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BANACH SPACES OF DISTRIBUTIONS WITH DOUBLE MODULE STRUCTURE AND TWISTED CONVOLUTION*

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A. Haar's discovery of the existence of an invariant measure on any locally compact group G opened the way to study these groups by looking at the corresponding group algebra $L^1(G)$ as a Banach algebra with convolution, or using the L^p -spaces on G (on which G acts by the left regular representation). However, even in the case $G = \mathbb{R}^m$ the study of multipliers or the Fourier transform (e.g. for L^p, $p > 2$) require the introduction of certain functional spaces, i.e. suitable distributions. An appropriate frame (for purposes of harmonic analysis) has been set up in [9] and [2].

In a previous paper [2] the authors have studied the family of Banach spaces $(B, \| \|_B)$ of distributions on a locally compact group G in 'standard situation', i.e. which are (besides some general assumptions) Banach modules with respect to convolution over some Beurling algebra $L^1_w(G)$ as well as under pointwise multiplication over a suitable (sufficiently large) Banach algebra $(A, \| \|_A)$ having bounded approximate units. It is the purpose of the present note to show that on these standard spaces there is a natural action of the Beurling–Leptin algebra $L^1_w(G,A)$ which is a Banach algebra under twisted convolution (cf. [15]). This action is a compact one (in fact, the action can be approximated by the action of suitable ele-

* This paper is in final form, and no version of it is, or will be, submitted for publication elsewhere.

ments operating as finite-dimensional operators), which allows, among others, a short proof of the compactness criterion for closed, bounded subsets of such spaces (cf. [8], [9]). The essential part and the module completion (with respect to this action) of a standard space B are identified with the minimal and the maximal space (B_0 and $\tilde{B}$, respectively) associated with B in the main diagram described in [2]. Relations between these spaces (including a characterization of dual or reflexive standard spaces) can be obtained by a direct application of simple Banach module theoretic results. Finally, minimal homogeneous Banach spaces (see for example [6], [10]) are discussed in this setting.

We start by fixing several notations and recalling basic facts concerning Leptin's twisted algebra $L^1_w(G,A)$ and Banach spaces in 'standard situation'.

Throughout this paper G denotes a locally compact group with a fixed left Haar measure dx, and Haar modulus Δ. For arbitrary functions f on G the following left and right translation operators are to be used:

$$(1) \qquad L_y f(x) := f(y^{-1}x) \ , \quad f_z(x) = f(xz) \qquad x,y,z \in G \ .$$

Given a continuous *weight function* w on G, i.e. a real valued continuous function w on G satisfying $w(x) \geqq 1$ and $w(xy) \leqq w(x)w(y)$ for all $x,y \in G$ (for convenience we assume that w is symmetric, i.e. $w(x) = w(x^{-1})$ for $x \in G$), one can form the *Beurling algebra* $L^1_w(G) := \{f \mid fw \in L^1(G)\}$. This is known (cf. [19]) to be a Banach algebra with respect to convolution, with norm $\|f\|_{1,w} := \|fw\|_1 = \int_G |f(x)|w(x)dx$. The space $K(G)$ of compactly supported continuous complex-valued functions is dense in $L^1_w(G)$. It follows therefrom that $(L^1_w, *, \| \ \|_{1,w})$ has bounded two-sided approximate units, i.e. there exists a bounded net (in the sequel denoted by $(e_\alpha)_{\alpha\in I}$) such that

(2) $\lim_\alpha \|e_\alpha * f - f\|_{1,w} = 0 = \lim_\alpha \|f * e_\alpha - f\|_{1,w}$ for all $f \in L^1_w(G)$.

One also has $\|\check{e}_\alpha * f - f\|_{1,w} \to 0$ for $\alpha \to \infty$, where $\check{g}(x) = g(x^{-1})$, if $(e_\alpha)_{\alpha\in I}$ has common compact support, for example. The reader is assumed to be familiar with the basic facts concerning Banach modules over a Banach algebra C (cf. [20], [3], [11], or § 1 of [2]). $H_C(B^1,B^2)$ denotes the space of (*left*) *module homomorphisms* between two (left) C-modules. B_C , the *essential part* of B is the closed linear span of $C \cdot B$ in B , and $B^C := H_C(C,B)$ denotes the C*-module completion* of B .

In our discussion below frequent use of a pointwise algebra A satisfying certain conditions will be made.

DEFINITION 1. A Banach algebra $(A,\| \|_A)$ will be called a *homogeneous (Banach) function algebra* on G if the following conditions are verified.

$\alpha 1)$ $(A,\| \|_A) \hookrightarrow (C^0(G),\| \|_\infty)$, i.e. A is continuously embedded into the Banach algebra of continuous functions on G which vanish at infinity (with the sup-norm $\| \|_\infty$);

$\alpha 2)$ $(A,\| \|_A)$ is a regular, selfadjoint (under complex conjugation) Banach algebra for pointwise multiplication;

$\alpha 3)$ $A_0 := A \cap K(G)$ is a dense subspace of A ;

$\alpha 4)$ A is isometrically left invariant (i.e. $\|L_y f\|_A = \|f\|_A$ for all $f \in A$, $y \in G$) and $y \mapsto L_y f$ is a continuous mapping from G to $(A,\| \|_A)$ for all $f \in A$;

$\alpha 5)$ Given any neighbourhood U of the identity there exists $k \in A$, $k \neq 0$, $\operatorname{supp} k \subset U$, $k \geq 0$, such that $k_z \in A$ for all $z \in G$ and such that $z \mapsto k_z$ is continuous.

REMARK 1. This set of conditions is essentially the same as in Theorem 4 of [17]. It is somewhat more general than the general assumption made in [9] or [2] inasmuch as $\alpha 5$) requires only the right invariance of certain elements in A.

Several immediate consequences are collected in the following lemma:

LEMMA 1. *Let* $(A,\| \|_A)$ *be a homogeneous function algebra. Then*

i) A *is a left Banach module over* $L^1(G)$ *with respect to convolution.*

ii) A_0 *is dense in* $K(G)$ *and therefore in* $C^0(G)$ *and* $L^1_w(G)$, *for any weight* w.

iii) *The subspace* $A_i := \{f \mid f \in A_0,\ f_z \in A$ *for all* $z \in G$ *and* $z \mapsto f_z$ *is continuous from* G *to* $(A,\| \|_A)\}$ *is dense in* A.

iv) *Given* $a_0 \in A_0$ *one has* $\Delta^{-1} a_0 \in A_0$.

v) *Given any compact set* $K \subseteq G$ *there exists* $a_K \in A_0$ *such that* $a_K(x) = 1$ *for all* $x \in K$. *In particular,* $A \cdot A$ *is dense in* A.

PROOF. i) follows from $(\alpha 1)+(\alpha 4)$, via vector-valued integration (cf. [23], §4) and ii) is essentially a consequence of $(\alpha 2)$. In order to verify iii), it will be sufficient to approximate $f \in A_0$ (due to $(\alpha 3)$). Given $\varepsilon > 0$ it follows from $(\alpha 4)$ and $(\alpha 5)$ that there exists $k \in A_1$ (normalized to $\int k(x)dx = 1$) such that $\|k*f-f\|_A < \varepsilon$. Observing now that $A_i \cap A_0$ is not only two-sided translation invariant, but also a two-sided ideal in the algebra $(K(G),*)$ one has $k*f \in A_i$ and the argument is complete. The proof of iv) follows from [17]. For $b \in A_0$ satisfying $\int b = 1$ the mapping $y \mapsto (L_{y^{-1}} b)a_0$ is continuous from G to A and has compact support. Consequently $v := \int (L_{y^{-1}} b)a_0 dy$ belongs to A. But

$$v(x) = \int_G b(yx)\ a_0(x)dy = \left(\int_G b(y)\,dy\right) \Delta^{-1}(x)a_0(x) = (\Delta^{-1}a_0)(x).$$

For a proof of the first part of v) cf. [6, Theorem 2]. It implies

$A_0 = A_0 \cdot A_0$.

Given a homogeneous (Banach) function algebra $(A, \| \ \|_A)$ we denote by $L^1_w = L^1_w(G,A)$ the space of all (equivalence classes of) measurable functions F with values in A, satisfying $\|F\|_{L^1_w} :=$ $:= \int_G \|F(x)\|_A w(x)\,dx < \infty$. Of course $L^1 = L^1_{w_0}$, with w_0 being the trivial weight $w_0(x) = 1$.

Since A is a function space it is clear that an element of L^1_w can be considered as a function on $G \times G$ via $F(x,y) := F(x)(y)$. It is well known that L^1_w is naturally isomorphic to the projective tensor product $L^1_w \underset{\gamma}{\otimes} A$ (cf. [15]). Basic facts concerning the Beurling–Leptin algebra L^1_w as an algebra with respect to *twisted convolution* are collected in the following proposition. They are stated without proof as they are proved in [15], [17] or can be obtained by minor modifications of the scalar case.

PROPOSITION 2. i) $\left(L^1_w(G,A), \| \ \|_{L^1_w}\right)$ *is a Banach algebra with respect to twisted convolution, given by*

$$(3) \quad F \star G(x) := \int_G L_y(F(xy))G(y^{-1})dy , \qquad F, G \in L^1_w(G,A)$$

which is a dense subalgebra of L^1 .

ii) *If* $(e_\alpha)_{\alpha \in I}$ *is an approximate unit in* $L^1_w(G)$ *and* $(u_\beta)_{\beta \in J}$ *is an approximate identity in* A *which is bounded in* A *(or the operator norm on* A *), then* $(e_\alpha \otimes u_\beta)_{(\alpha,\beta) \in I \times J}$ *is a two-sided approximate identity in* L^1_w *which is bounded in* L^1_w *(or of bounded action on* L^1_w *respectively).*

Next we discuss the existence of sufficiently many elements of a particular form. The following lemma of Kugler and Leptin will be essential in the sequel.

DEFINITION 2. For $f,g \in C^0(G)$ let the function $f \circ g : G \to C^0(G)$ be given by

(4) $\quad f \circ g(x)(y) := f(xy)g(y) \ , \quad x,y \in G \ .$

We denote by E the linear span of the set $\{f \circ g \mid f,g \in A_0\}$.

THEOREM 3. *Let* A *be a homogeneous function algebra. Then* E *is a dense subspace of* $L^1_w(G,A)$.

For a proof cf. [14] or [17].

Before spaces in 'standard situation' can be described we have to recall a few facts about A_0 and its dual. We shall use the fact that A_0 is a locally convex topological vector space, if it is endowed with the natural inductive limit topology. Its topological dual is denoted by A_0' . The duality will be written as $\langle \sigma, a_0 \rangle$, for $\sigma \in A_0'$, $a_0 \in A_0$. A_0' will be endowed with the weak, i.e. the $\sigma(A_0',A_0)$ -topology. The fact that it reduces to the ordinary vague topology on $R(G) = K(G)' = A_0'$ for $A = C^0(G)$ hopefully sufficiently justifies our 'abuse de langage' of calling it the *vague topology* and of writing $f = \text{vag-lim}_\alpha f_\alpha$ if the net is $\sigma(A_0',A_0)$ convergent (in contrast to other weak types of convergence appearing later). It will be an important fact that A_0 , considered as a subspace of A_0' (the elements of A_0 acting via integration) is vaguely dense in A_0' . In fact, the topological dual of $(A_0',\sigma(A_0',A_0))$ is naturally identified with A_0 (cf. [21, Ch. IV, 1.2]). Since a functional $x \in (A_0')'$ of the form $x(\mu) = \mu(a_0)$ for some $a_0 \in A$ and vanishing on A_0 (i.e. with $\int a(x)a_0(x)dx = 0$ for all $x \in A_0$) is trivial, the assertion follows from the Hahn—Banach theorem. Finally we observe that various operations on A_0 extend in a natural way to A_0' , by the following conventions:

(5) $\langle L_y\mu, a_0\rangle := \langle \mu, L_y^{-1} a_0\rangle$; $\langle h\mu, a_0\rangle = \langle \mu, h a_0\rangle$

and

(6) $\langle k*\mu, a_0\rangle := \langle \mu, \check{k}*a_0\rangle$, for $\mu \in A_0'$, $a_0 \in A_0$, $h \in A$, $k \in K(G)$.

For convenience we take up the convention of interpreting $k*h\sigma$ always as $k*(h\sigma)$.

It is now possible to describe spaces with double module structure in a very general way (cf. [9], [2]).

DEFINITION 3. A Banach space $(B,\| \ \|_B)$ will be called a space *in standard situation* (w.r.t. A) , or simply an A-L_w^1 *double module* if it satisfies:

β1) $(B,\| \ \|_B) \hookrightarrow A_0'$;

β2) $(B,\| \ \|_B)$ is a Banach module over $(A,\| \ \|_A)$ with respect to pointwise multiplication;

β3) $(B,\| \ \|_B)$ is a Banach module over $(L_w^1,\| \ \|_{1,w})$ with respect to convolution, i.e. $\|k*b\|_B \le \|k\|_{1,w}\|b\|_B$ for $k \in K(G)$, $b \in B$.

B is called an *essential standard space* if it is an essential Banach module for both actions.

REMARK 2. It will follow automatically (cf. Corollary 5) that $A_0 \hookrightarrow B$; therefore our definition is equivalent to those given earlier. A list of Banach spaces of distributions in standard situation or at least selected examples are given in [9] and [2]. It is not necessary to repeat it here.

The following result, showing that the two module actions may be combined to a compact action of the Leptin algebra $L_w^1(G,A)$, will be of fundamental importance throughout this paper. For technical reasons, first it is formulated only for standard spaces with continuous translations (i.e. for $B = B_G := L^1_w(G)*B$, cf. [2]).

THEOREM 4. *Let* $(B,\| \|_B)$ *be a standard space w.r.t.* $(A,\| \|_A)$, *which is an essential* $L^1_w(G)$ *-module. Then* B *is a Banach-module over* $L^1_w(G,A)$ *, the action being given by the formula*

$$F \cdot b := \int_G L_y(F(y)b)dy \ , \quad F \in L^1_w(G,A) \ , \quad b \in B \ . \tag{7}$$

The action of an elementary tensor $g \otimes a \in L^1_w(G)\hat{\otimes}A = L^1_w$ *can be described by*

$$(g \otimes a) \cdot b = g * ab \tag{8}$$

and the elements $u \circ v$, $u,v \in A_0$ *act according to*

$$(u \circ v) \cdot b = \langle \Delta^{-1} v, b \rangle u \tag{9}$$

as a one-dimensional operator on B . *Consequently* $L^1_w(G,A)$ *acts as an algebra of compact operators on* $(B,\| \|_B)$.

PROOF. The first formula is just the realization of the general concept (cf. [15], p. 127, for example). The integral representing $F \cdot b$ is in fact well defined, the integrand being continuous with values in B , and having integrable majorant (since any essential L^1_w -module is translation invariant with continuous translation). The validity of (8) follows immediately from the fact that one has for $F(x) = g \otimes a(x) = g(x)a$:

$$F \cdot b = \int_G L_y[g(y)ab]dy = \int_G L_y(ab)g(y)dy = g * ab \ .$$

In order to verify (9) let us observe that one has for $u,v,a_0 \in A_0$ and $b \in B$:

$$\langle (u \circ v) \cdot b, a_0 \rangle = \int_G \langle u \, L_y(vb), a_0 \rangle dy = \int_G \langle L_y b, (L_y v) a_0 u \rangle dy =$$

$$= \int_G \langle b, v\left(L_{y^{-1}} a_0 u\right)\rangle dy =: \langle b,w\rangle, \text{ where } w := \int_G L_{y^{-1}}(a_0 u)\cdot v\, dy .$$

The integrand being continuous from G to A and having compact support it is clear that the mapping $S : a_0 \mapsto Sa_0 := w$ is in fact a continuous mapping to A_0. Therefore the mapping $b \mapsto (u\circ v)\cdot b$ is $\sigma(A_0', A_0)$-continuous on B. The same being true for the mapping $b \mapsto \langle\Delta^{-1}v,b\rangle u$ it will be sufficient to check (9) for $b \in A_0$ (by the $\sigma(A_0',A_0)$-density of A_0 in any standard space). The validity of (9) for $b \in A_0$ follows now by using Fubini's theorem:

$$\langle (u\circ v)\cdot b, a_0\rangle = \int_G \langle u\, L_y(vb), a_0\, dy\rangle =$$

$$= \int_G u(z)a_0(z)\left(\int_G v(y^{-1}z)b(y^{-1}z)dy\right)dz = \langle u,a_0\rangle\,\langle\Delta^{-1}v,b\rangle .$$

COROLLARY 5. *Any non-trivial standard space* $(B,\| \|_B)$ *contains* A_0.

PROOF. (9) above implies $A_0 \subseteq B_G$ if $B_G \neq \{0\}$. Since $B_G = \{0\}$ would imply $\langle f,a_0\rangle = \lim_\alpha \langle f,\check{e}_\alpha * a_0\rangle = \lim_\alpha \langle e_\alpha * f, a_0\rangle = 0$ for all $a_0 \in A_0$, i.e. $B = \{0\}$. Thus the Corollary is proved.

The first question arising from the L^1_w-module structure on B concerns a characterization of B_L (the closed linear span of $L^1_w\cdot B$ in B) the *essential* part of B with respect to this algebra.

PROPOSITION 6. i) $B_L = B_0 := \overline{A_0}^B$;

ii) B *is an essential* L*-module if and only if it is an essential standard space, i.e.* $B_L = B_A \cap B_G = B_0$.

PROOF. i) Since (9) shows that $E\cdot b = A_0$ for any $b \in B$, $b \neq 0$, it follows (using the density of E in L^1_w) that $A_0 \subseteq L^1_w \cdot B \subseteq B_0$, and consequently $B_L = B_0$.

ii) If $B = B_L = B_0$ by i) then B is apparently an essential standard space.

(In fact, left translation is continuous in A_0, hence in B_0, i.e. $B_0 = (B_0)_G$. The existence of local units in A (cf. Lemma 1 v) shows that $A_0 = A_0 A_0$ and therefore AB_0 is dense in B_0.) Let now B be an essential standard space. For some $C > 0$ it is then possible to find for any $b \in B$, $\varepsilon > 0$ elements $g,h \in A_0$ such that $\|g\|_{1,w} \leq C$, $\|b-g*b\|_B < \varepsilon$ and $\|b-hb\|_B < \varepsilon$. It follows that $\|b-g*hb\|_B \leq (C+1)\varepsilon$. In view of (8) the proof is complete.

For a treatment of the case of general standard space B as well as for the identification of the L-module completion $B^L = H_L(L,B)$ of B the notion $\tilde{B}$ is required.

DEFINITION 4. Given a standard space $(B,\| \|_B)$ its *vague relative completion* $\tilde{B}$ is given by

$$\tilde{B} := \{\mu \mid \mu \in A_0', \ \mu = \text{vag-lim}\, f_\alpha \ \text{for some bounded net in } B\} \ ;$$

$\|\mu\|_{\tilde{B}} = \inf\{\sup_\alpha \|f_\alpha\|, \ldots\}$, the infimum being taken over all bounded nets in B having μ as a vague limit, i.e. with $f_\alpha \to \mu$ in $\sigma(A_0', A_0)$.

Several basic facts are obvious:

LEMMA 7. *$(\tilde{B},\| \|_{\tilde{B}})$ is again a standard space, and $(B,\| \|_B)$ is contractively embedded in $(\tilde{B},\| \|_{\tilde{B}})$. Moreover, $(\tilde{B},\| \|_{\tilde{B}})$ coincides with $((B_G)^\sim,\| \|_{\tilde{B_G}})$ as a normed space.*

PROOF. The first part is routine (cf. [2], Lemma 3.1). To show $\tilde{B} \subseteq (B_G)^\sim$ let $\mu \in \tilde{B}$ be given, with $\mu = \text{vag-lim}_\gamma\, b_\gamma$, $\sup_\gamma \|b_\gamma\|_B \leq \|\mu\|_{\tilde{B}} + \varepsilon$. Then $\mu = \text{vag-lim}_{\alpha,\gamma}\, e_\alpha * b_\gamma$. Since $e_\alpha * b_\gamma \in B_G$, and $\sup_{\alpha,\gamma} \|e_\alpha * b_\gamma\|_B \leq C(\|\mu\|_{\tilde{B}} + \varepsilon)$ the second assertion follows.

THEOREM 8. i) *Any standard space $(B,\| \|_B)$ is an $L^1_w(G,A)$-module in a natural way. The action of $F \in L^1_w(G,A)$ is the unique* *vaguely continuous (on bounded sets) extension of the action on* B_G *as given by* (7). *The extended action still satisfies* (8) *and* (9).

ii) *Actually, this extension maps bounded, vaguely convergent nets* $(b_\gamma)_{\gamma \in L}$ *in* B *to norm convergent nets in* $(B_0, \| \ \|_B)$.

iii) *The action is compact from* $\tilde{B}$ *to* B_0 . *In particular it is compact as an action on* B .

PROOF. Consider any B-bounded, vaguely convergent net $(b_\gamma)_{\gamma \in L}$ in B_G , with $\text{vag-}\lim_\gamma b_\gamma = \mu \in \tilde{B}$ (cf. Lemma 7). Then it is clear from formula (9) that $(F_1 \cdot b_\gamma)$ is a net in A_0 , convergent in the norm $\| \ \|_B$, for $F_1 \in E$. The density of E in $L^1_w(G,A)$ and the boundedness of $(b_\gamma)_{\gamma \in L}$ in B_G imply that $(F \cdot b_\gamma)_{\gamma \in L}$ is a Cauchy net in $(B_0, \| \ \|_B)$ for any $F \in L^1_w(G,A)$. The definition $F \cdot \mu = \lim_\gamma F \cdot b_\gamma$ is therefore justified and one checks that it is uniquely determined. Moreover, one derives (using Theorem 4 and Lemma 7) that

$$\|F \cdot \mu\|_B \leq \|F \cdot\|_{B_G \to B_G} \|\mu\|_{(B_G)^\sim} C \|F\|_L \|\mu\|_{\tilde{B}} \quad (\leq C \|F\|_L \|\mu\|_B) .$$

The persistence of formulas (8) and (9) is then clear. The continuity and compactness assertions now follow once more from Theorem 4.

REMARK 3. The above proof shows that one has

$$(10) \qquad \|F \cdot \mu\|_B \leq C \|F\|_{L^1_w} \|\mu\|_{\tilde{B}} \quad \text{for } \mu \in \tilde{B} .$$

As an immediate consequence of the above result we obtain a variant of the well known compactness criterion for L^p-spaces which applies to standard spaces (cf. [9], where such a result is proved by different methods, using the right invariance of A).

THEOREM 9. i) *A closed, bounded subset* M *of a standard space* B *is compact if one of the following two conditions is satisfied:*

a) *for* $\varepsilon > 0$ *there exists* $F \in L^1_w(G,A)$ *such that*

$\|F\cdot b-b\|_B < \varepsilon$ *for all* $b \in M$;

b) *for* $\varepsilon > 0$ *there exist* $a \in A$, $g \in L^1_w(G)$ *such that*

$\|ab-b\|_B < \varepsilon$ *and* $\|g*b-b\|_B < \varepsilon$ *for all* $b \in M$.

ii) *If* $B = B_0$ *and if* A *contains an approximate identity* $(u_\beta)_{\beta\in J}$ *of (uniformly) bounded action on* B *each of these conditions is also necessary, hence they are equivalent in this case.*

PROOF. i) Since it follows immediately from Theorem 8 that a) implies compactness of M it will be sufficient to verify b) ⇒ a). One may assume $g,a \in A_0$ (cf. Lemma 1). The estimate $\|g*ab-b\|_B \leq$ $\leq (\|g\|_{1,w}+1)\varepsilon$, together with (8) shows that one only has to verify that g may be replaced by g' with $\|g'\|_{1,w} \leq C$, if necessary. That this is possible follows from $\|e_\alpha*b-b\|_B \leq \|e_\alpha*g-g\|_{1,w}\|b\|_B +$ $+ \|g*b-b\|_B < \varepsilon$ for $\alpha \geq \alpha_0$. Since $e_{\alpha_0} =: g'$ may be chosen in A_0 the argument is complete.

ii) It is a matter of routine (cf. [2]) to check that condition b), hence a) are necessary in that case.

REMARK 4. The typical example of a family $(u_\beta)_{\beta\in J}$ as in b) above is a family of trapezoid functions $(\tau_\alpha)_{\alpha\in I}$ in A_0 , bounded in the operator norm on B , as in the description of the 'standard situation' as given in § 1 of [9]. The assumptions made in that paper therefore are sufficient for the validity of the above results (if A_0 is dense in B), but also for Proposition 11 below.

The combination of Theorems 3, 4, and 9, iii) actually yields another important property for essential standard spaces:

COROLLARY 10. *Let* $(B,\| \|_B)$ *be an essential standard space. Assume that there is an approximate unit* $(u_\beta)_{\beta\in J}$ *in* A_0 *of uniformly bounded action on* B *(of operator norm* ≤ 1 *on* B *). Then* $(B,\| \|_B)$ *has the* C*-approximation property (metric approximation* *property), i.e. there exists* $C > 1$ *such that for each compact set* $K \subseteq B$ *there exists a finite dimensional operator* S *on* B *, with* $\|Sb-b\|_B < 1$ *for all* $b \in K$ *, with* $\|S\|_{B\to B} \leq C(\|S\|_{B\to B} \leq 1)$.

The same condition on $(A,\| \|_A)$ as above also implies that the relations between $\tilde{B}$ and B_0 are as good as one may wish. Under stronger assumptions such results are shown in [2, Proposition 3.3]. One has:

PROPOSITION 11. *If* $(A,\| \|_A)$ *has approximate units, bounded in the operator norm on* B *(e.g. norm bounded in* A *) one has besides the formulas*

(11a) $(B_0)_0 = B_0$ (11b) $(\tilde{B})^{\sim} = \tilde{B}$

also

(12a) $(B_0)^{\sim} = B^{\sim}$ (12b) $(\tilde{B})_0 = B_0$.

PROOF. (11a) is obvious, and (11b) is easily checked. In view of (11b) it will be sufficient to verify the embedding $B \hookrightarrow (B_0)^{\sim}$. For $b \in B$, $a_0 \in A_0$ one has $\langle b,a_0\rangle = \lim_{\alpha,\beta} \langle b,u_\beta(\check{e}_\alpha * a_0)\rangle =$ $= \lim_{\alpha,\beta} \langle e_\alpha * u_\beta b,a_0\rangle$. The boundedness of $(u_\beta)_{\beta\in J}$ on B implying the boundedness of the net $(e_\alpha * u_\beta b)_{(\alpha,\beta)}$ in B the inclusion follows. In order to show (12b) we observe that one only has to verify equivalence of the norms $\| \|_B$ and $\| \|_{\tilde{B}}$ on A_0 , i.e. $\|b\|_B \leq C\|b\|_{\tilde{B}}$. Choosing any net $(b_\gamma)_{\gamma\in I}$ in B , with $\sup_\gamma \|b_\gamma\|_B \leq$ $\leq \|b\|_{\tilde{B}} + \varepsilon$, and α_0, β_0 such that $\|e_{\alpha_0} * u_{\beta_0} b-b\|_B < \varepsilon$, one has $\|e_{\alpha_0} * u_{\beta_0} b_\gamma - e_{\alpha_0} * u_\beta b\|_B < \varepsilon$ for $\gamma \geq \gamma_0$, in view of Theorem 8. Since $\|e_{\alpha_0} * \mu_\beta b_\gamma\|_B \leq C'\|b\|_{\tilde{B}} + \varepsilon$ (for some $C' > 0$) it follows $\|b\|_B \leq$ $\leq \|b\|_{\tilde{B}}$, as was required.

Having a compact action of the algebra $L^1_w(G,A)$ on a standard space B there are various assertions concerning L-module completions available. Since they do not depend on the particular structure of L we state them in the general setting of Banach modules for later reference (cf. [20] or [11] for the definition of module tensor products).

LEMMA 12. *Let* $(B,\|\ \|_B)$ *be a left Banach module over a Banach algebra* $(C,\|\ \|_C)$ *. Assume that the action of* C *on* B *is weakly compact. Then one has:* i) *If* C^2 *is a total subset of* C *then*

$$(13)\qquad H_C(C,B) = H_C(C,B'') = \left(C \underset{C}{\otimes} B'\right)' .$$

ii) *If* C *has a left approximate identity, bounded in the operator norm on* C , *then*

$$(14)\qquad H_C(C,B) = (C \underline{\otimes} B')' ,$$

where $C \underline{\otimes} B' := \left\{b', b' \in B', \ b' = \sum_n b'_n c_n \ , \text{ with } \sum_n \|c_n\|_C \|b'_n\|_{B'} < \infty\right\}$, *with the usual norm.*

iii) *If* C *has furthermore a bounded left approximate identity, then*

$$(15)\qquad B^C := H_C(C,B) = (B')_C{}'$$

and

$$(16)\qquad B = B'_C{}'_C = B_C{}'_C{}'_C .$$

PROOF. i) The natural inclusion $H_C(C,B) \subseteq H_C(C,B'')$ is obvious. To prove the converse inclusion take $T \in H_C(C,B'')$. It then follows for $u,v \in C$ $T(uv) = uT(v) = L_u''T(v)$, where L_u'' denotes the bidual action of C on B" . The action of C being weakly compact one has $L_u''b'' \in B$ for all $b'' \in B''$ (cf. [5], p. 482), and consequently $T(C^2) \subseteq B$. Therefore T maps C (as the closed linear span of C^2) into B , proving the other inclusion. The formula

$H_C(C,B'') = \left(C \underset{C}{\otimes} B'\right)'$ is well known (cf. [20]). The assertions stated in ii) and (15) are immediate consequences of the results given in [12] (cf. in particular Corollary 2.3, which is applicable as a consequence of Proposition 2).

(16) follows from (15) using the formulas $B^C = {B_C}^C$ and $B_C = {B^C}_C$ ([2], § 1, cf. also [13], Proposition 1.1).

For convenience let us collect some further simple facts on Banach modules. We shall call a left [right] Banach module B over C *nondegenerate* if $Cb = \{0\}$ [$bC = \{0\}$] if and only if $b = 0$, or equivalently, if the canonical mapping $B \to B^C = H_C(C,B)$ [respectively the right completion $H^C(C,B)$] is an injection. We only formulate these facts in a one-sided version.

LEMMA 13. i) *If* C *has left approximate units, then any essential left* C*-module is non-degenerate;*

ii) *If* C *has bounded left approximate units, then the following conditions for a left module* B *are equivalent (* B' *is a right* C*-module):*

a) B *is an essential* C*-module;*

b) B' *is a non-degenerate right* C*-module;*

c) B' *is a complete* C*-module (i.e.* $B' = B'^C$ *).*

PROOF. i) is obvious. In ii) the equivalence of (a) and (b) is an immediate consequence of the Hahn–Banach theorem. That a complete C-module (i.e. for which B can be identified with B^C) is non-degenerate is again clear, hence (c) implies (b). Using Rieffel's formulas (see [20]) one finally verifies that a) implies

(c) $$B'^C = H^C(C,B') = (B \otimes_C C)' = B_C' = B' .$$

COROLLARY 14. *Let* C *be a Banach algebra with bounded two-sided* *approximate units, and let* B *be a non-degenerate left* C*-module, with weakly compact action of* C . *Then* B *is reflexive as a Banach space if and only if* B *is a complete* C*-module and* B' *is an essential* C*-module.*

PROOF. i) If both conditions are satisfied, it follows from (15) that one has $B = B^C = B'_C{}' = B''$. This identification being the natural one it follows that B is a reflexive Banach space.

ii) Assuming that $B = B''$ is nondegenerate it follows that B' is an essential, hence also nondegenerate C-module, and consequently B itself is essential. As dual spaces of essential C-modules the spaces B' and $B'' = B$ are complete as well, and the proof is finished.

REMARK 5. Since one can show that solid BK-spaces or Banach function spaces on $\mathbb{N}$ have absolutely continuous norm (cf. [23], Chap. 15 for the terminology used in this remark) or the Fatou property if and only they are essential or complete Banach modules over the Banach algebra c_0 (of null-convergent sequences, with the supnorm) one recognizes similarities between the above Corollary and Theorem 2 of § 73 in [23].

If A has a bounded approximate identity we can now identify the L^1_w-module completion of B with $\tilde{B}$. The module theoretic results of the above Lemma have then immediate implications concerning $\tilde{B}$. (ii) below has been derived by different means in [2].

THEOREM 15. *Assume that* B *is a standard space for a homogeneous function algebra* $(A,\|\ \|_A)$ *having bounded approximate units and let* $(B,\|\ \|_B)$ *be a standard space w.r.t.* A . *Then one has:*

i) $B^L \cong \tilde{B}$, *i.e. there is a Banach isomorphism between these two spaces, given by* $\mu \in \tilde{B} \hookrightarrow T_\mu \in B^L$, $T_\mu(F) := F\cdot\mu$.

ii) $B = B_0{}'_0{}'$;

iii) B *is* L^1_w *-complete if and only if it is the Banach dual of an essential standard space.*

iv) B *is reflexive as a Banach space if and only if* $(B_0 =)$ $B = \tilde{B}$ *and* B' *is* L *-essential.*

PROOF. i) By Remark 3 (cf. Theorem 8) $\mu \in \tilde{B}$ defines an operator $T_\mu \in B^L$, with $\|T_\mu\|_{B^L} \leq C\|\mu\|_{\tilde{B}}$. Let, conversely, $T \in B^L$ be given. Since $L^1_w(G,A)$ has a bounded two-sided approximate identity $(E_\delta)_{\delta \in M}$ we may consider the net $(T(E_\delta))_{\delta \in M}$ as a bounded net in $B \subseteq B''$. Replacing it by a subnet (if necessary) we may assume that it is $\sigma(B'',B')$ -convergent. But $B \hookrightarrow A_0'$ implies that $A_0 \hookrightarrow B'$ in a natural way. In particular, this shows that $T(E_\delta)$ is a bounded, vaguely convergent net in B. Write μ for its limit, then $\mu \in \tilde{B}$. Furthermore one has by Theorem 8: $T_\mu(F) = F \cdot \mu = F \cdot \lim_\delta T(E_\delta) = \lim_\delta T(F \cdot E_\delta) = T(F)$, q.e.d.

ii) It follows from Lemma 12 iii) that $B^L = (B_L)^L = B_L{}'_L{}'$. In view of i) above and Proposition 6 i) this is the required assertion.

iii) If $B = B^L$ the above formula gives us the required predual. Conversely, any dual of an essential Banach module is complete (cf. Lemma 13, ii)).

iv) This is an immediate consequence of Corollary 14, the action of L on a standard space being always nondegenerate (e.g. as a consequence of (9)).

We conclude this note with some observations concerning minimal constructions as considered in [6], [10] and [18]. Recall (cf. [6]) that for a homogeneous function algebra A there is a minimal homogeneous Banach space A_{min} with the property of being a pointwise Banach A-module. It can be characterized as follows: Given any compact set Q with nonvoid interior, one has

$$A_{min} = \left\{ f \mid f = \sum_{n \geq 1} L_{y_n} f_n,\ y_n \in G,\ f_n \in A_0 \text{ for all } n, \right.$$
$$\left. \operatorname{supp} f_n \subseteq Q,\ \sum_{n \geq 1} \|f_n\|_A < \infty \right\}.$$

It is a Banach space with the usual infimum norm.

Using the results of this note a different characterization (and an alternative existence proof) of this minimal space can be given:

THEOREM 16. *Given a homogeneous function algebra there is a smallest space* A_m *in the family of all homogeneous Banach spaces* B *which are pointwise* A*-modules and satisfy* $A_0 \cap B \neq \{0\}$. A_m *can be characterized as follows. Given* $a_0 \in A_0$, $a_0 \neq 0$, *one has*

$$(17) \qquad A_m = L^1(G,A) \cdot a_0 \,,$$

endowed with the norm

$$\|a\|_{A_m} := \inf \left\{ \|F\|_L : F \cdot a_0 = a,\ F \in L \right\} .$$

PROOF. It is routine to verify that A_m is a homogeneous Banach space and a pointwise A-module. Furthermore one has $\|a\|_{A_m} \leq \|a_0\|_A \|a\|_{A_m}$ for $a \in A_m$, and therefore A_m is continuously embedded in A, hence A_0'. It follows from Corollary 5 that $A_0 \subseteq A_m$.

Let now A_1 be any other space of this kind, with $0 \neq a_1 \in A_1 \cap A_0$ follows from (9) and Theorem 4 that $L^1(G,A) \cdot a_1 \subseteq A_1$, with $\|F \cdot a_1\|_{A^1} \leq \|F\|_L \|a_1\|_{A^1}$, hence A_m is embedded in A_1. It also follows that A_m is independent of the choice of a_0 (e.g. because (9) implies that L^1 acts transitively on A_0).

COROLLARY 17. i) A_{min} *coincides with* A_m *(as a set, and as a normed space, for any* $a_0 \in A_0$, $a_0 \neq 0$ *).*

ii) *Given any* $a_m \in A_m$, $a_m \neq 0$ *one has* $A_m = L^1 \cdot a_m$ *; in particular,* L^1 *acts transitively on* A_{min}.

PROOF. i) This follows immediately from the different characterizations of the minimal space.

ii) A_m being a module over L^1 one has $L^1 \cdot a_m \subseteq A_m$. The fact that $L^1 \cdot a_m$ is in a natural way a module over L^1 implies $A_0 \subseteq L^1 \cdot a_m$ (cf. Corollary 5). The minimality of A_m implies then equality of the two spaces.

Next we derive therefrom the 'continuous' characterization of A_{min} (cf. [18]). In [16] (p. 129/130) Leptin has studied the Banach space (for $a_0 \neq 0$, $a_0 \in A_0$ fixed)

$$A_1 := \{a \mid a \in A, a_0 \circ a \in L^1(G,A)\} = \\ = \left\{a \mid a \in A, \int_G \left\| \left(L_{x^{-1}} a_0\right) a \right\|_A dx =: \|a\|_{A_1} < \infty\right\}.$$

It is clear that $\|a\|_{A_1} < \infty$ for any $a \in A_0$ (the integrand having compact support).

PROPOSITION 18. *For any* $a_0 \in A_0$, $a_0 \neq 0$ *the space* $A_1 = A_1(a_0)$ *defined above coincides with* $L^1 \cdot a_0$, *hence with* A_{min}.

PROOF. i) One has for $F \in L^1$, the following estimate:

$$\left\| \left(L_{x^{-1}} a_0\right) F \cdot a_0 \right\|_A \leq \int \|F(y)\|_A \left\| \left(L_{x^{-1}} a_0\right) (L_y a_0) \right\|_A dy \Rightarrow$$

$$\Rightarrow \int \left\| \left(L_{x^{-1}} a_0\right) F \cdot a_0 \right\|_A dx \leq \int\int \|F(y)\|_A \left\| \left(L_{(yx)^{-1}} a_0\right) a_0 \right\|_A dxdy =$$

$$= \int \|F(y)\|_A dy \cdot \int \left\| \left(L_{s^{-1}} a_0\right) a_0 \right\|_A ds = \|F\|_{L^1} \|a_1\|_{A_1}.$$

Therefore:

$$\|F \cdot a_0\|_{A_1} \leq \|a_0\|_{A_1} \|F \cdot a_0\|_{A_m}$$

and consequently A_m is continuously embedded into A_1.

ii) Let conversely $a \in A_1$ be given. Then $(a \circ a_0) \cdot a_0 \in L^1 \cdot a_0$. But (9) shows that $(a \circ a_0) \cdot a_0 = \langle \Delta^{-1} a_0, a_0 \rangle a$ and further

$$\|a\|_{A_m} \leq |\langle \Delta^{-1} a_0, a_0 \rangle|^{-1} \|a \circ a_0\|_L \leq C_{a_0} \|a\|_{A_1} ,$$

as was required.

REMARK 6. The above result could also be proved in a different setting, using the so-called Wiener-type spaces as introduced in [7].

REFERENCES

[1] G. F. Bachelis - J. E. Gilbert, Banach spaces of compact multipliers and their dual spaces, *Math. Z.* 125 (1972), 285-297.

[2] W. Braun - H. G. Feichtinger, Banach spaces of distributions having two module structures, *J. Funct. Anal.* 51 (1983), 174-212.

[3] J. Cigler - V. Losert - P. Michor, *Banach modules and functors on categories of Banach spaces*, Lect. Notes Pure Appl. Math. vol. 46, Marcel Dekker, Inc., 1979.

[4] R. S. Doran - J. Whichmann, *Approximate identities and factorization in Banach modules*, Lect. Notes Math. 768, Springer Verlag (Berlin - Heidelberg - New York, 1979).

[5] N. Dunford - J. T. Schwarz, *Linear operators*, Vol. 1, Pure and Appl. Math. Vol. 7, Interscience (New York, 1958).

[6] H. G. Feichtinger, A characterization of minimal homogeneous Banach spaces, *Proc. Amer. Math. Soc.* 81 (1981), 55-61.

[7] H. G. Feichtinger, Banach convolution algebras of Wiener's type, *Proc. Conf. "Functions, Series, Operators"*, Budapest, August 1980. Colloquia Math. Soc. J. Bolyai 35., North Holland Publ. Co. (Amsterdam - Oxford - New York, 1984), 509-524.

[8] H. G. Feichtinger, A compactness criterion for translation invariant Banach spaces of functions, *Analysis Math.* 8 (1983), 165-172.

[9] H. G. Feichtinger, Compactness in translation invariant Banach spaces of distributions and compact multipliers, *J. Math. Anal. Appl.* 102 (1984), 289-327.

[10] H. G. Feichtinger, Minimal Banach spaces and atomic representations, *Publ. Math. Debrecen*, to appear.

[11] M. Grosser, *Bidualräume und Vervollständigungen von Banach-moduln*, Lect. Notes Math. 717, Springer-Verlag (Berlin - Heidelberg - New York, 1979).

[12] D. L. Johnson - Ch. D. Lahr, Weak approximate identities and multipliers, *Studia Math.* 74 (1982), 1-15.

[13] H. E. Krogstad, Multipliers on homogeneous Banach spaces on compact groups, *Ark. Mat.* 12 (1974), 203-212.

[14] W. Kugler, Über die Einfachheit gewisser verallgemeinerter L^1-Algebren, *Arch. der Math.* 26 (1975), 82-88.

[15] H. Leptin, Verallgemeinerte L^1-Algebren und projektive Darstellungen lokal kompakter Gruppen, *Invent. Math.* 3 (1967), 257-281; 4 (1967), 68-86.

[16] H. Leptin, On onesided harmonic analysis in non-commutative locally compact groups, *J. reine angew. Math.* 306 (1979), 122-153.

[17] H. Leptin - D. Poguntke, Symmetry and nonsymmetry of locally compact groups, *J. Funct. Anal.* 33 (1979), 119-134.

[18] D. Poguntke, Gewisse Segalsche Algebren auf lokalkompakten Gruppen, *Arch. Math.* 33 (1980), 454-460.

[19] H. Reiter, *Classical Analysis and Locally Compact Groups*, Oxford University Press, 1968.

[20] M. Rieffel, Induced Banach representations of Banach algebras and locally compact groups, *J. Funct. Anal.* 1 (1967), 443-491.

[21] H. H. Schäfer, *Topological Vector Spaces*, Graduate Texts in Math. Vol. 3, Springer-Verlag (Berlin - Heidelberg - New York, 1971).

[22] A. C. Zaanen, *Integration*, North Holland (Amsterdam, 1967).

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