb{R}^n x\{0\}$). For better understanding of this result let us mention that for a discrete subgroup H (lattice) in $\mathbb{R}^m$ $S_0'(H)$ should be understood as the space of all discrete measures $\sum_{n\in\mathbb{Z}^m} a_n\delta_n$, with $\sup_n|a_n| < \infty$.

Theorem 3.5. Let H be a closed subgroup of $\mathbb{R}^m$. Then a distribution $\sigma \in S_0'(\mathbb{R}^m)$ is concentrated on H (i.e. supp $\sigma \subseteq H$) if and only if there exists $\tau \in S_0'(H)$ such that $\sigma(f) = \tau(f|_H)$ for all $f \in S_0(\mathbb{R}^m)$.

As a combination of the last two theorems (or of course by other arguments to be found in the literature) one can find the following result, which is equivalent to the classical Poisson formula (cf. Ref. StW , p. 252, or Ref. REI , p. 120).

Corollary 3.6. The Shah-distribution Ш is invariant under Fourier transform. In fact, it is - up to normalization - the only $\mathbb{Z}^m$-invariant distribution concentrated on the lattice $\mathbb{Z}^m$.

The proof of Thm. 3.5 is based on the fact that a closed subgroup is a set of synthesis for $S_o'(\mathbb{R}^m)$. The question of spectral synthesis can be described as follows: As we have seen, spec σ contains the relevant frequencies. On the other hand (combining Theorems 3.3 and 3.4) we know that any $\sigma \in S_o'(\mathbb{R}^m)$ arises as the w*-limit of trigonometric polynomials (approximate $\hat{\sigma}$ by finite, discrete measures and take the inverse FT). Now the question arises, whether it is sufficient to use the "relevant" frequencies for this synthesis. In view of the invariance of our spaces under Fourier transform the following definition (which can be introduced without theory of FT) turns out being useful but less complicated:

Definition. A distribution $\sigma \in S_o'(\mathbb{R}^m)$ admits synthesis, if it can be approximated in the w*-sense by finite discrete measures $\Sigma_{i \in F}\, a_i \delta_{x_i}$, with $x_i \in \operatorname{supp} \sigma$ for all $i \in F$. A closed subset $M \subseteq \mathbb{R}^m$ is called a set of synthesis if it is true that any element $\sigma \in S_o'(\mathbb{R}^m)$ with supp $\sigma \subseteq M$ can be approximated in the w*-sense by finite discrete measures $\Sigma_{i \in F}\, a_i \delta_{x_i}$, with $x_i \in M$ for all .

Without going into details we mention here that this notion is actually equivalent to the classical notion of sets of synthesis (cf. Refs. BE, REI). As an argument in this direction we mention the usual produre of retranslation questions of synthesis to the uniqueness of a closed ideal (for pointwise multiplication) with cospectrum (= set of common zeros) equal to M in $S_o(\mathbb{R}^m)$ or the Fourier algebra. However, in view of the ideal theorem for Segal algebras (cf. Ref. REI , Chap. 6, §2.4) this uniqueness holds true for $A(\mathbb{R}^m)$ if and only it holds true for $S_o(\mathbb{R}^m)$. Referring to Ref. REI , Chap. 2 §5.2 we thus have that the empty set and any discrete set is a set of synthesis. Via Fourier transform this tells us that in case supp σ consists of isolated points only σ is of the form $\sigma = \sum_{i \in I} a_i \delta_{x_i}$.

Theorem 3.3. is also of practical value. As a typical application we have the following situation: Given some $\sigma \in S_o'(\mathbb{R}^m)$ of which we want to obtain the Fourier transform. Then any w^*-approximation can be taken (either in S_o or a discrete measure), of which the FT can be obtained in the classical way (or in a simple way resulting in a trigonometric function) and going to the w^*-limit finally. In this way the connection between classical FT and the theory of Fourier series (on most texts treated in a formal way, especially as far as the motivation for the inversion formula ...) can be explained in more detail. Thus, e.g. any $\mathbb{Z}^m$-periodic L^1-function (this means f is periodic and locally integrable) can be obtained as a w^*-limit of integrable functions. In fact, we have $f = \sum_{n \in \mathbb{Z}^m} L_n f_o$ (f_o representing the basic period), which may be seen as a w^*-limit of finite partial sums $\sum_{n \in F} L_n f_o$ (F finite in $\mathbb{Z}^m$). Thus the convergence of $(\sum_{n \in F} L_n f_o)\hat{}$ takes place in the w^*-topology. As expected, one may verify that the limit is of the form $\sum_{n \in \mathbb{Z}^m} \hat{f}(n) \delta_n$.

Conversely, it is possible to come from the theory of Fourier series (means representation of periodic functions through series, or at least as w^*-limit of suitable means for their series) the Fourier transform by observing that for $f \in L^1$ (we assume for simplicity that it has compact support) that $f = \lim_{n \to \infty} F_n$, with $F_n = \sum_{k \in \mathbb{Z}^m} L_{nk} f$. Using our interpretation of the Fourier transform it is clear that spec F_n is a discrete lattice of order $1/n$. It is not difficult to verify that the discrete measures $\hat{F}_n$ actually are w^*-convergent to some continuous function, in fact, to $\hat{f}$, as defined at the beginning. (cf. Ref. PA1 for typical arguments presented usually).

Another situation, where our approach might bring some clarification is the following one. What about the inversion of the Fourier transform for functions (e.g. $f \in L^1 \cap C^b(\mathbb{R}^m)$) having a non-integrable FT $\hat{f}$. A natural way to appoximate $\hat{f}$ by functions which are integrable is to multiply it with some cut-down function to make it integrable. At the and it turns out that considering $\hat{f}$ as a limit of $D_\alpha h\cdot\hat{f}$ for some $h\in S_0(\mathbb{R}^m)$ with $h(0)=1$ is quite reasonable, since one has (assuming $h=\hat{g}$ for $g\in S_0$) $D_\alpha h\cdot\hat{f}=(St_\alpha g * f)^\wedge \in S_0 * L^1 \subseteq S_0(\mathbb{R}^m)$, thus the inversion theorem 1.7 is applicable. Finally, we note that $St_\alpha g * f \to f$ uniformly (since $f\in C^0$) (we would also have approximation in the L^p-norm if $f\in L^p$). This applies to most situations where classical kernels (the so-called means of Cesaro, Abel, De La Valle-Poussin ...) are involved.

§4. FURTHER APPLICATIONS.

In this section we shall indicate various further applications of our space $S_0'(\mathbb{R}^m)$ (beyond that of understanding the generalized FT). As a first result we discuss the characterization of translation invariant operators. These operators arise in system theory under the name of LTIS (linear translation invariant operators), i.e. essentially mappings which commute with the family of all translations operators L_x and some kind of superposition principle, together with mild continuity properties. In classical notation one has the following threee results (the first is called Wendel's theorem, and the second one is an immediate consequence of Plancherel's theorem.

Theorem 4.1. Let T be a linear operator satisfying $TL_x=L_xT$ for $x\in\mathbb{R}^m$. Then additional continuity assumptions imply that T can be represented as convolution operator (i.e. $Tf(x)=\sigma(L_x f^\vee)$ for $f\in S_0$) where:

i) continuity of T: $\mathcal{K}(\mathbb{R}^m) \longmapsto L^1(\mathbb{R}^m)$ for $\|\ \|_1$ implies $\sigma\in M$;

ii) continuity of T: $\mathcal{K}(\mathbb{R}^m) \longmapsto C_0(\mathbb{R}^m)$ for $\|\ \|_\infty$ implies $\sigma\in M$;

iii) continuity of T: $\mathcal{K}(\mathbb{R}^m) \longmapsto L^2(\mathbb{R}^m)$ for $\|\ \|_2$ implies $\sigma\in PM(\mathbb{R}^m)$, or in other words, $(Tf)^\wedge = h\hat{f}$ for some $h\in L^\infty(\mathbb{R}^m)\subseteq S_0'(\mathbb{R}^m)$ and $\sigma\in S_0'(\mathbb{R}^m)$ satisfies $\hat{\sigma}=h$.

For the representation of operators which are just continuous from one L^p-space to another the space $PM(\mathbb{R}^m)$ of so-called pseudomeasures is too small and Gaudry invented the so-called quasimeasures in order to derive representation theorems for operators as above (called multipliers in view of iii) above, cf. Ref. LA). However, using a characterization quasimeasures due to Ref. COW allows to show that

$S_0'(\mathbb{R}^m)$ is a subspace of the space of quasimeasures (having the same local structure), thus the essential content of Chap.4 of Ref. LA is contained in the following result (that we have $G=\mathbb{R}^m$ is only for simplicity). Recall that we have $S_0(\mathbb{R}^m) \subseteq L^p(\mathbb{R}^m) \subseteq S_0'(\mathbb{R}^m)$ for all $p\geq 1$.

<u>Theorem 4.2.</u> Let T be a bounded linear operator commuting with translations. Then there exists a unique $\sigma \in S_0'(\mathbb{R}^m)$ such that $Tf = \sigma * f$ for all $f \in S_0(\mathbb{R}^m)$. Equivalently we may describe T through the existence of some $h \in S_0'(\mathbb{R}^m)$ such that $(Tf)\hat{} = h\hat{f}$ for all $f \in S_0(\mathbb{R}^m)$. Of course one has $h = \hat{\sigma}$.

<u>Remark 4.1.</u> Following the terminology of signal analysis one should call σ the *impulse response* of the LTIS T and h the *transfer* (or characteristic) *function*. Actually, the observation (made by applied people) is that one has for Dirac sequences such as $f_n := St_{1/n}f$ (with $f \in S_0(\mathbb{R}^m)$ and $\int f(x)dx = 1$) , which is convergent to the unit impulse δ_0, the result $\sigma = \lim_{n\to\infty} Tf_n$, at least in the w*-sense, since $Tf_n(f\check{}) = Tf_n * f(0) = (\sigma * f_n) * f(0) = \sigma * (f_n * f)(0) \longrightarrow \sigma * f(0)$ for $n \longrightarrow \infty$.

In a somewhat more general context the last result can be seen as a special case of a more general observation: Under certain conditions the w*-approximation of the input signal (e.g. approximation of smooth signal by a discrete sequence of impulses, in other words, of a continuous function by discrete measures) is sufficient to approximate the output in some stronger sense (at least in the w*-sense). For band-limited systems (i.e. if spec σ is compact) a complete immitation of arbitrary inputs through discrete ones is possible (cf. Ref. F9).

<u>Remark 4.2.</u> As a consequence it is easy to characterize those linear operators on $S_0(\mathbb{R}^m)$ having periodic output, i.e. with $L_y(Tf) = Tf$ for all $y \in H \leq \mathbb{R}^m$. They must have periodic impulse response σ, or their transfer distribution h is concentrated on $H^\perp$.

For more general linear operators from $S_0(\mathbb{R}^m)$ to $S_0'(\mathbb{R}^m)$ we have the following kernel theorem (resulting from the tensor product property for S_0), showing that in some sense we might think of operators as of matrices, but now not just with continuous "kernels" (as a first continuous version of a matrix), but with a distributional kernel. In detail we have:

Theorem 4.3. (Kernel Theorem) The bounded linear operators $S_o(\mathbb{R}^m) \longrightarrow S_o'(\mathbb{R}^m)$ are precisely characterized through the existence of some $\sigma \in S_o'(\mathbb{R}^m \times \mathbb{R}^m)$ such that

$$Tf(g) = \sigma(f \otimes g) \text{ for all } f, g \in S_o(\mathbb{R}^m) \ .$$

Indication of the proof. Again the utility of the w*-topology becomes appearent. Remembering the extraction of matrix elements from a given linear mapping we would like to set $\sigma(x,y) := T(\delta_x)(\delta_y)$, which should be expected to coincide with the limit for $n \longrightarrow \infty$ of $T(L_x St_{1/n} f)(L_y St_{1/n} f) =: h_n(x,y)$. Now these functions h_n are well defined, forming a bounded sequence in $S_o'(\mathbb{R}^m)$. It is thus at least plausible that (a subsequence) is w*-convergent to some $\sigma \in S_o'(\mathbb{R}^m \times \mathbb{R}^m)$, which is of course the "correct" kernel.

Remark 4.3. There are many immediate consequences that can be derived from this general result. For example, using the so-called partial FT one can establish in a fairly direct way the connection between the kernel and the symbol of a pseudo-differential operator (cf. Ref. PET). Also, operators which don't increase supports are easily recognized as those having a kernel σ concentrated on the diagonal, or (by the synthesis property) multiplication operators $f \longrightarrow \tau f$ for some $\tau \in S_o'(\mathbb{R}^m)$.

As another application we mention the following Tauberian theorem (cf. Ref. REI , Chap 1, §4.5 for the classical version):

Theorem 4.4. Let $\sigma \in S_o'(\mathbb{R}^m)$ be given, and assume that for some $f_o \in S_o(\mathbb{R}^m)$ one has $\lim_{x\to\infty} \sigma * f_o(x) = 0$ then $\lim_{x\to\infty} \sigma * f(x) = 0$ for all $f \in S_o(\mathbb{R}^m)$.

The general rules and the convolution - multiplication correspondence under the Fourier transform allows to derive from Cor.3.5 the following generalized sampling principle (cf. Ref. BSS for a survey in this direction):

Theorem 4.5. Assume that for a given band-limited function g with $\hat{g}(z) = 1$ on P and spec g ⊆ Q for compact sets P, Q an m-tuple $\alpha := (\alpha_1, \dots, \alpha_m)$ of positive numbers has the property that the family $\alpha^{-1}k := (\alpha_1^{-1}k_1, \dots, \alpha_m^{-1}k_m) + Q$, with $k = (k_1, \dots, k_m) \in \mathbb{Z}^m$ consists of pairwise disjoint sets. Then any $f \in S_0'(\mathbb{R}^m)$ with spec $\sigma \subseteq P$ actually belongs to $C^b(\mathbb{R}^m)$ and can be reconstructed from the sampling values at αk, with $k \in \mathbb{Z}^m$, through the formula (sampling principle):

$$\sigma(x) = \sum_{k \in \mathbb{Z}^m} \sigma(\alpha k) L_{\alpha k} g(x) \tag{SP}$$

the sum being absolutely and uniformly convergent over compact sets.

Proof. Since g belongs to $S_0(\mathbb{R}^m)$ we have $\hat{\sigma} \cdot \hat{g} = \hat{\sigma}$ we have $\sigma \in S_0' * S_0 \subseteq C^b(\mathbb{R}^m)$. The condition on the lattice $\alpha^{-1}\mathbb{Z}^m$ to be sufficiently separated (which in turn means that $\alpha\mathbb{Z}^m$ is sufficiently dense ("oversampling") implies that we have $(\hat{\sigma} * St_\alpha ⧢) \cdot \hat{g} = (\sum_{k \in \mathbb{Z}^m} L_{\alpha^{-1}k}\hat{\sigma}) \cdot \hat{g} = \hat{\sigma}$, since $\hat{g}$ "cuts out" just one copy out of the periodic variant of $\hat{\sigma}$. Using Cor.3.5 and inverse Fourier transform one derives therefrom the $\sigma = (\sigma \cdot ⧢) * g$. It remains to verify that the expression on the right side conincides with the expression in (SP). This can be done by observing first that the sum convergent as stated (because $g \in S_0$) and coincides with $(\sigma \cdot ⧢) * g$ (by a w^*-limit argument).

Unfortunately we do not have enough place to discuss in detail the theory of generalized stochastic processes here. We simply want to mention, that the relationship between ordinary functions and distributions is quite the same as between classical stochastic processes (we think here just of a mapping from $\mathbb{R}^m$ into the Hilbert space $H = L_0^2(\Omega, \Sigma, P)$) (satisfying suitable continuity conditions) and the generalized ones, which we may define as bounded linear operators from $S_0(\mathbb{R}^m)$ into H .This approach lets stochastic proccesses appear as natural generalizations of linear functionals, since instead of the one-dimensional complex space $\mathbb{C}$ on has an abstract Hilbert space now. It is at least worth being mention here that the basic properties of $S_0(\mathbb{R}^m)$ and $S_0'(\mathbb{R}^m)$ are sufficient in order to overcome the usual technical problems (usually solved by integration theory with respect to bimeasures) without problems. The basic notions are the following (details are given in Ref. H , for example):

Definition. A generalized stochastic process $\rho\colon S_o(\mathbb{R}^m) \longrightarrow H$ is called stationary if one has $\langle\rho(f), \rho(g)\rangle_H = \langle\rho(L_x f), \rho(L_x g)\rangle_H$ for all $f, g \in S_o(\mathbb{R}^m)$ and $x \in \mathbb{R}^m$. It is called orthogonally scattered if one has $\langle\rho(f), \rho(g)\rangle_H = 0$ if $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$.

One of the basic results in the theory of stationary stochastic processes can be formulated (and proved) in our setting, keeping the (natural) definition of a Fourier transform (spectral process) : $\hat{\rho}(f) := \rho(f)$. As in the deterministic setting one of the main advantages of this approach is its symmetry with respect to Fourier transform, i.e. it is not neccesary to consider to different kinds of objects (point processes and vector measures), but considers both as linear functionals on the test space $S_o(\mathbb{R}^m)$.

Theorem 4.6. (Spectral representation theorem)
A generalized stochastic process $\rho\colon S_o(\mathbb{R}^m) \longrightarrow H$ is stationary if and only it arises as the Fourier transform of an orthogonally scattered stochastic process.

At the end of this note we have to mention some generalizations. As we have seen the spaces $S_o(\mathbb{R}^m)$ and $S_o'(\mathbb{R}^m)$ are minimal and maximal resp. with respect to the property of being isometrically invariant under translation and invariant under the Fourier transform. It is evident that replacing $S_o(\mathbb{R}^m)$ by some smaller space with similar properties allows to show similar properties for the dual space. A typical space suitable for this purpose would be the space $S_s(\mathbb{R}^m)$, defined through the condition of integrability of $STFT_{g_o}(f)\cdot w_s$, where we set (for $s\in\mathbb{R}$) $w_s(x,t) := (1+|x|^2+|t|^2)^{s/2}$. It is not difficult to verify that $S_s(\mathbb{R}^m) \subseteq S_o(\mathbb{R}^m)$ for any $s\geq 0$. Using the submultiplicativity of w_s on $\mathbb{R}^m \times \mathbb{R}^m$ one can show that one has an atomic decomposition, where the condition on the coefficients has to be $\sum_{n,k} |a_{n,k}|(1+|n|+|k|)^s < \infty$. Actually, in order to get reasonable (small) Banach spaces of test functions (and thus large spaces of distributions or "ultradistributions") one may take weights satisfying the so-called Beurling-Domar condition on $\mathbb{R}^m \times \mathbb{R}^m$. The theory arising in this way can be compared with that presented by Björck (Ref. Bj). One way to proof such results are given in Rocky. In a much more general frame related results are are contained in the work Ref. FG2 (a first summary giving the basic approach is given in Ref. FG1) on general atomic decompositions. As a typical application the elements of $S_o'(\mathbb{R}^m)$ can be characterized through Gabor representations with bounded coefficients.

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