

Banach Spaces Related to Integrable Group Representations and Their Atomic Decompositions. Part II

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(Received 9 June 1989)

Abstract. We continue the investigation of coorbit spaces which can be attached to every integrable, irreducible, unitary representation of a locally compact group \mathcal{G} and every reasonable function space on \mathcal{G} . Whereas Part I was devoted to atomic decompositions of such spaces, Part II deals with general properties of these spaces as Banach spaces. Among other things we show that inclusions, the quality of embeddings, reflexivity and minimality and maximality of coorbit spaces can be completely characterized by the same properties of the corresponding sequence spaces. In concrete examples (cf. Part III) one recovers several and often difficult theorems with ease.

Introduction

In this part we continue our investigation of coorbit spaces. Whereas in Part I the main theme was the atomic decomposition for these spaces, in Part II we aim at a thorough study of their properties as Banach spaces.

Let us recall briefly how this family of spaces is constructed. We start with an irreducible, unitary representation π of a locally compact group \mathcal{G} on a Hilbert space \mathcal{H} which is at least integrable. Then there exist suitable elements $g \in \mathcal{H}$ such that the representation coefficients (or generalized wavelet transforms) $V_g(f)(x) := \langle \pi(x)g, f \rangle$ are not only defined for $f \in \mathcal{H}$, but also for all f in a larger space $(\mathcal{H}_w^1)^\top \supseteq \mathcal{H}$. For a translation-invariant solid BF-space Y on the group \mathcal{G} , the coorbit of Y under the representation π is defined as the subspace of all $f \in (\mathcal{H}_w^1)^\top$ for which the continuous function $V_g(f)$

¹ *Acknowledgement.* The second author gratefully acknowledges the substantial support by the Österreichische Forschungsgemeinschaft (project nr. 09/0010). Major parts of the paper were prepared while the second author held a position at McMaster University (Hamilton/Canada).

belongs to Y . The orthogonality relations for irreducible, square-integrable representation coefficients lead to a reproducing formula for the extended representation coefficients and furthermore to a one-to-one correspondence between the elements in coorbit spaces and functions on the group which satisfy the reproducing formula.

Thus all the work may be done on the group level, and the results concerning the coorbit spaces are obtained by a transfer of statements on function spaces on \mathcal{G} to statements about coorbits. In particular, the atomic decomposition of the coorbit spaces involved a fairly careful study of convolution operators and their discretization. Whereas the hitherto existing theories of atomic decompositions, e.g. for spaces of Besov—Triebel—Lizorkin type, for the family of modulation spaces or for Bergman spaces, had to be based on a case-by-case study, the new technique of transferring all questions to the group level provides a single method for a unified treatment of such questions (cf. [LRP], Section 8 for examples).

From another point of view the atomic decomposition establishes a relation between these coorbit spaces $\mathcal{C}_o Y$ and the sequence space Y_d , from which $\mathcal{C}_o Y$ inherits many properties. For example, $\mathcal{C}_o Y$ inherits reflexivity from Y_d , and the behaviour of the family of coorbit spaces under interpolation is the same as that of the sequence spaces (Cor. 6.3., [FG]).

Since sequence spaces are undoubtedly easier to handle than both function spaces on the group and the abstract coorbit spaces, we want to understand the properties of the coorbit spaces as far as possible by those of their associated sequence spaces. This is our goal for Part II, where we continue the numeration of sections and formulas from Part I.

In Section 8 we show that the extended representation coefficients exhibit a local behavior similar to that of analytic functions. The two theorems on restrictions and pointwise interpolation of extended representation coefficients demonstrate the advantage of these functions on \mathcal{G} , in contrast to the general elements (“distributions”) in the reservoir $(\mathcal{H}_w^1)^\top$, which do not have a particular local behavior as functions/distributions on \mathbb{R}^n in some of the examples. As a

consequence, inclusions between coorbit spaces can be characterized by the inclusions of their sequence spaces.

In Section 9 the double retract property for $\mathcal{C}_0 Y$ and $Y_d(X)$ is proved. It states that the structure of the Banach spaces $\mathcal{C}_0 Y$ and Y_d is essentially the same and that they share many properties such as reflexivity, RN-property, etc. The same is true for the quality of embeddings between different coorbits and their sequence spaces respectively.

Part II ends with a discussion of minimal and maximal coorbit spaces and equivalent characterizations (Section 10).

As to the methods, we shall be faithful to the principles of Part I. Because of the one-to-one correspondence between reproducing functions on the group and elements in the coorbit spaces we may argue entirely on the group. Only at the end these assertions will be interpreted in terms of the coorbits. Some necessary techniques on the group level — new convolution theorems and a pointwise interpolation principle — are contained in the opening Section 7.

We shall assume that the reader is familiar with the results of Part I. We keep the same definitions and notations and repeat only the basic assumptions (at the beginning of Sections 7 and 8).

Absolute constants are denoted by C, C', C_1, \dots . They may take different values at different occurrences, their precise value is always apparent from the proofs.

7. Some Preparations

In Part I we defined a family of Banach spaces, called coorbit spaces, related to an integrable representation of a locally compact group \mathcal{G} . It is parametrized by a natural family of function spaces on \mathcal{G} . By means of representation theory a one-to-one correspondence between elements of these coorbit spaces and well-behaved functions on \mathcal{G} was established (§4). Thus all questions about coorbit spaces could be transferred to related questions about these function spaces on \mathcal{G} . For instance, the atomic decomposition of coorbit spaces could be derived from results on the discretization of convolution operators (§5/6).

We start by recalling the general conventions made in Part I. π denotes an integrable irreducible unitary representation of a lc.

group \mathcal{G} on a Hilbert space \mathcal{H} , Y is a solid, translation-invariant BF-space on \mathcal{G} and w is a weight function on \mathcal{G} such that

$$Y * (L_w^1)^\vee = Y * L_{w^*}^1 \subseteq Y \quad (7.1)$$

where the weight w^* is given by $w^*(x) := w(x^{-1}) \Delta^{-1}(x)$.

We also use the modified right translation A_x given by

$$A_x F(y) := F(y x^{-1}) \Delta^{-1}(x) = \Delta^{-1}(x) R_{x^{-1}} F(y).$$

Observe that then the following estimate holds:

$$w^*(x) \geq C \| \| A_x \| \| \quad \text{for all } x \in \mathcal{G}. \quad (7.2)$$

The set of analyzing vectors $\mathcal{A}_w = \{g \in \mathcal{H} : V_g(g) \in L_w^1\}$ is supposed to be non-trivial. Then the set of "better vectors"

$$\mathcal{B}_w = \{g \in \mathcal{H} : V_g(g) \in W^R(C^0, L_w^1)\} \quad (7.3)$$

which served as the appropriate set of atoms for the decomposition of the coorbit spaces, is non-trivial as well and even dense in \mathcal{H} (cf. [LRP], Prop. 4.3). In the sequel all analyzing vector will be taken in \mathcal{B}_w because in this case the convolution relation (7.5) in Theorem 7.1 will provide some additional information on the representation coefficients.

In order to continue the programm started in Part I we need some auxiliary results in addition to those given in Sections 3 and 5.

Theorem 7.1. *For w as above the following convolution relations (together with corresponding norm estimates) hold true:*

$$\text{a) } Y * W(L^\infty, L_{w^*}^1) \subseteq W(L^\infty, Y) \quad (7.4)$$

$$\text{b) } Y * W(C^0, L_{w^*}^1) \subseteq W(C^0, Y). \quad (7.5)$$

$$\text{c) } Y * W^R(L^\infty, L_w^1) \subseteq W(L^1, Y) * W^R(L^\infty, L_w^1) \subseteq Y. \quad (7.6)$$

Proof. We argue similar to the proof of [FG], Proposition 3.10. Any $G \in W(L^\infty, L_{w^*}^1)$ has a decomposition $G = \sum_{n=1}^{\infty} L_{z_n} G_n$ with $G_n \in L^\infty$, $\text{supp } G_n \subseteq Q = Q^{-1}$ (compact) and $\sum_{n=1}^{\infty} \|G_n\|_\infty w^*(z_n) \cong \cong \|G\| W(L^\infty, L_{w^*}^1) < \infty$.

Proof of (a): Step 1. For $F \in Y$ we estimate first the control function of $F * L_{z_n} G_n = (A_{z_n} F) * G_n$, i.e.

$$\begin{aligned}
K(x) &:= \|L_x k(F * L_{z_n} G_n)\|_\infty = \|L_x k(A_{z_n} F * G_n)\|_\infty = \\
&= \|L_x k \cdot (L_x k_1 \cdot A_{z_n} F) * G_n\|_\infty \leq \|k\|_\infty \| (L_x k_1 \cdot A_{z_n} F) * G_n \|_\infty \leq (7.7) \\
&\leq \|k\|_\infty \cdot \|G_n\|_\infty \cdot \|L_x k_1 \cdot A_{z_n} F\|_1 \leq \|k\|_\infty \|G_n\|_\infty (A_{z_n} |F|) * k_1^\vee(x),
\end{aligned}$$

where $k_1 \in \mathcal{K}(\mathcal{G})$ satisfies $k_1 \geq 0$, and $k_1(x) \equiv 1$ on $(\text{supp } k) \cdot Q$ (compare Proposition 3.10). Therefore

$$\begin{aligned}
\|F * L_{z_n} G_n | W(L^\infty, Y)\| &= \|K | Y\| \leq \|k\|_\infty \|G_n\|_\infty \| (A_{z_n} |F|) * k_1^\vee | Y\| \leq \\
&\leq \|k\|_\infty \|G_n\|_\infty \|k_1 | L_w^1\| \|A_{z_n} |F| | Y\| \leq \text{Const} \cdot \omega^*(z_n) \|G_n\|_\infty \|F | Y\|. (7.8)
\end{aligned}$$

Step 2. Putting together the pieces $L_{z_n} G_n$ we obtain

$$\begin{aligned}
&\|F * G | W(L^\infty, Y)\| = \\
&= \|\sum_{n=1}^\infty A_{z_n} F * G_n | W(L^\infty, Y)\| \leq \sum_{n=1}^\infty \|A_{z_n} F * G_n | W(L^\infty, Y)\| \leq \\
&\text{(by (7.8))} \\
&\leq C \sum_{n=1}^\infty \|G_n\|_\infty \omega^*(z_n) \|F | Y\| \leq C' \|F | Y\| \|G | W(L^\infty, L_w^1)\|,
\end{aligned}$$

which finishes the proof of (a).

(b) Since $Y * \mathcal{K}(\mathcal{G})$ consists of continuous functions and $W(C^0, L_w^1)$ is the closure of $\mathcal{K}(\mathcal{G})$ in $W(L^\infty, L_w^1)$, (7.5) follows immediately from (a).

(c) Given $F \in W(L^1, Y)$ and $G \in W^R(L^\infty, L_w^1)$ choose a decomposition $G = \sum_{n=1}^\infty R_{z_n} G_n$, with $\text{supp } G_n \subseteq Q = Q^{-1}$, $G_n \in L^\infty$ and $\sum_{n=1}^\infty \|G_n\|_\infty \omega(z_n) \cong \|G | W^R(L^\infty, L_w^1)\|$. Alternatively one might combine Proposition 3.10 with Lemma 3.5.c).

Step 1. For any n we obtain

$$\begin{aligned}
|F * G_n(y)| &= \left| \int F(\xi) G_n(\xi^{-1} y) d\xi \right| \leq \left(\int |F(\xi)| c_Q(\xi^{-1} y) d\xi \right) \|G_n\|_\infty = \\
&= \|G_n\|_\infty \cdot \|L_y c_Q \cdot F\|_1 = \|G_n\|_\infty K(y)
\end{aligned}$$

(where the convolution can be written as an integral because $F \in L_{loc}^1$).

Thus

$$\|F * G_n | Y\| \leq \|G_n\|_\infty \|K | Y\| = \|G_n\|_\infty \|F | W(L^1, Y)\|.$$

Step 2. Adding over the building blocks G_n , one obtains

$$\begin{aligned} \|F * G | Y\| &\leq \sum_{n=1}^{\infty} \|R_{z_n}(F * G_n) | Y\| \leq \sum_{n=1}^{\infty} w(z_n) \|F * G_n | Y\| \leq \\ (\text{by Step 1}) &\leq \sum_{n=1}^{\infty} w(z_n) \|G_n\|_{\infty} \|F | W(L^1, Y)\| \leq \\ &\leq C \|F | W(L^1, Y)\| \|G | W^R(L^{\infty}, L_w^1)\|. \end{aligned}$$

Combining this with Lemma 3.9.a) (7.6) is proved. \square

Remark 1. The above proof requires only slight modifications to obtain the following general convolution result for Wiener type spaces: Assume that a *Banach convolution triple* is given, i.e. three Banach spaces such that $B_1 * B_2 \subseteq B_3$ (where convolution need not be defined pointwise) holds, with the norm estimate

$$\|g * f\|_{B_3} \leq \|g\|_{B_1} \|f\|_{B_2}.$$

Set $v_1(x) := \| |A_x | W(B_1, Y) \|$ and $v_2(x) := \| |R_x | W(B_2, Y) \|$. Then

$$W(B_1, Y) * W(B_2, L_{v_1}^1) \subseteq W(B_3, Y) \quad (7.9)$$

$$W(B_1, Y) * W^R(B_2, L_{v_2}^1) \subseteq W(B_3, Y) \quad (7.10)$$

together with the corresponding norm estimates. In fact, one observes similarly to (7.8) that

$$\|F * G_n | W(B_3, Y)\| \leq C \|F | W(B_1, Y)\| \|G_n\|_{B_2}$$

whenever $G_n \in B_2$ and $\text{supp } G_n \subseteq Q$. For general $G \in W(B_2, L_{v_1}^1)$ (or $W^R(B_2, L_{v_2}^1)$ respectively) the decomposition $G = \sum_{n=1}^{\infty} L_{z_n} G_n$ (or $G = \sum_{n=1}^{\infty} R_{z_n} G_n$) and a synthesis argument as in Step 2 yield the conclusion.

For IN-groups — a too restricted class for our purposes (it would apply only to example 7.1 of [LRP] — where there exists a compact neighbourhood U of the identity satisfying $Ux = xU$ for all $x \in \mathcal{G}$, more general convolution relations are already known (cf. [F1], Thm. 3). Since we shall need only Theorem 7.1. here we shall not go into a more detailed discussion of related questions. We want to mention here at least that the convolution relation

$$W(L^1, Y) * W(C^0, L_w^1) \subseteq W(C^0, Y). \quad (7.11)$$

already holds true under the condition (7.1).

In the applications the conditions for \mathcal{B}_w are not as serious as it might appear. Furthermore in the case of IN-groups the following is a consequence of (7.11).

Lemma 7.2. *If \mathcal{G} is an IN-group, then $\mathcal{A}_w = \mathcal{B}_w$.*

Proof. We fix some $g_0 \in \mathcal{B}_w$, then any $g \in \mathcal{A}_w$ is of the form $\pi(F)g_0 = \int F(y)\pi(y)g_0 dy$ for some $F \in L_w^1$ (apply [LRP], Lemma 4.2). Therefore

$$\begin{aligned} V_g(g) &= V_{\pi(F)g_0}(\pi(F)g_0) = \\ &= F * V_{g_0}(g_0) * F^\nabla \in L_w^1 * W(C^0, L_w^1) * L_w^1 \subseteq W(C^0, L_w^1) \end{aligned}$$

according to ([F1], Thm. 3). \square

Theorem 7.3. *(Pointwise interpolation). Assume that Y and w satisfy (7.1). Then for any $G \in W(C^0, L_w^1)$ with $G(e) \neq 0$ there exists a compact set $K \subseteq \mathcal{G}$ such that the following is true: For the K -separated family $X = (x_i)_{i \in I}$ in \mathcal{G} the restriction mapping $F \mapsto (F * G(x_i)_{i \in I})$ is surjective from Y onto $Y_d(X)$, i.e. given $\lambda \in Y_d(X)$ there exists $F \in Y$ such that $F * G(x_i) = \lambda_i$ for all $i \in I$ and*

$$\|F|Y\| \leq C \| \lambda | Y_d(X) \|. \tag{7.12}$$

Proof. Since $G(e) \neq 0$ we may fix a compact set Q with $c_Q * G(e) \neq 0$ and $k \in \mathcal{X}(\mathcal{G})$, $k \geq 0$, with $k(z) \equiv 1$ on Q . The norms in $Y_d(X)$ and in $W(C^0, \cdot)$ will be taken with respect to the same Q and k .

(a) With the above choice the family of restriction operators R_X from $W(C^0, Y)$ into $Y_d(X)$, given by $R_X F := (F(x_i)_{i \in I})$, is uniformly bounded, where X runs through the Q -separated families $X = (x_i)_{i \in I}$ in \mathcal{G} .

For $F \in W(C^0, Y)$ the norm $\|R_X F|Y_d(X)\|$ equals

$$\begin{aligned} \|(F(x_i)_{i \in I})|Y_d(X)\| &= \|\sum_{i \in I} |F(x_i)| c_{x_i Q}(y) | Y\| = \|\sum_{i \in I} |F(x_i)| | Y\| \leq \\ &\leq \| \|L_y k \cdot F\|_\infty | Y\| = \|F|W(C^0, Y)\| \end{aligned}$$

where $I_y := \{i: x_i \in yQ\}$ as usual (and $\text{card } I_y \leq 1$ by assumption). In particular,

$$|F(e)| \leq \|F|W(C^0, Y)\| \tag{7.13}$$

(b) On the other hand, the family of synthesizing operators $T_{X,G}: Y_d(X) \rightarrow \mathcal{W}(C^0, Y)$, given by $T_{X,G}(A) := (\sum_{i \in I} \lambda_i c_{x_i Q}) * G$ is uniformly bounded by $\|G\| \mathcal{W}(C^0, L_{w^*}^1)$ by Theorem 7.1.b:

$$\begin{aligned} \|T_{X,G}(A) \mid \mathcal{W}(C^0, Y)\| &= \|(\sum_{i \in I} \lambda_i c_{x_i Q}) * G \mid \mathcal{W}(C^0, Y)\| \leq \\ &\leq \|\sum_{i \in I} \lambda_i c_{x_i Q} \mid Y\| \|G \mid \mathcal{W}(C^0, L_{w^*}^1)\| = \|A \mid Y_d(X)\| \|G \mid \mathcal{W}(C^0, L_{w^*}^1)\|. \end{aligned} \quad (7.14)$$

By an analogous argument the mapping $F \mapsto (F * G(x_i))_{i \in I}$ is bounded from Y into $Y_d(X)$, more precisely

$$\|(F * G(x_i))_{i \in I} \mid Y_d\| \leq \|G \mid \mathcal{W}(C^0, L_{w^*}^1)\| \|F \mid Y\| \quad (7.15)$$

(c) We know now that for every Q -separated family $X = (x_i)_{i \in I}$ the operator $R_X T_{X,G}$ maps $Y_d(X)$ into $Y_d(X)$ with a norm $\leq \|G \mid \mathcal{W}(C^0, L_{w^*}^1)\|$.

(d) The theorem will be proved if we can find a compact set $K \supseteq Q$ and $\lambda \neq 0$, $\lambda \in \mathbb{C}$ such that for all K -separated families X

$$\| |\lambda| Id_{Y_d(X)} - R_X T_{X,G} \| < |\lambda| \quad (7.16)$$

Then $R_X T_{X,G}$ can be inverted. Given $A \in Y_d(X)$, set

$$\Gamma = (\gamma_i)_{i \in I} := (R_X T_{X,G})^{-1} A \in Y_d(X) \quad \text{and} \quad F := \sum_{i \in I} \gamma_i c_{x_i Q} \in Y.$$

Consequently

$$(F * G(x_i))_{i \in I} := R_X T_{X,G} (R_X T_{X,G})^{-1} A = A$$

and thus $F \mapsto (F * G(x_i))_{i \in I}$ is surjective. Moreover

$$\|F \mid Y\| = \|\Gamma \mid Y_d(X)\| \leq \| (R_X T_{X,G})^{-1} \| \|A \mid Y_d(X)\|.$$

(e) In order to accomplish (7.16) we use the density of $\mathcal{K}(\mathcal{G})$ in $\mathcal{W}(C^0, L_{w^*}^1)$ to find some $G_1 \in \mathcal{K}(\mathcal{G})$ such that for $H := G_1 - G \in \mathcal{W}(C^0, L_{w^*}^1)$

$$\|H \mid \mathcal{W}(C^0, L_{w^*}^1)\| < \min(3/4, (4\|c_Q \mid L_{w^*}^1\|)^{-1}) \|c_Q * G(e)\| \quad (7.17)$$

This choice of G_1 guarantees that

$$\begin{aligned} &|c_Q * G_1(e)| - |c_Q * G(e)| - |c_Q * H(e)| \geq \\ &\geq |c_Q * G(e)| - \|c_Q * H \mid \mathcal{W}(C^0, L_{w^*}^1)\| \geq \\ &\geq |c_Q * G(e)| - \|c_Q \mid L_{w^*}^1\| \|H \mid \mathcal{W}(C^0, L_{w^*}^1)\| > 3/4 \|c_Q * G(e)\| > 0. \end{aligned}$$

Then $K := \text{supp}(c_Q * G_1) \subseteq Q \cdot (\text{supp } G_1)$ has the desired properties. For any K -separated family $X = (x_i)_{i \in I}$ and $\lambda \in Y_d(X)$ one obtains $c_Q * G_1(x_k^{-1} x_i) = \delta_{ik} c_Q * G_1(e)$ and

$$\begin{aligned} & c_Q * G_1(e) \lambda_i - (R_X T_{X,G}(\lambda))_i = \\ & = \sum_k \lambda_k L_{x_k} c_Q * G_1(x_i) - \sum_k \lambda_k L_{x_k} c_Q * G(x_i) = \\ & = \sum_k \lambda_k L_{x_k} c_Q * H(x_i) = (R_X T_{X,H}(\lambda))_i \end{aligned} \quad (7.18)$$

Combining (7.18), (c) and (7.17) this yields

$$\begin{aligned} & ||| c_Q * G_1(e) Id_{Y_d(X)} - R_X T_{X,G} | Y_d(X) ||| = ||| R_X T_{X,H} | Y_d(X) ||| \leq \\ & \leq \|H\| \mathcal{W}(C^0, L_{w^*}) \leq 3/4 |c_Q * G(e)| < |c_Q * G_1(e)|. \quad \square \end{aligned}$$

8. The Local Behaviour of Extended Representation Coefficients and Inclusions of Coorbit Spaces

We first recall the definition of a coorbit space. The representation π and the translation-invariant solid BF-space Y on \mathcal{G} are as in Section 7. In the context of coorbit spaces the weight w associated to Y always satisfies (cf. [FG] (4.10)):

$$w(x) \geq C \max \{ ||| L_x | Y |||, ||| L_{x^{-1}} | Y |||, ||| R_x | Y |||, ||| A_x | Y ||| \}.$$

\mathcal{A}_w is assumed to be non-trivial, by polarization of \mathcal{A}_w we obtain a space of "test functions" \mathcal{H}_w^1 (see [LRP], chap. 4). Then $\mathcal{C}_o Y$ is defined by

$$\mathcal{C}_o Y = \{ f \in (\mathcal{H}_w^1)^\dagger : V_g(f) \in Y \}$$

with norm

$$\|f\|_{\mathcal{C}_o Y} := \|V_g(f) | Y\|.$$

In Part I only the global properties of $V_g(f)$ and the reproducing property were involved. For the purposes of this part we need still more information on the extended representation coefficients. The next theorems describe precisely the local behaviour of the wavelet transforms $V_g(f)$ and its relation to the global properties.

Theorem 8.1. *Assume that $g \in \mathcal{B}_w$. Then $f \in \mathcal{C}_o Y$ implies $V_g(f) \in \mathcal{W}(C^0, Y)$. In particular, for every relatively separated family $X = (x_i)_{i \in I}$ the evaluation $f \mapsto (V_g(f)(x_i))_{i \in I}$ maps $\mathcal{C}_o Y$ into $Y_d(X)$ and*

$$\| (V_g(f)(x_i))_{i \in I} | Y_d(X) \| \leq C \|f\|_{\mathcal{C}_o Y}. \quad (8.1)$$

Proof. It is no loss of generality to assume $\|Ag\| = 1$. Since $V_g(g) \in W(C^0, L^1_w)$ the assertion is then an immediate consequence of the reproducing formula for $V_g(f)$ and (7.5). Moreover (8.1) follows from Lemma 3.8 (b) [FG]. \square

Proposition 8.2. (*Pointwise interpolation of wavelet transforms*). Assume that Y and w satisfy (7.2). Then for every normalized $g \in \mathcal{B}_w$ there is a compact set $K \subseteq \mathcal{G}$ and $C > 0$ such that for any K -separated family $(x_i)_{i \in I}$ in \mathcal{G} and any sequence $\Lambda \in Y_d$ there exists $f \in \mathcal{C}_0 Y$ satisfying $\|f\|_{\mathcal{C}_0 Y} \leq C \| \Lambda | Y_d \|$ and $V_g(f)(x_i) = \lambda_i$ for all $i \in I$.

Proof. Since $G = V_g(g)$ satisfies the assumptions of Theorem 7.3 one can find a compact set K such that for every K -separated family X the mapping $F \mapsto (F * G(x_i))_{i \in I}$ is surjective from Y onto $Y_d(X)$. Thus, given $\Lambda \in Y_d(X)$ there exists $F \in Y$ with $F * G(x_i) = \lambda_i$ for all $i \in I$ and the norm estimate $\|F\|_Y \leq C \| \Lambda | Y_d \|$ (cf. (7.12)). By Theorem 4.3 and the reproducing property for $F * G$ there exists $f \in \mathcal{C}_0 Y$ with $V_g(f) = F * G$. \square

Remark 2. The reader should compare these statements on evaluations, equivalence of various norms on spaces of extended coefficients (cf. below) with similar ones about analytic functions known as theorems of Plancherel—Polya—Nikolskij-type (cf. [TR], Chap. 1.3). There are also similarities to questions of pointwise interpolation. Although the properties of representation coefficients and of analytic functions seem to stem from completely different arguments (if one looks for the classical proofs) the similarity might be seen as a consequence of the existence of an *integrable* reproducing convolution kernel (e.g. $g \in \mathcal{S}(\mathbb{R}^m)$ with Fourier transform identical one on a sufficiently large set) in both cases. At least for us this analogy was a source of inspiration and helped us to understand better various problems.

Theorem 8.3. (*Independence of local properties of Y*). Let Z be a solid, translation-invariant BF-space on \mathcal{G} such that

$$W(C^0, Y) \hookrightarrow Z \hookrightarrow W(L^1, Y).$$

Then, $\mathcal{C}_0 Y = \mathcal{C}_0 Z$, with equivalent norms.

Proof. If g is an analyzing vector in \mathcal{B}_w , then $G = V_g(g) \in \mathcal{W}^R(C^0, L_w^1) \cap \mathcal{W}(C^0, L_w^1)$. For $f \in \mathcal{C}_0 Z$, $V_g(f) = V_g(f) * G \in \mathcal{Z} * \mathcal{W}^R(C^0, L_w^1) \subseteq \mathcal{W}(L^1, Y) * \mathcal{W}^R(C^0, L_w^1) \subseteq Y$ and

$$\begin{aligned} \|f| \mathcal{C}_0 Y\| &= \|V_g(f)| Y\| \leq C_1 \|V_g(f)| \mathcal{W}(L^1, Y)\| \leq \\ &\leq C_1 \|V_g(f)| \mathcal{Z}\| = C_1 \|f| \mathcal{C}_0 Z\| \end{aligned}$$

by (7.6). It follows $\mathcal{C}_0 Z \subseteq \mathcal{C}_0 Y$. On the other hand, for $f \in \mathcal{C}_0 Y$, (7.5) yields

$$\begin{aligned} \|f| \mathcal{C}_0 Z\| &= \|V_g(f)| \mathcal{Z}\| \leq C_2 \|V_g(f)| \mathcal{W}(C^0, Y)\| \leq \\ &\leq C_2 \|V_g(f)| Y\| = C_2 \|f| \mathcal{C}_0 Y\|. \end{aligned}$$

Therefore $\mathcal{C}_0 Y \subseteq \mathcal{C}_0 Z$ and $\|f| \mathcal{C}_0 Y\| \cong \|f| \mathcal{C}_0 Z\|$. \square

The above theorem expresses that $\mathcal{C}_0 Y$ depends only on the “global behaviour” of the functions in Y (e.g. summability, decay conditions), but not on local properties (such as continuity or local p -summability).

Theorem 8.4. (*Inclusions*). $\mathcal{C}_0 Y \subseteq \mathcal{C}_0 Z$ if and only if $Y_d \subseteq Z_d$. In particular, two coorbit spaces coincide if and only if the corresponding sequence spaces are the same.

Proof. Given Y, Z the assumption $Y_d \subseteq Z_d$ is actually unambiguous, i.e. it is independent of X (cf. [FG], Lemma 3.5.c)) and implies $\mathcal{W}(L^\infty, Y) \subseteq \mathcal{W}(L^\infty, Z)$. Since the implication $Y_1 \subseteq Z_1 \Rightarrow \mathcal{C}_0 Y_1 \subseteq \mathcal{C}_0 Z_1$ is trivial. Thm. 8.3 yields $\mathcal{C}_0 Y = \mathcal{C}_0 \mathcal{W}(L^\infty, Y) \subseteq \mathcal{C}_0 \mathcal{W}(L^\infty, Z) = \mathcal{C}_0 Z$, as desired.

For the converse it has to be shown that a proper inclusion $Y_d(X) \subseteq Z_d(X)$ for any well-spread family X implies a proper inclusion for the corresponding coorbit spaces. To this end we split $(x_i)_{i \in I}$ according to Lemma 3.3 (b) into a finite family of K -separated subfamilies $(x_i)_{i \in I_r}$, $1 \leq r \leq s$, where K is chosen corresponding as in Proposition 8.2. The inclusion $Y_d \subseteq Z_d$ being proper, there exists some sequence λ of coefficients and some r such that $P_r \lambda \in Z_d \setminus Y_d$. By Prop. 8.2 there exists $f \in \mathcal{C}_0 Z$ such that $V_g(f)(x_i) = \lambda_i$ for $i \in I_r$. Therefore $(V_g(f)(x_i))_{i \in I_r}$ does not belong to Y_d . In view of Thm. 8.1 and Lemma 3.8(a) this implies $f \notin \mathcal{C}_0 Y$.

9. The Double Retract Property, Banach Space Structure

We have seen that $\mathcal{C}_o Y$ is a retract of $Y_d(X)$ as a consequence of the main result of [FG], namely the atomic decomposition for $\mathcal{C}_o Y$. In other words, $\mathcal{C}_o Y$ is isomorphic to a closed, complemented subspace of the sequence space $Y_d(X)$, hence $\mathcal{C}_o Y$ shares many properties of $Y_d(X)$, and many questions about $\mathcal{C}_o Y$ can be reduced to questions on the corresponding sequence space, which is of course much easier to handle in general. In order to obtain exact characterizations of the properties of coorbit spaces through corresponding properties of $Y_d(X)$, we sharpen Corollary 6.2 of [FG] and prove the so-called *double retract property* of $\mathcal{C}_o Y$.

Theorem 9.1. *$\mathcal{C}_o Y$ is a retract of $Y_d(X)$ and conversely, $Y_d(X)$ is a retract of a finite direct sum of copies of $\mathcal{C}_o Y$. Consequently, $\mathcal{C}_o Y$ shares all properties (as a Banach space) with the sequence space $Y_d(X)$ which are inherited by closed subspaces and finite direct sums of Banach spaces.*

Proof. Choose a compact set K which satisfies Proposition 8.2. and split the index set I into finitely many subfamilies $I_r, i = 1, \dots, s$, such that $\{x_i, i \in I_r\}$ is K -separated for all r (Lemma 3.3 (ii)). Denote by P_r the canonical projection on Y_d (i.e. $(P_r \Lambda)_i = \lambda_i$ for $i \in I_r$ and $= 0$ otherwise) and by $S_r: Y_d \rightarrow \mathcal{C}_o Y$ the “pointwise interpolation operator” on $I_r: S_r(\lambda)_{i \in I} = f$ with $V_g(f)(x_i) = \lambda_i$ for $i \in I_r$, whose existence is asserted in Proposition 8.2. By means of P_r and S_r we construct the following bounded mappings between $Y_d(X)$ and $\bigoplus_{r=1}^s \mathcal{C}_o Y$:

$$\bar{A}: Y_d \mapsto \bigoplus_{r=1}^s \mathcal{C}_o Y, \quad \bar{A}(\Lambda) := (S_1 \circ P_1 \Lambda, \dots, S_s \circ P_s \Lambda) \tag{9.1}$$

and

$$\bar{B}: \bigoplus_{r=1}^s \mathcal{C}_o Y \mapsto Y_d, \quad \bar{B}(f_1, \dots, f_s) := \sum_{r=1}^s P_r(V_g(f_r)(x_i)). \tag{9.2}$$

Now it follows immediately that $\bar{B} \circ \bar{A} = \text{Id}_{Y_d}$ and that $Y_d(X)$ is a retract of $\bigoplus_{r=1}^s \mathcal{C}_o Y$.

Corollary 9.2. *$\mathcal{C}_o Y$ is a reflexive Banach space if and only if $Y_d(X)$ is reflexive. Also, $\mathcal{C}_o Y$ has the Radon—Nikodym property, the metric approximation property, is of a certain type or cotype if and only if the same is true for $Y_d(X)$.*

The above results indicate that the Banach space structure of $\mathcal{C}_0 Y$ is closely related to that of the sequence space $Y_d(X)$. However, we do not know whether $\mathcal{C}_0 Y$ is always isomorphic to $Y_d(X)$ or whether $\mathcal{C}_0 Y$ always possesses an unconditional basis if this is true for $Y_d(X)$ (cf. [GR] for results in this direction). However, for the most important cases we can give a positive answer to that question:

Proposition 9.3. $\mathcal{C}_0 L_m^p$ is isomorphic to l^p and therefore admits an unconditional basis for $1 \leq p < \infty$ and any moderate function m on \mathcal{G} .

Proof. By [LRP], Theorem 6.2, $\mathcal{C}_0 L_m^p$ is isomorphic to a closed, infinite-dimensional, complemented subspace of l_m^p which, in turn, is isomorphic to l^p . Since l^p is a prime Banach space (cf. [LT], Vol. I, p. 54—57), every complemented, closed subspace is isomorphic to l^p .

Remark 3. It is known that the coorbit spaces of the $ax + b$ -group \mathcal{G}_0 (taken with respect to its unique irreducible infinite-dimensional representation) are all isomorphic to their associated sequence spaces $Y_d(X)$, for a certain family $X = (x_i)_{i \in I}$ in \mathcal{G}_0 , because there exist “coherent” unconditional bases for $\mathcal{C}_0 Y$ (or “wavelet bases”), i.e. a basis of the form $\{\pi(x_i)g_0, i \in I\}$ for carefully chosen elements $g_0 \in \mathcal{H}$ (cf. [M]).

As a further consequence of the double retract property we obtain a powerful criterion, allowing a classification of the quality of inclusion mapping between coorbit spaces. The relevant tool for such a refined description of embedding mappings is provided by the theory of operator ideals. Operator ideals are functors defining a class of compact mappings for each pair of Banach spaces, which is closed with respect to composition by bounded linear mappings. The best-known classes are the nuclear operators, the Hilbert—Schmidt operators, or more generally the Schatten—von Neumann classes (cf. [P] for details).

Theorem 9.4. i) The inclusion mapping $J: \mathcal{C}_0 Y \hookrightarrow \mathcal{C}_0 Z$ between coorbit spaces is automatically continuous. The same is true for $J_d: Y_d \hookrightarrow Z_d$.

ii) J is compact if and only if J_d is compact.

iii) Let \mathcal{J} be an operator ideal of compact operators. Then $J \in \mathcal{J}$ if and only if $J_d \in \mathcal{J}$.

Proof. i) The automatic continuity is an immediate consequence of the closed graph theorem, since coorbit spaces are continuously embedded into $(\mathcal{H}_w^1)^\top$ by Theorem 4.2.i), and solid BK-spaces Y_d are continuously embedded into the topological vector space of all sequences with pointwise convergence.

ii + iii) Let A and B be the retract mappings as defined in Corollary 6.2. They are defined on all of $(\mathcal{H}_w^1)^\top$ and $l_{1/w}^\infty$ respectively, and are bounded from $\mathcal{C}_o Y$ into Y_d and from Y_d onto $\mathcal{C}_o Y$ for general Y . As has been shown the inclusion mapping factorizes as follows:

$$J = B \circ J_d \circ A. \quad (9.3)$$

Using the bounded operators \bar{A} , \bar{B} of (9.1) and (9.2), the double retract property yields a similar factorization for J_d :

$$J_d = \bar{B} \circ \left(\bigoplus_{r=1}^s J \right) \circ \bar{A} \quad (9.4)$$

where

$$Y_d \xrightarrow{\bar{A}} \bigoplus_{r=1}^s \mathcal{C}_o Y \xrightarrow{\bigoplus_{r=1}^s J} \bigoplus_{r=1}^s \mathcal{C}_o Z \xrightarrow{\bar{B}} Y_d.$$

In (9.3) and (9.4) we have implicitly used Theorem 8.4.

Thus by the defining properties of operator ideals J belongs to \mathcal{F} (in particular to the space of compact operators) provided that J_d is in this operator ideal (compact resp.). Conversely, if J belongs to some operator ideal (or is just compact), then the same is true for $\bigoplus_{r=1}^s J$, because the sequence of s -numbers of this operator is the same as that of J (up to r -fold repetitions). By the same argument as above (9.4) implies that J_d belongs to \mathcal{F} (is compact).

The following Corollary is a simple application of Theorem 9.4. For concrete examples it includes new embedding theorems which have not yet been derived by direct methods.

Corollary 9.5. *For $p > 1$, $\mathcal{C}_o L_w^p$ is (compactly) embedded into \mathcal{H}^1 if and only if $w^{-1} \in L^{p'}(\mathcal{G})$, with $1/p + 1/p' = 1$.*

Proof. Hölder's inequality implies that $\ell_w^p \subseteq \ell^1$ if and only if $\sum_{i \in I} w^{-p'}(x_i) < \infty$. Because the weight w is submultiplicative this is equivalent to the condition $w^{-1} \in L^{p'}(\mathcal{G})$.

The embedding is even compact because — again by Hölder — the unit ball of ℓ_w^p can be approximated in the ℓ^1 -norm by suitable bounded subsets of finite-dimensional subspaces of ℓ^1 . \square

We conclude this section with the following observation.

Proposition 9.6: *Assume that w is a weight function on \mathcal{G} such that $w^{-1} \in L^r(\mathcal{G})$ for some $r > 1$, and that $\bigcap_{s \geq 0} \mathcal{A}_w s \neq \emptyset$. Then the following spaces coincide and form a nuclear Frechet-space with respect to their natural systems of seminorms ($1 \leq p, q \leq \infty$):*

$$\bigcap_{s \geq 0} \mathcal{C} \circ L_{w^s}^p = \bigcap_{s \geq 0} \mathcal{C} \circ L_{w^s}^q = \bigcap_{k=1}^{\infty} \mathcal{C} \circ L_{w^k}^p.$$

Proof. By the assumption these spaces are well defined. Furthermore, sufficiently large powers of w^{-1} belong to any given L^s -space for $s > 0$. It follows therefrom that the embedding $\ell_{w^k}^p \subseteq \ell_{w^s}^p$ is a nuclear mapping if $k - s$ is large enough. The assertion now follows using Theorem 9.4.

10. Minimality and Maximality. Relative Completions of Coorbit Spaces

Although the norm of a Banach space which is continuously embedded into $(\mathcal{H}_w^1)^\top$ (endowed with the w^* -topology) is unique up to equivalence (by the closed graph theorem) it may happen that for a given norm $\|\cdot\|$ in $(\mathcal{H}_w^1)^\top$ there are several subspaces of $(\mathcal{H}_w^1)^\top$ which become Banach spaces with respect to that norm. In the sequel we characterize the minimal and maximal coorbit spaces:

Since any coorbit space contains some \mathcal{H}_w^1 for a suitable weight function w the following notations appear as the natural ones:

Definition. A coorbit space $\mathcal{C} \circ Y$ is called *minimal* (with respect to the property of defining a given coorbit norm on \mathcal{H}_w^1) if \mathcal{H}_w^1 is norm dense in $\mathcal{C} \circ Y$; it is called *maximal*, if it is not properly contained in a coorbit space defining the same norm on \mathcal{H}_w^1 as the given space (thus $\mathcal{C} \circ Y$ cannot be contained as a closed subspace in another coorbit space).

Proposition 10.1. *The following properties are equivalent:*

- i) $\mathcal{C}_0 Y$ is a minimal coorbit space.
- ii) The partial sums in the atomic decomposition are norm convergent for any $f \in \mathcal{C}_0 Y$.
- iii) The finite sequences are dense in Y_d (i.e. is minimal as a BK-space).
- iv) $(Y_d)' = (Y_d)^\alpha$.

Proof. The equivalence of iii) and iv) follows immediately from the theorem of Hahn—Banach. If iii) is satisfied, it is clear from the proof of the atomic decomposition that ii), hence i) hold true.

In order to derive iii) from i) we show first that a given sequence $\Lambda = (\lambda_i)_{i \in I} \in Y_d(X)$ can be approximated by sequences in ℓ_w^1 . For that purpose we take up the notations of Theorem 9.1. Consider the image $\bar{A}(\Lambda) = (f_1, \dots, f_r) \in \bigoplus_{i=1}^r \mathcal{C}_0 Y$, where $V_g(f_j)(x_i) = \lambda_i$ for $i \in I_j$. By assumption i) there are $h_1, \dots, h_r \in \mathcal{H}_w^1$ such that $\|f_i - h_i\|_{\mathcal{C}_0 Y} < \varepsilon/rC$, for $i = 1, \dots, r$ which implies the estimate $\|(\sum_{k=1}^r V_g(f_k - h_k)(x_i))_{i \in I} \|_{Y_d} < \varepsilon$ by an application of (8.1).

An application of the inverse operator $\bar{B}: \bigoplus_{i=1}^r \mathcal{C}_0 Y \rightarrow Y_d$ and the double retract property allow to conclude that $\Lambda_1 := \bar{B}(h_1, \dots, h_r)$ satisfies the inequality $\|\Lambda - \Lambda_1\|_{Y_d} < \varepsilon \|\bar{B}\|$. Since $\Lambda_1 \in \ell_w^1$ it may be approximated by a finite sequence Λ_2 in ℓ_w^1 (in the norm of ℓ_w^1 , hence that of Y_d), the proof is complete. \square

Remark 4. Recall that the above conditions are true, if the functions in Y with compact support are dense in Y (cf. Lemma 3.5 (a)). Under the additional assumption that the right translation in $(Y, \|\cdot\|_Y)$ is continuous it can be shown that iii) is equivalent to the density of $\mathcal{K}(\mathcal{G})$ in Y .

Remark 5. It follows immediately from condition iii) of Proposition 10.1 above that the family of minimal coorbit spaces is closed with respect to finite intersections, sums or arbitrary interpolation methods.

Next we investigate relative completions of normed spaces within the context of coorbit spaces (cf. [BF] for related investigations). Let

$(B, \|\cdot\|_B)$ be any normed subspace of $(\mathcal{H}_w^1)^\dagger$, then we define the relative completion \tilde{B} of B as

$$\tilde{B} := \{\sigma \in (\mathcal{H}_w^1)^\dagger : \sigma = w^*\text{-}\lim_n f_n \text{ for some bounded sequence } (f_n) \text{ in } B\}.$$

With the natural norm

$$\|\sigma\|_{\tilde{B}} := \inf \{\overline{\lim} \|f_n\| : \sigma = w^*\text{-}\lim_n f_n\}$$

\tilde{B} is a Banach space continuously embedded into $(\mathcal{H}_w^1)^\dagger$.

The following theorem and its corollary describe the relative completions of coorbit spaces again as certain coorbit spaces;

Theorem 10.2. *Under the usual assumption*

$$(\mathcal{C}_\sigma Y)^\sim = \mathcal{C}_\sigma Y^{\alpha\alpha}.$$

Proof. We shall use an appropriate characterization of $Y^{\alpha\alpha}$ which can be easily read off from [LT], Vol. II, p. 30: A function F belongs to $Y^{\alpha\alpha}$ if and only if it is the pointwise limit a.e. of a sequence (F_n) which is bounded in Y . Moreover

$$\|F\|_{Y^{\alpha\alpha}} \cong \inf \{\overline{\lim} \|F_n\|_Y\}$$

where the infimum is taken over all such sequences.

i) Now suppose $\sigma \in (\mathcal{C}_\sigma Y)^\sim$. We choose a bounded w^* -convergent sequence (f_n) with

$$\sigma = w^*\text{-}\lim_n f_n \text{ and } \overline{\lim} \|f_n\|_{\mathcal{C}_\sigma Y} \leq 2\|\sigma\|_{(\mathcal{C}_\sigma Y)^\sim}.$$

By Theorem 4.1. (v) this implies $V_g(f_n)(x) \rightarrow V_g(\sigma)(x)$ for all $x \in \mathcal{G}$. By means of the preceding discussion we obtain that $V_g(\sigma) \in Y^{\alpha\alpha}$.

$$\begin{aligned} \|\sigma\|_{\mathcal{C}_\sigma Y^{\alpha\alpha}} &= \|V_g(\sigma)\|_{Y^{\alpha\alpha}} \leq \\ &\leq \overline{\lim} \|V_g(f_n)\|_Y = \overline{\lim} \|f_n\|_{\mathcal{C}_\sigma Y} \leq 2\|\sigma\|_{(\mathcal{C}_\sigma Y)^\sim}, \end{aligned}$$

and therefore $(\mathcal{C}_\sigma Y)^\sim \hookrightarrow \mathcal{C}_\sigma Y^{\alpha\alpha}$.

ii) On the other hand, if $\sigma \in \mathcal{C}_\sigma Y^{\alpha\alpha}$ we choose an exhausting sequence of compact sets $(K_n)_{n=1}^\infty$ for \mathcal{G} and define $F = V_g(\sigma)$, and $F_n := (F \chi_{K_n}) * G$. Thus $F \in Y^{\alpha\alpha} \cap L_{1/w}^\infty$ and $(F_n)_{n=1}^\infty$ is a sequence in $L_w^1 \cap Y$, bounded in the norm of Y , because $\|F_n\|_Y = \|F_n\|_{Y^{\alpha\alpha}} \leq \|G\|_{L_w^1} \|F\|_{Y^{\alpha\alpha}}$, where we have used Theorem 7.1, the solidity of $Y^{\alpha\alpha}$ and the fact that the embedding from Y into

$Y^{\alpha\alpha}$ is always isometric. Since $G \in L_w^1(\mathcal{G})$ it follows that $F(x) - F_n(x) = (F - F c_{K_n}) * G(x) = \langle L_x G, F - F c_{K_n} \rangle$ converges to 0 for every fixed $x \in \mathcal{G}$. Using the reproducing property for F_n we can find elements $f_n \in \mathcal{H}_w^1 \subseteq \mathcal{C}_o Y$ with $\|f_n|_{\mathcal{C}_o Y}\| \leq \|F|_{Y^{\alpha\alpha}}\|$ and convergent in the w^* -sense to σ . Thus $\sigma \in (\mathcal{C}_o Y)^\sim$ and

$$\|\sigma|_{(\mathcal{C}_o Y)^\sim}\| \leq \|F|_{Y^{\alpha\alpha}}\| = \|\sigma|_{\mathcal{C}_o Y^{\alpha\alpha}}\|.$$

As a corollary we have the following characterizations of relatively complete coorbit spaces.

Corollary 10.3. *The following conditions are equivalent:*

- i) $\mathcal{C}_o Y$ is relatively complete, i.e. $\mathcal{C}_o Y = (\mathcal{C}_o Y)^\sim$.
- ii) Any function F on \mathcal{G} which is the pointwise limit of a bounded sequence of functions $(V_g(f_n))$ is of the form $V_g(f)$, for some $f \in \mathcal{C}_o Y$.
- iii) $\mathcal{C}_o Y = \mathcal{C}_o Z^\alpha$ for some solid translation invariant BF-space Z .
- iv) $\mathcal{C}_o Y$ is a maximal coorbit space.
- v) $Y_d = Z_d^\alpha$ for some (minimal) solid BK-space Z_d .

Proof. (i) \Leftrightarrow (ii) This is implicit in the proof of Thm. 10.2.

(i) \Rightarrow (iii) If $\mathcal{C}_o Y = (\mathcal{C}_o Y)^\sim = \mathcal{C}_o Y^{\alpha\alpha}$, we take Z to be Y^α .

(iii) \Rightarrow (i) If $\mathcal{C}_o Y = \mathcal{C}_o Z^\alpha$, then

$$(\mathcal{C}_o Y)^\sim = \mathcal{C}_o Y^{\alpha\alpha} = \mathcal{C}_o Z^{\alpha\alpha\alpha} = \mathcal{C}_o Z^\alpha = \mathcal{C}_o Y.$$

(i) \Leftrightarrow (v) is shown similarly. Taking Z_d to be the closure of the finite sequences in the $(Y_d)^\alpha$ -norm, we see that Z_d can be chosen a minimal sequence space.

(i) \Rightarrow (iv) Assume that $\mathcal{C}_o Y$ is relatively complete and that $\mathcal{C}_o Y$ is a closed subspace of some coorbit space $\mathcal{C}_o Z$. By Banach's theorem the norms $\|\cdot\|_{\mathcal{C}_o Y}$ and $\|\cdot\|_{\mathcal{C}_o Z}$ are equivalent on $\mathcal{C}_o Y$. Given $f \in \mathcal{C}_o Z$, we proceed as in the proof of Thm. 10.2 to find a sequence (f_n) in $\mathcal{H}_w^1 \subseteq \mathcal{C}_o Y \cap \mathcal{C}_o Z$ which converges to f in the w^* -sense and which is bounded in $\mathcal{C}_o Z$ and thus bounded in $\mathcal{C}_o Y$. Therefore $f \in (\mathcal{C}_o Y)^\sim = \mathcal{C}_o Y$ and thus $\mathcal{C}_o Y = \mathcal{C}_o Z$.

(iv) \Rightarrow (i) Because Y is isometrically embedded into $Y^{\alpha\alpha}$, $\mathcal{C}_o Y$ is a closed subspace of $\mathcal{C}_o Y^{\alpha\alpha}$. If $\mathcal{C}_o Y$ is maximal it must therefore coincide with $\mathcal{C}_o Y^{\alpha\alpha}$; in other words, $\mathcal{C}_o Y$ is relative complete. \square

Remark 6. Again it follows from the equivalence of these conditions (especially v) in connection with Remark 5 that the family of maximal coorbit spaces is closed with respect to sums, intersections and (upper) complex interpolation.

Corollary 10.4. *A coorbit space $\mathcal{C}_\sigma Y$ is reflexive if and only if the following holds true:*

i) *If $\sigma \in (\mathcal{H}_w^1)^\vee$ is the w^* -limit of a sequence from \mathcal{H}_w^1 , bounded in $\mathcal{C}_\sigma Y$, then $\sigma \in \mathcal{C}_\sigma Y$. Even more, it can be approximated in the norm by elements in \mathcal{H}_w^1 .*

ii) *Any $\tau \in (\mathcal{C}_\sigma Y)'$ can be approximated in the norm by elements of \mathcal{H}_w^1 .*

Proof. By the above results, (i) is equivalent to the fact that $\mathcal{C}_\sigma Y$ is both minimal and maximal, or equivalently Y_d is both minimal and maximal. The dual space is then automatically a maximal coorbit space. Condition (ii) is then equivalent to the minimality of $(\mathcal{C}_\sigma Y)'$ or $(Y_d)' = Y_d^*$. Thus the conditions stated are equivalent to the conditions characterizing the reflexivity of a solid BF-space Y_d as stated in [Z], §75. Now apply Corollary 9.2. \square

As a consequence of the reflexivity criterion we derive the following

Corollary 10.5. *The family of all reflexive coorbit spaces is closed with respect to duality, intersections, sums and complex interpolation.*

Proof. This results follows from Remarks 5 and 6, using the fact that for two minimal coorbit spaces:

$$(\mathcal{C}_\sigma Y^1 \cap \mathcal{C}_\sigma Y^2)' = (\mathcal{C}_\sigma Y^1)' + (\mathcal{C}_\sigma Y^2)' \quad (10.1)$$

and

$$(\mathcal{C}_\sigma Y^1 + \mathcal{C}_\sigma Y^2)' = (\mathcal{C}_\sigma Y^1)' \cap (\mathcal{C}_\sigma Y^2)' \quad (10.2)$$

hold true. The stability with respect to interpolation follows from [BL], Corollary 4.5.2, which shows that duality is compatible with complex interpolation. \square

There is also a couple of further results corresponding precisely Corollaries 3.9 to 3.12 given in [BF]. We shall not state these results here explicitly.