

## ITERATIVE RECONSTRUCTION OF MULTIVARIATE BAND-LIMITED FUNCTIONS FROM IRREGULAR SAMPLING VALUES\*

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**Abstract.** This paper describes a real analysis approach to the problem of complete reconstruction of a band-limited, multivariate function  $f$  from irregularly spaced sampling values  $(f(x_i))_{i \in I}$ . The required sampling density of the set  $X = (x_i)_{i \in I}$  depends only on the spectrum  $\Omega$  of  $f$ . The proposed reconstruction methods are iterative and stable and converge for a given function  $f$  with respect to any weighted  $L^p$ -norm  $1 \leq p \leq \infty$ , for which  $f$  belongs to the corresponding Banach space  $L^p_b(\mathbb{R}^m)$ . It is also shown that any band-limited function  $f$  can be represented as a series of translates  $L_{y_j} g$  (with complex coefficients) for a given integrable, band-limited function  $g$  if the Fourier transform satisfies  $\hat{g}(t) \neq 0$  over  $\Omega$  and the family  $Y = (y_j)_{j \in J}$  is sufficiently dense. Moreover, the behavior of the coefficients (such as weighted  $p$ -summability) corresponds precisely to the global behavior of  $f$  (i.e., membership in the corresponding weighted  $L^p$ -space). The proofs are based on a careful analysis of convolution relations, spline approximation operators, and discretization operators (approximation of functions by discrete measures). In contrast to Hilbert space methods, the techniques used here yield pointwise estimates. Special cases of the algorithms presented provide a theoretical basis for methods suggested recently in the engineering literature. Numerical experiments have demonstrated the efficiency of these methods convincingly.

**Key words.** irregular sampling, iterative reconstruction, approximation of convolutions, multivariate band-limited functions, function spaces

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**1. Introduction.** According to the classical sampling theorem attributed to Whittaker, Shannon, Kotel'nikov, and several others, a band-limited function can be reconstructed from sampling values over any sufficiently fine lattice. Because of its great importance in information theory, electrical engineering, signal processing, and other applications, a lot of work has been carried out on improvements and extensions, e.g., to lattices in higher dimensions [PM], [Me], [DM]. Reviews and extensive references on these investigations are given in [Bu], [Je], [Hi2], [BERS], [BSS], and [Ma] (some of which mention the irregular sampling problem).

The regular sampling theorem is based on Fourier series and Poisson's formula. This limits the possible sampling sets to lattices, in other words, to discrete sets that arise from the standard lattice  $\mathbb{Z}^m \subseteq \mathbb{R}^m$  through application of an invertible real  $n \times n$  matrix (cf. [Co]). However, in many applications, e.g., optics, tomography, synthetic aperture radar, computer graphics, signal processing, meteorology, and geophysics it is necessary to deal with situations where sampling values are *not* available on a regular grid [SA], [Ce], [PK], [St], [So].

In this paper a new *iterative approach* to the irregular sampling problem is described which represents a *constructive* solution to this question. The proposed algorithms use only standard operations which are also well suited for numerical implementation.

**1.1. The requirements for an irregular sampling theory.** As a simple model for sampling theory let us recall the regular sampling theorem: Given a band-limited function (of finite energy)  $f \in L^2(\mathbb{R})$  with spectrum in  $[-\pi W, \pi W]$ ,  $f$  can be represented as a cardinal series

$$f(t) = \sum_{k \in \mathbb{Z}} f(k/W) \operatorname{sinc}(\pi(Wt - k))$$

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where  $\text{sinc}(x) := x^{-1} \sin(x)$  for  $x \neq 0$  and  $\text{sinc}(0) = 1$ ; cf. [Bu], [BSS], [Pa]. The series is convergent in the  $L^2$ -sense and uniformly. In the case of oversampling, i.e., if sampling values  $(f(\alpha k))_{k \in \mathbb{Z}}$  for some  $\alpha < 1/W$  are known, the nonintegrable sinc-function can be replaced by other “windows” (cf. [Sch], [BERS]) with a more rapid decay. Then the series converges with respect to weighted  $L^p$ -norms as well, whenever  $f$  belongs to such a weighted  $L^p$ -space (cf. [F4, Thm. 2]). Such alternative windows with rapid decay are also required if band-limited tempered distributions must be reconstructed from their regular sampling values (cf. [Ca], [BERS], [Se]).

The regular sampling theorem combines *two aspects*:

(a) Any band-limited function can be *completely reconstructed* from its sampled values over a *sufficiently fine* lattice by means of a simple series expansion with the sampled values as coefficients.

(b) Any band-limited function can be *expanded into a series with translates of a single function  $g$*  as building blocks.

For the irregular sampling problem we will discuss these two aspects separately; see Theorems 3.1 and 3.2. Practical considerations impose the following requirements on an irregular sampling theory. They are satisfied in the case of regular sampling, at least for fast decaying kernels. The theory should be

(1) *Constructive*, i.e., a possibly iterative algorithm should allow numerical reconstruction;

(2) *Multidimensional*, so that it can be used in signal and image processing or for the interpolation of sequences of images;

(3) *Local*, so that the value of a band-limited function at a point is essentially determined by the adjacent sampling values, and more distant sampling values have no influence. Estimates with respect to weighted  $L^p$ -norms are a suitable tool to describe decay conditions and the locality properties of the reconstruction operators.

(4) *Stable*, so that small perturbations of the parameters cause only small errors in the reconstruction.

**1.2. The real analysis approach of this paper.** To achieve these objectives we choose a real variable approach. We start with the observation that a band-limited function satisfies a convolution equation of the form  $f = f * g$ , and then analyze this convolution equation carefully. The main tools are *pointwise estimates* (a) for spline type approximations of smooth functions and (b) for the approximation of a convolution by a weighted sum of translates, see Lemmas 4.1–4.4.

Iterative algorithms arise through repeated application of these approximation operators to the remainder term. The resulting sequence converges to the original signal at a geometric rate. These reconstruction methods satisfy all the requirements stated above.

In contrast to many previous papers on irregular sampling, we do not use methods from analytic function theory. This is one of the reasons why the results extend easily to multivariate irregular sampling. Our techniques allow for a treatment of irregular sampling in a very general class of Banach spaces.

Discretization operators which are dual to the spline operators lead to non-orthogonal series expansions of band-limited signals in terms of translates of a single function. The coefficients depend in a linear way on the expanded function, and the coefficient mapping is continuous with respect to any of the norms under consideration. Similar techniques can be used on nonabelian locally compact groups to derive very general results on atomic decompositions for function spaces [FG1], [FG2], [Gr1].

**1.3. Review of the literature.** Most papers concerning the irregular sampling theorem deal with signals of finite energy, i.e., work in the Hilbert space  $L^2(\mathbb{R})$  [DS], [Be1,2], [Wi]. In general, such arguments are not applicable to the important class of almost periodic functions or even trigonometric polynomials. Unweighted  $L^p$ -spaces are treated in [Go] for regular sampling and in the important but nonconstructive paper by Beurling [B]. Only in a few cases [SA], [BH2], [PM] are two-dimensional problems discussed, however, with additional conditions on the sampling set.

In the mathematical literature very strong uniqueness theorems are found for band-limited functions, both in one dimension [BM], [Wa] and in higher dimensions [La], [B]. These results, which use complex analysis and operator theory, are of high theoretical interest, but they have had no practical implications because they are not constructive.

For small deviations from regular sampling the perturbation theory of orthonormal bases in Hilbert spaces yields irregular sampling theorems [Hi1], [Ye], [Ka], [Yo], [Ra1,2], [BH1,2]. In principle, these methods are constructive, but they seem to be too difficult to use for numerical computations. For instance, a reconstruction through a Lagrange type interpolation involves functions which are given as infinite products. A reconstruction by means of the biorthogonal system requires that these functions be computed first. Besides the computational complexity of this task, it is also known that the biorthogonal system depends on the sampling set in an unstable way, so that a small change of a single sampling point affects all functions of the biorthogonal system in an unpredictable way [Sp]. Moreover, sampling theorems of this kind are restricted to the Hilbert space  $L^2(\mathbb{R}^m)$ , and they cannot treat sampling sets with strong variations of the local density.

The results in [CPL], [So] are based on the idea of transforming the irregular sampling set into a regular one. They give exact reconstruction, but only for certain classes of functions which are not band-limited and which depend heavily on the sampling geometry in a nontransparent way.

For numerical computations *iterative methods* are most useful for recovering a band-limited function from irregularly sampled values [Wi], [SA], [MA]. They are all derived from the fundamental paper of Duffin and Shaeffer [DS] on nonharmonic Fourier series and a theorem in [Sa]. The convergence of these methods is known only for the Hilbert space  $L^2$ . The description of convergence therefore lacks the much desired locality.

The numerical implementation of [SA] is quite successful in image restoration, although for their third method no proof of convergence is known.

**1.4. A short overview.** The plan of this paper is as follows: Section 2 begins with a description of a family of Banach spaces of functions and measures on  $\mathbb{R}^m$ , which will allow us to describe convergence of the iterative methods with respect to a variety of norms. Together with these spaces, suitable operators such as spline type approximations and approximations through discrete measures are introduced. They are not well defined for  $L^p$ -spaces, but bounded on the auxiliary spaces introduced in this section. These concepts are also crucial in order to adapt the approach described in [F3] for  $L^1$ -spaces to  $L^2(\mathbb{R}^m)$  or to weighted  $L^p$ -spaces. The main results of this paper are stated in § 3. In contrast to known results, we need not assume a positive minimal distance of the sampling points and can thus treat local variations of the density. For numerical applications this means that all information in regions of high sampling density can be used. For  $L^p$ -spaces and with positive minimal distance between the sampling points, the results allow a much simpler and more accessible formulation. Therefore

we state them explicitly as corollaries. Section 4 contains the technical parts and the proof of the results. The underlying estimates are formulated in a series of lemmas.

**2. Function spaces and operators.** First let us fix some notation. We denote the space of all Radon measures (regular Borel measures on  $\mathbb{R}^m$ ) by  $R(\mathbb{R}^m)$ . By the Riesz representation theorem we identify it with the topological dual of  $\mathcal{H}(\mathbb{R}^m)$ :  $\mathcal{H}(\mathbb{R}^m) := \{k|k \text{ continuous, complex-valued, with compact support}\}$ . A point measure  $\delta_x$  is characterized by  $\delta_x(f) = \int f(y) d\delta_x(y) := f(x)$ . For the uniform norm we use the following symbol:  $\|f\|_\infty := \sup_{z \in \mathbb{R}^m} |f(z)|$ .

Submultiplicative weight functions, i.e., continuous functions satisfying  $w(x) \geq 1$  for all  $x \in \mathbb{R}^m$  and  $w(x+y) \leq w(x)w(y)$  for all  $x, y \in \mathbb{R}^m$ , are important because the weighted  $L^1$ -spaces  $L^1_w(\mathbb{R}^m) := \{f|f w \in L^1(\mathbb{R}^m)\}$ , with the norm  $\|f\|_{1,w} := \int_{\mathbb{R}^m} |f(x)|w(x) dx$ , are Banach algebras under convolution (cf. [Rei], [F1] for details)

$$(2.1) \quad f * g(x) := \int_{\mathbb{R}^m} f(x-y)g(y) dy \quad \text{for } f, g \in L^1_w(\mathbb{R}^m).$$

If  $w = w_a : y \rightarrow (1 + |y|)^a$ , for some  $a \geq 0$  we write  $L^1_a(\mathbb{R}^m)$  instead of  $L^1_w(\mathbb{R}^m)$ .

We shall describe our approach in the setting of solid BF-spaces (or Banach lattices). Formally we make the following general assumptions:

(B1)  $(B, \|\cdot\|_B) \hookrightarrow L^1_{loc}(\mathbb{R}^m)$  is a Banach space of locally integrable functions on  $\mathbb{R}^m$ , and for any compact set  $Q \subseteq \mathbb{R}^m$  there exists  $C_Q > 0$  such that

$$\int_Q |f(x)| dx \leq C_Q \|f\|_B \quad \text{for all } f \in B.$$

(B2)  $(B, \|\cdot\|_B)$  is a Banach module over  $(C^0(\mathbb{R}^m), \|\cdot\|_\infty)$  with respect to pointwise multiplication, i.e.,  $hf \in B$  and  $\|hf\|_B \leq \|h\|_\infty \|f\|_B$  for  $h \in C^0$  and  $f \in B$ .

(B3)  $(B, \|\cdot\|_B)$  is translation invariant in the following sense: The translation operators  $L_y, y \in \mathbb{R}^m$ , given by  $L_y f(x) := f(x-y)$ , map  $B$  into itself and for some  $a \geq 0$  we have  $\|L_y f\|_B \leq C_B (1 + |y|)^a \|f\|_B$  for all  $f \in B$ .

(B3')  $(B, \|\cdot\|_B)$  is a Banach convolution module over  $L^1_a(\mathbb{R}^m)$ , i.e., we assume that, for  $f \in B$  and  $g \in L^1_a, g * f \in B$  and  $\|g * f\|_B \leq C_B \|g\|_{1,a} \|f\|_B$ .

*Remark 2.1.* It follows from (B3) that  $B$  is a space of tempered distributions, i.e.,  $(B, \|\cdot\|_B) \hookrightarrow \mathcal{S}'(\mathbb{R}^m)$ . Consequently, the Fourier transform  $\hat{f}$  and the spectrum  $\text{spec } f := \text{supp } \hat{f}$  (equals the support of  $\hat{f}$ ) are well defined for  $f \in B$ . For any closed set  $\Omega \subseteq \mathbb{R}^m$  the set

$$B^\Omega := \{f \in B, \text{spec } f \subseteq \Omega\}$$

is a closed subspace of  $B$ .

*Examples.* The most natural examples satisfying (B1)-(B3') are the spaces

$$L^p_b := \{f|fv \in L^p(\mathbb{R}^m)\}, \quad \text{with norm } \|f\|_{v,p} := \left( \int_{\mathbb{R}^m} |f(x)v(x)|^p dx \right)^{1/p}$$

for  $1 \leq p < \infty$  (and a sup norm for  $p = \infty$ ); cf. [F1]. The function  $v$  is assumed to be a continuous and positive function which is *moderate* with respect to the *weight function*  $w_a$ , i.e., it satisfies  $v(x+y) \leq C_1 w_a(x)v(y)$  for all  $x, y \in \mathbb{R}^m$ . Note that obviously  $w_b$  is a moderate function (with respect to  $w_a$ ) for any  $b \in [-a, a]$ . Our general approach also includes (weighted) mixed norm spaces in the sense of Benedek-Panzone, or weighted variants of rearrangement invariant Banach spaces such as Lorentz or Orlicz spaces [LT] or spaces of bounded  $p$ -mean [F5].

In order to extend the results stated in [F3] to  $L^p$ -spaces (with  $p > 1$ ) the Wiener type spaces  $W(M, B)$  and  $W(C^0, B)$  are an important tool [F2]. For  $B = L^p$  they coincide with amalgam spaces in the sense of [FSt]. We shall describe them briefly, using the symbols  $MB$  and  $CB$ , and give a new simple proof of a convolution theorem.

DEFINITION 2.1. For a fixed open, bounded subset  $Q \subseteq \mathbb{R}^m$  we define the *local maximal function*  $x \mapsto f^*(x)$  by  $f^*(x) := \sup_{z \in Q+x} |f(z)|$ . Then

$$(2.2) \quad CB := \{f \mid f \text{ continuous, } f^* \in B\}.$$

defines a Banach space with the norm

$$(2.3) \quad \|f\|_{CB} := \|f^*\|_B.$$

We shall denote the space  $CL_a^1(\mathbb{R}^m)$  by  $C_a^1(\mathbb{R}^m)$ . Note that the Schwartz space  $\mathcal{S}(\mathbb{R}^m)$  is continuously embedded into these spaces:  $\mathcal{S}(\mathbb{R}^m) \hookrightarrow C_a^1(\mathbb{R}^m)$  for any  $a \in \mathbb{R}$ . The second space associated with  $B$  in a natural way allows us to deal with discrete measures with a certain global behavior.

DEFINITION 2.2. For  $Q$  as above we set

$$(2.4) \quad MB = \{\mu \in R(\mathbb{R}^m), \text{ with } q_\mu : x \mapsto |\mu|(x+Q) \in B\},$$

$$(2.5) \quad \|\mu\|_{MB} := \|q_\mu\|_B.$$

For a discrete set  $X = (x_i)_{i \in I}$  in  $\mathbb{R}^m$  we shall write  $MB_X$  for the closed subspace  $\{\mu = \sum_{i \in I} \lambda_i \delta_{x_i}, \mu \in MB\}$  of measures in  $MB$  supported on  $X$ .

In the last two definitions different bounded sets  $Q_1, Q_2$  generate the same space and equivalent norms; hence  $g \in CB$  if and only if  $g^{**} \in B$ . For any positive  $k \in \mathcal{H}(\mathbb{R}^m)$ ,  $\|\cdot\|_{|\mu|*k}$  is also an equivalent norm for  $MB$ . These facts are used in the sequel (e.g., for (2.9) below) without notice.

Next we describe basic properties of these spaces.

THEOREM 2.1. *Let  $(B, \|\cdot\|_B)$  satisfy the general assumptions (B1)-(B3'). Then the following hold:*

(i)  *$CB$  and  $MB$  are Banach spaces satisfying properties (B1)-(B3'), with the following continuous embeddings:*

$$(2.6) \quad CB \hookrightarrow B \hookrightarrow MB.$$

(ii)  *$MB * C_a^1 \subseteq CB$ , and  $\|\mu * g\|_{CB} \leq C_0 \|\mu\|_{MB} \cdot \|g\|_{C_a^1}$  for all  $\mu \in MB$  and  $g \in C_a^1$ .*

(iii) *The spaces  $MB^\Omega$ ,  $B^\Omega$ , and  $CB^\Omega$  coincide, and the respective norms are equivalent for any compact subset  $\Omega \subseteq \mathbb{R}^m$ , i.e., any  $\mu \in MB^\Omega$  is represented by a function  $f \in CB$ , and there exists a constant  $C_\Omega > 0$  such that  $\|f\|_{CB} \leq C_\Omega \|f\|_B \leq C_\Omega \|f\|_{MB}$  for all  $f \in MB^\Omega$ .*

(iv) *If  $\mathcal{H}(\mathbb{R}^m)$  is dense in  $B$ , then it is dense in  $CB$ .*

Remark 2.2. In view of the examples above we assume that (2.6) holds in the form

$$\|f\|_{MB} \leq \|f\|_B \leq \|f\|_{CB} \quad \text{for all } f \in CB.$$

*Proof of Theorem 2.1.* The verification of (i) and (iv) is left to the interested reader; cf. [F2]. For (ii) we have to verify  $(\mu * f)^* \in B$ . By direct computation the following two pointwise estimates can be obtained:

$$(2.7) \quad (\mu * f)^* \leq |\mu| * f^*,$$

and for any  $k \in \mathcal{H}(\mathbb{R}^m)$ ,  $k \geq 0$ , with  $\text{supp } k \subseteq Q = -Q$  and  $\int_{\mathbb{R}^m} k(y) dy = 1$ ,

$$(2.8) \quad h^* * k \geq |h|.$$

Combining these two estimates we obtain

$$(2.9) \quad (\mu * g)^* \leq |\mu| * g^* \leq |\mu| * (k * g^{**}) = (|\mu| * k) * g^{**} \in B * L_a^1 \subseteq B$$

whenever  $\mu \in MB$  and  $g^{**} \in L_a^1$ , i.e.,  $g \in C_a^1$ , and the proof of (ii) is complete.

Since  $\text{spec } \mu \subseteq \Omega$ ,  $\mu = \mu * g$  holds for any  $g \in C_a^1$  with  $\hat{g}(t) \equiv 1$  on  $\Omega$ . Consequently,  $\mu = \mu * g \in MB * C_a^1 \subseteq CB$  by (ii). This also gives the required estimate (cf. Remark 2.2)

$$(2.10) \quad \|\mu\|_{MB} \leq \|\mu\|_B \leq \|\mu\|_{CB} = \|\mu * g\|_{CB} \leq \|(\mu * g)^*\|_B \leq C_B \|\mu\|_{MB} \|g^*\|_{1,a}. \quad \square$$

Besides these *convolution relations* two special types of *operators* on *CB* and *MB* are needed. Both involve so-called  $\delta$ -PUs.

**DEFINITION 2.3.** We call a family of nonnegative (measurable) functions  $\Psi = (\psi_i)_{i \in I}$  a  $\delta$ -partition of unity ( $\delta$ -PU for short) if the following is satisfied:  $\sum_{i \in I} \psi_i(x) \equiv 1$  on  $\mathbb{R}^m$ , and  $\text{supp } \psi_i \subseteq K_\delta(x_i)$  for  $i \in I$  for some discrete family  $X = (x_i)_{i \in I}$  in  $\mathbb{R}^m$ . Here  $K_\delta(x)$  denotes the open ball of radius  $\delta$  centered at  $x$ . We shall use the symbol  $|\Psi|$  for the infimum over all numbers  $\delta$  such that  $\Psi$  is a  $\delta$ -PU.

Note that  $X$  has to be  $\delta$ -dense in  $\mathbb{R}^m$ , i.e.,  $\mathbb{R}^m = \cup_{i \in I} K_\delta(x_i)$  for any  $\delta$ -PU  $\Psi$ .

*Examples of  $\delta$ -partitions of unity.* (a) If  $X$  is any  $\delta$ -dense family and  $(P_i)_{i \in I}$  is a partition of  $\mathbb{R}^m$  such that  $P_i \subseteq K_\delta(x_i)$ , then the family of indicator functions  $(c_{P_i})_{i \in I}$  is a  $\delta$ -PU. The use of Voronoi regions is most natural for our task (cf. [FG3]).

(b) If  $\psi_0(x) = 1 - |x|/\delta$  for  $|x| \leq \delta$  and  $\psi_0(x) = 0$  for  $|x| > \delta$ , then  $L_{\delta n} \psi_0$ ,  $n \in \mathbb{Z}$  is a continuous  $\delta$ -partition for  $\mathbb{R}$ .

(c) For irregular sequences in  $\mathbb{R}$  we can take triangular functions with  $\psi_i(x_i) = 1$  and supported by  $[x_{i-1}, x_{i+1}]$ . The natural analogue for  $\mathbb{R}^2$  can be described as follows. Starting with a triangulation induced from the set  $X$ , choose the functions to satisfy  $\psi_i(x_i) = 1$  and to be piecewise linear over the triangles having  $x_i$  as a vertex; cf. [SA].

(d) For any  $\delta$ -dense family  $X$  in  $\mathbb{R}^m$  we can find smooth  $\delta$ -PUs. In the regular case smooth PUs in  $\mathbb{R}^m$  of the form  $L_{\delta n} \psi_0$ ,  $n \in \mathbb{Z}^m$ , can be obtained using *B-splines*.

**Remark 2.3.** In many cases it is convenient to work with families  $X = (x_i)_{i \in I}$  which are *well spread* in the following sense:  $X$  is  $\delta$ -dense and *relatively separated* in  $\mathbb{R}^m$ , i.e.,  $X$  is a *finite union* of subfamilies, such that  $|x_i - x_j| \geq \delta_0 > 0$  for all  $i \neq j$  in the same subfamily, for some  $\delta_0 > 0$ . In this case the covering through the balls  $K_\delta(x_i)$  is of finite height. Of course a sequence is relatively separated in  $\mathbb{R}$  if  $|x_n - n| \leq C$  for all  $n \in \mathbb{N}$  (cf. [BH2] for a two-dimensional version).

**Remark 2.4.** If the family  $Y$  is well spread and  $v$  is a moderate function, then  $\mu = \sum_{j \in J} c_j \delta_{y_j} \in (ML_v^p)_Y$  if and only if  $(\sum_{j \in J} |c_j|^p v(y_j)^p)^{1/p} < \infty$ . For well-spread families it is also easy to check that  $\sum_{j \in J} |f(y_j)|^p v(y_j)^p \leq C_Y^p \|f\|_{CL_v^p}^p$  for all  $f \in CL_v^p$ . The estimate given in Theorem 2.1(iii) is thus a generalization of Nikolskij's inequality to general  $p$  and  $m$  dimensions (cf. [Ni, pp. 123-125], or [BH2, Thm. 1] for special cases with  $p = 2$ ).

Using  $\delta$ -PUs we define the following operators.

**DEFINITION 2.4.** Given a  $\delta$ -PU  $\Psi$  associated with a family  $X$  we denote by  $Sp_\Psi$  (actually we should write  $Sp_{\Psi, X}$ ) the operator defined for continuous  $f$ :

$$(2.11) \quad Sp_\Psi(f) := \sum_{i \in I} f(x_i) \psi_i.$$

We also use the same symbol  $Sp_\Psi$  in order to describe a related operator which maps sequences  $\Lambda = (\lambda_i)_{i \in I}$  into functions on  $\mathbb{R}^m$ :

$$Sp_\Psi(\Lambda) := \sum_{i \in I} \lambda_i \psi_i.$$

*Remark 2.5.* Note that  $\|Sp_{\Psi}(f)\|_{\infty} \leq \|f\|_{\infty}$  for  $f \in C^b(\mathbb{R}^m)$ , and  $Sp_{\Psi}(f) \in C^b(\mathbb{R}^m)$  if in addition  $\Psi$  consists of continuous functions. If  $\Psi$  consists of piecewise polynomials of fixed order, then  $Sp_{\Psi}(f)$  is a spline approximation (or quasi interpolant) of  $f$ , which explains our notation. If  $\Psi$  is a system of triangular functions on  $\mathbb{R}$ , then  $Sp_{\Psi}f$  is the piecewise linear interpolation of the sampling values of  $f$  at  $X$ . Basic properties of  $Sp_{\Psi}$  are collected in the following proposition.

*Proposition 2.2.* Assume that  $(B, \|\cdot\|_B)$  satisfies (B1) and (B2).

(i) There is a constant  $C_S > 0$  such that  $\|Sp_{\Psi}f\|_B \leq C_S \|f\|_{CB}$  for all  $f \in CB$ ; The family  $Sp_{\Psi}$  of all spline operators with continuous  $\Psi$  and  $|\Psi| \leq 1$  acts uniformly bounded on any space  $CB$ .

(ii)  $\text{supp}(Sp_{\Psi}f) \subseteq \text{supp}(f) + K_1(0)$ ; hence  $Sp_{\Psi}(\mathcal{H}(\mathbb{R}^m)) \subseteq \mathcal{H}(\mathbb{R}^m)$  for  $\Psi$  in  $\mathcal{H}(\mathbb{R}^m)$ .

(iii)  $Sp_{\Psi}f \rightarrow f$  with respect to  $\|\cdot\|_{\infty}$  for  $|\Psi| \rightarrow 0$ , for any  $f \in \mathcal{H}(\mathbb{R}^m)$ , and consequently,  $Sp_{\Psi}f \rightarrow f$  in  $CB$  for any  $f \in CB$ , whenever  $\mathcal{H}(\mathbb{R}^m)$  is dense in  $B$ .

(iv) For fixed  $X$  the operator  $Sp_{\Psi}$  may be considered as a bounded operator from  $MB_X$  into  $B$ , i.e., we have

$$\|Sp_{\Psi}(\Lambda)\|_B \leq C \cdot \left\| \sum_{i \in I} \lambda_i \delta_{x_i} \right\|_{MB}.$$

*Proof.* (i) It is easily verified that  $\sup_{z \in X+Q} |Sp_{\Psi}f(z)| \leq \sup_{z \in X+Q_1} |f(z)|$  for  $Q_1 := Q + K_1(0)$ , because  $\text{supp} \psi_i \subseteq K_1(x_i)$  for all  $i \in I$ . Taking the  $B$ -norm with respect to  $x$  on both sides proves both statements of (i). The simple proof of (ii) and (iii) is left to the interested reader. (iv) follows from

$$|Sp_{\Psi}(\Lambda)(x)| \leq \sum_{i \in I} |\lambda_i| \psi_i(x) \leq \sum_{\psi_i(x) \neq 0} |\lambda_i| \leq q_{\mu}(x)$$

with  $\mu = \sum_{i \in I} \lambda_i \delta_{x_i}$  and  $Q := K_1(0)$ , if  $|\Psi| \leq 1$ , by taking the  $B$ -norm on both sides.

Next we introduce an operator which replaces a given function by a discrete measure. In order to avoid confusion with the PU  $\Psi$  used above we now write  $\Phi = (\varphi_j)_{j \in J}$  for a PU associated with  $Y = (y_j)_{j \in J}$  in  $\mathbb{R}^m$ .

**DEFINITION 2.5.** The discrete measure obtained from  $f \in L^1_{\text{loc}}(\mathbb{R}^m)$  through concentration of mass by means of  $\Phi$  is denoted by

$$D_{\Phi}f := \sum_{j \in J} \langle f, \varphi_j \rangle \delta_{y_j},$$

with  $\langle f, \varphi_j \rangle = \int_{\mathbb{R}^m} \varphi_j(x) f(x) dx$ .

The following properties of these operators are of interest to us.

**PROPOSITION 2.3.** Assume that  $(B, \|\cdot\|_B)$  satisfies (B1)-(B3').

(i) There is a constant  $C_D > 0$  such that for all  $f \in B$  and all  $\Phi$  with  $|\Phi| \leq 1$ ,

$$\|D_{\Phi}f\|_{MB} \leq C_D \|f\|_B;$$

(ii) Assume that  $\mathcal{H}(\mathbb{R}^m)$  is dense in  $B$ . Then  $D_{\Phi}f * h \rightarrow f * h$  in  $CB$ , hence in  $B$  and uniformly over compact sets, for  $h \in C^1_a$ , as  $|\Phi| \rightarrow 0$ .

*Proof.* In order to determine the norm of  $D_{\Phi}f$  in  $MB$  we observe that

$$\begin{aligned} |D_{\Phi}f|(x+Q) &= \sum_{\{j|y_j \in x+Q\}} |\langle f, \varphi_j \rangle| \\ &\leq \int_{\mathbb{R}^m} \left( \sum_{\{j|y_j \in x+Q\}} |f(y) \varphi_j(y)| \right) \leq |f| * c_{Q_1}(x) \end{aligned}$$

where  $c_{Q_1}$  is the indicator function of  $Q_1 := Q + K_1(0)$ . Since  $|f| * c_{Q_1} \in B * L^1_a \subseteq B$  by (B3') we obtain  $\|D_{\Phi}f\|_{MB} \leq C_1 \|\mu\|_{MB}$ . The proof of (ii) is left to the interested reader.

*Remark 2.6.* Note that both  $Sp_\Psi$  and  $D_\Phi$  are not bounded on  $B$  itself, and that the auxiliary spaces  $CB$  and  $MB$  are essential for our approach.

**3. The main results.** With the notations of § 2 we now describe the results about reconstruction and series expansions of irregularly sampled band-limited functions. The main theorems will contain two parts. The first part asserts the existence of a reconstruction operator. This formulation reveals the stability of the reconstruction and the correct norm estimates. The second part realizes the reconstruction operator as an iterative procedure. This form emphasizes the algorithmic aspect of the reconstruction. The corollaries make clear that for well-spread families the results can be described using natural Banach spaces of sequences instead of  $MB$ .

The first theorem deals with the complete reconstruction of band-limited functions from their sampling values.

**THEOREM 3.1** (General sampling theorem for band-limited functions). *Let  $\Omega$  be a compact subset of  $\mathbb{R}^m$ . Then there exist  $\delta = \delta(\Omega) > 0$  and  $C = C(\delta, \Omega) > 0$  such that for any  $\delta$ -PU  $\Psi$  on  $\mathbb{R}^m$  there is a bounded operator  $\mathbf{B}: B \rightarrow B^\Omega$  with*

$$(3.1) \quad \|\mathbf{B}(\phi)\|_B \leq C \cdot \|\phi\|_B,$$

which inverts  $Sp_\Psi$  on  $B^\Omega$ , i.e.,  $f = \mathbf{B}(Sp_\Psi f)$  for all  $f \in B^\Omega$ . Consequently,  $f$  can be completely recovered from the sampling values  $(f(x_i))_{i \in I}$ .

The operator  $\mathbf{B}$  can be realized by the following iterative algorithm: Fix a pair of band-limited functions  $g, h \in L^1_\alpha$  such that  $\hat{g}(t) \equiv 1$  on  $\Omega$  and  $\hat{h}(t) \equiv 1$  on  $\text{spec } g$ . Set

$$(3.2) \quad \phi_0 := \phi \quad \text{and} \quad \phi_{k+1} := \phi_k * h - Sp_\Psi(\phi_k * h);$$

Then

$$(3.3) \quad \mathbf{B}(\phi) = \left( \sum_{k=0}^{\infty} \phi_k \right) * g.$$

Formula (3.1) expresses the stability of the reconstruction. The algorithm satisfies all the other natural requirements discussed in the introduction. For more precise statements about the locality, see Theorem 3.4 and [FG4].

The operations involved can be easily implemented and our first numerical experiments have shown that the algorithm works efficiently. As has been shown in [Gr2], the required sampling density (at least for the one-dimensional case and a natural choice of  $\Psi$ ) is just the Nyquist rate.

If  $\Omega = [-1, 1]$ ,  $g = h = \text{sinc}$ , and  $\Psi$  is a system of triangular (or pyramid) type functions on  $\mathbb{R}$  or  $\mathbb{R}^2$ , then the algorithm can be shown to be equivalent to method (2) suggested in [SA] (without proof for the irregular case there). Since  $\text{sinc} \notin L^1$ , our argument for the convergence of the algorithm cannot be applied directly, but the arguments given in § 4 are easily adjusted to this case.

**COROLLARY A.** *Given a compact set  $\Omega \subseteq \mathbb{R}^m$  there exists  $\delta = \delta(\Omega) > 0$  such that for any  $\delta$ -dense, well-spread family  $X$  in  $\mathbb{R}^m$  the following is true: There exists a bounded linear operator  $\mathbf{R}$  from the sequence space  $l^p_\nu := \{\Lambda | (\sum_{i \in I} |\lambda_i|^p \nu(x_i)^p)^{1/p} < \infty\}$  into  $L^p_\nu(\mathbb{R}^m)$  such that  $\mathbf{R}$  provides a complete reconstruction of  $f$  from its sampling values, i.e.,  $\mathbf{R}(f(x_i)_{i \in I}) = f$  for all  $f \in (L^p_\nu)^\Omega$ . For given  $\Omega, X$  the same  $\mathbf{R}$  and  $\delta$  work for all  $a$ -moderate weights  $\nu$  and  $1 \leq p \leq \infty$ .*

A "dual" variant of Theorem 3.1 yields series expansions in terms of translates of a single function.

**THEOREM 3.2** (Series expansions for band-limited functions). *For any  $g \in L^1_\alpha$  with  $\hat{g}(s) \neq 0$  on a compact set  $\Omega \subseteq \mathbb{R}^m$  there is a positive number  $\gamma = \gamma(\Omega, g) > 0$  such that*

for any  $\gamma$ -dense family  $Y = (y_j)_{j \in J}$  there is a bounded linear operator  $\mathbf{D}$  from  $B^\Omega$  into  $MB_Y$  satisfying

$$f = D(f) * g \quad \text{for all } f \in B^\Omega.$$

Writing  $D(f) = \sum_{j \in J} c_j \delta_{y_j}$ , this means

$$(3.4) \quad f = \sum_{j \in J} c_j L_{y_j} g.$$

The coefficients are obtained as  $c_j := \sum_{k=0}^{\infty} \langle \varphi_j, f_k \rangle$ , where the sequence  $(f_k)_{k=0}^{\infty}$  is given iteratively (using an auxiliary function  $g_1$ ) by

$$(3.5) \quad f_0 := f * g_1 \quad \text{and} \quad f_{k+1} := (f_k - D_0 f_k) * h,$$

*Remark 3.1.* Since the sampling sets in  $\mathbb{R}^m$  do not have any natural order, we understand convergence of a series  $h = \sum_{i \in I} h_i$  in the following sense: for any exhausting sequence  $F_n \subseteq I$  of finite subsets of  $I$ , i.e.,  $F_n \subseteq F_{n+1}$  and  $\bigcup_{n=1}^{\infty} F_n = I$ , the sequence of partial sums  $\sum_{i \in F_n} h_i$  converges to  $h$ . Consequently, these series converge unconditionally, i.e., they converge for any fixed enumeration of  $I$ . If  $I = \mathbb{Z}^m$ , the interpretation as a multiple iterated sum is also admissible.

**COROLLARY B.** *Under the assumptions of Theorem 3.2 for any  $\eta$ -dense family  $Y$  in  $\mathbb{R}^m$  there is a bounded linear operator  $\mathbf{C}: (L_0^p)^\Omega \rightarrow l_0^p$  such that  $f(x) = \sum_{j \in J} (Cf)_j g(x - y_j)$  holds for every  $f \in (L_0^p)^\Omega$ . The series converges uniformly over compact sets and for  $1 \leq p < \infty$  also in the norm of  $L^p$ .*

The next theorem offers an alternative reconstruction algorithm which is easier to implement numerically and computationally less intensive, but the required sampling density for this algorithm may be higher.

**THEOREM 3.3** (Method of adaptive weights). *Given  $\Omega \subseteq \mathbb{R}^m$  compact,  $g \in L_a^1$  with  $\hat{g}(t) \equiv 1$  on  $\Omega$  and  $h \in L_a^1$  with  $\hat{h}(t) \equiv 1$  on  $\text{spec}(g)$ , there exists  $\eta = \eta(\Omega, g) > 0$  such that  $f \in B^\Omega$  can be reconstructed from its sampled values  $(f(x_i)_{i \in I})$  on any  $\eta$ -dense family  $X$  by the following algorithm: Set  $w_i = \int \psi_i(x) dx$  and*

$$(3.6) \quad \begin{aligned} \phi_0 &= \sum_{i \in I} f(x_i) \cdot w_i \cdot \delta_{x_i} \in MB_X, \\ \phi_{k+1} &= \phi_k * h - \sum_{i \in I} \phi_k(x_i) \cdot w_i \cdot L_{x_i} h. \end{aligned}$$

Then  $f = \sum_{n=0}^{\infty} \phi_n * g$  and the right side depends only on the sampling values  $(f(x_i)_{i \in I})$ .

*Remark 3.2.* The proof will show that the partial sums

$$(3.7) \quad f^{(n)} := \sum_{i \in I} w_i \sum_{k=0}^n \phi_k(x_i) \cdot L_{x_i} g$$

are convergent to  $f$  at a geometric rate.

The following variant of Theorem 3.1 is of interest if many functions with the same spectrum are to be reconstructed from samples taken over the same family  $X$  or if  $X$  and  $\Omega$  are given in advance.

**THEOREM 3.4.** *Under the conditions of Theorem 3.1 there is a family  $(e_i)_{i \in I}$  in  $C_n^1$  such that  $f \in B^\Omega$  can be written as  $f = \sum_{i \in I} f(x_i) e_i$ . The series converges uniformly over compact sets and, if  $\mathcal{H}(\mathbb{R}^m)$  is  $a$ -dense in  $B$ , in the norm of  $CB$ .*

Theorem 3.4 expresses the locality of the reconstruction. In contrast to the sinc-functions in the classical cardinal series the  $e_i$ 's have much better decay properties. In applications the collection of functions  $(e_i)_{i \in I}$  may be calculated in advance, using only the knowledge of  $\Omega$  and the sampling set  $X$ . Given the sampling values  $(f(x_i))_{i \in I}$ , the reconstruction of  $f$  is then obtained quickly by ordinary summation.

*Remark 3.3.* The representation of  $f$  as a series in  $(e_i)_{i \in I}$  is not unique. In contrast in Kadec's  $\frac{1}{4}$ -theorem the functions  $(e_i)_{i \in I}$  are not linearly independent in general and thus do not constitute a basis. On the other hand, these series expansions work simultaneously for all  $p$ ,  $1 \leq p < \infty$ . The nonuniqueness of the expansion is closely related to numerical stability.

In our last theorem we combine the two aspects of the regular sampling theorem and show how for a given family  $Y$  a suitable sequence of coefficients for a series expansion of the form (3.4) can be computed directly from the sampled values of  $f$  alone.

**THEOREM 3.5** (Combined sampling and expansion). *Given  $g \in L^1_\alpha$  with  $\hat{g}(x) \equiv 1$  on a compact set  $\Omega \subseteq \mathbb{R}^m$ , there exist two constants  $\delta = \delta(\Omega, g) > 0$  and  $\gamma = \gamma(\Omega, g) > 0$  such that for any two families  $(x_i)_{i \in I}$  and  $(y_j)_{j \in J}$  which are  $\delta$  and  $\gamma$ -dense, respectively, there is a linear mapping  $\mathbf{M}$  from the space of sampling values  $\{(f(x_i))_{i \in I}, f \in B^\Omega\}$  into  $MB_Y$  satisfying*

$$(3.8) \quad f = \mathbf{M}(f(x_i)) * g = \sum_{j \in J} c_j L_{y_j} g.$$

The coefficients can be obtained from a sequence defined iteratively by

$$(3.9) \quad f_0 := f, f_{k+1} := f_k * h - (D_\Phi S p_\Psi f_k) * h,$$

through

$$c_j := \sum_{k=0}^{\infty} \langle \phi_j, S p_\Psi f_k \rangle.$$

**COROLLARY C.** *If in the situation of Theorem 3.5 the sets  $X$  and  $Y$  are well spread, there exists a bounded linear operator  $\mathbf{N}$  from  $l^p_\nu(I)$  into  $l^p_\nu(J)$  such that for  $f \in (L^p_\nu)^\Omega$ ,*

$$f(x) = \sum_{j \in J} \mathbf{N}(f(x_i)_{i \in I})_j g(x - y_j),$$

with convergence in  $L^p_\nu$  for  $1 \leq p < \infty$ .

**4. Proofs.**

**LEMMA 4.1.** *For any compact subset  $\Xi \subseteq \mathbb{R}^m$  there exists some constant  $C_\Xi^1 = C(\Xi, a) > 0$  such that uniformly for all  $\delta$ -PUs  $\Psi$  and spaces  $B$  satisfying (B1)–(B3')*

$$\|f - S p_\Psi f\|_B \leq C_\Xi^1 \cdot \|f\|_B \cdot \delta \quad \text{for all } f \in B^\Xi.$$

*Proof.* We discuss the one-dimensional case first and check that the condition  $\text{spec } f \subseteq \Xi$  implies  $f' \in CB$ . Since  $\hat{f}'(t) = 2\pi i t \hat{f}(t)$  for all  $t \in \mathbb{R}$  it is sufficient to choose some  $u \in \mathcal{S}(\mathbb{R})$  such that  $\hat{u}$  is in  $\mathcal{D}$  (infinitely differentiable with compact support) and satisfies  $\hat{u}(t) = 2\pi i t$  on  $\Xi$ . Then  $f' = f * u$ , and since  $\mathcal{S}(\mathbb{R}^m) \subseteq C^1_a$  this implies  $f' \in CB$ , and by Theorem 2.1(ii),

$$(4.1) \quad \|f'\|_{CB} \leq C_0 \|f\|_B \|u\|_{C^1_a}.$$

The mean value theorem implies

$$|f(x) - f(y)| \leq \sup_{z \in K_\delta(x,y)} |f'(z)| \cdot \delta \quad \text{for } x, y \in \mathbb{R}^m \quad \text{with } |x - y| \leq \delta.$$

It follows therefore that (note that  $\text{supp } \psi_i \subseteq K_\delta(x_i)$ )

$$(4.2) \quad |(f(x) - f(x_i)) \psi_i(x)| \leq \delta \cdot \sup_{z \in K_\delta(x)} |f'(z)| \cdot \Psi_i(x).$$

By summation over  $i \in I$ , using the properties of a  $\delta$ -PU we obtain

$$|(f - Sp_{\Psi} f)(x)| \leq \sum_{i \in I} |(f(x) - f(x_i))\psi_i(x)| \leq \delta \cdot \sup_{z \in K_{\delta}(x)} |f'(z)|,$$

and, after taking  $B$ -norms on both sides,

$$(4.3) \quad \|Sp_{\Psi} f - f\|_B \leq \delta \cdot \|f'\|_{CB} \leq \delta \cdot C_{\Xi}^1 \|f\|_B,$$

for  $C_{\Xi}^1 := C_0 \cdot \|u\|_{C_a^1}$ . This completes the proof in the one-dimensional case.

In the case  $m \geq 2$  we use that the partial derivatives on  $B^{\Xi}$  can be represented as convolution operators, i.e.,  $\partial f / \partial x_k = f * u_k$  for all  $f \in B^{\Xi}$  if  $u_k \in \mathcal{S}(\mathbb{R}^m)$  satisfies  $\hat{u}_k(t) = 2\pi i t_k$  on  $\Xi$ . Thus

$$|\text{grad } f(x)| = \left( \sum_{k=1}^m (\partial f / \partial x_k(x))^2 \right)^{1/2} \leq \sum_{k=1}^m |\partial f / \partial x_k(x)| = \sum_{k=1}^m |f * u_k(x)|.$$

Setting  $u(x) := \sum_{k=1}^m |u_k(x)|$ , we have  $u \in C_a^1$  and

$$(4.4) \quad |\text{grad } f(x)| \leq (|f| * u)(x) \quad \text{for all } x \in \mathbb{R}^m.$$

Invoking the mean value theorem we obtain for any  $y \in B_{\delta}(x)$

$$|f(x) - f(y)| \leq |x - y| \sup_{z \in K_{\delta}(x)} |\text{grad } f(z)| \leq \delta \cdot \sup_{z \in K_{\delta}(x)} (|f| * u)(z).$$

By Theorem 2.1(ii) we have  $|f| * u \in CB$  and the same arguments as above apply (with  $f'$  being replaced by  $|f| * u$ ).  $\square$

*Remark 4.1.* Observe that in the above situation we have the inequality

$$(4.5) \quad |\text{grad } f|^{\#} \leq |f| * u^{\#}.$$

Using (2.7) and (4.4) we therefore derive the estimate

$$(4.6) \quad \|\text{grad } f\|_{CB} \leq C_{\Xi}^2 \cdot \|f\|_B \quad \text{for } f \in B^{\Xi}.$$

This result is a generalization of Bernstein's inequality from  $L^p$  to general function spaces  $B$ .

*Proof of Theorem 3.1.* Given  $\Omega \subseteq \mathbb{R}^m$  compact, we fix a pair of band-limited functions  $g, h \in L_a^1(\mathbb{R}^m)$  such that  $\hat{g}(t) = 1$  on  $\Omega$ , and  $\hat{h}(t) = 1$  on  $\text{spec } g$  (any pair of Schwartz functions  $\hat{g}, \hat{h} \in \mathcal{S}$  satisfying the condition will do). Consequently,

$$(4.7) \quad h * g = g, \quad f * g = f = f * g * h \quad \text{for all } f \in B^{\Omega}.$$

In order to define the operator  $\mathbf{B}$  on the Banach space  $(B, \|\cdot\|_B)$  we define for  $\phi \in B$  a sequence, starting with  $\phi_0 = \phi$ :

$$(4.8) \quad \phi_{k+1} := \phi_k * h - Sp_{\Psi}(\phi_k * h).$$

Since  $h$  is band-limited the functions  $\phi_k * h$  are band-limited and Lemma 4.1 is applicable with  $\Xi := \text{spec } h$ . Hence (using (B3'))

$$(4.9) \quad \|\phi_{k+1}\|_B \leq \delta \cdot C_{\Xi}^1 \|\phi_k * h\|_B \leq \delta \cdot C_{\Xi}^1 \cdot C_B \|h\|_{1,a} \|\phi_k\|_B.$$

If  $|\Psi| \leq \delta$ , such that  $\delta \cdot C_{\Xi}^1 C_B \|h\|_{1,a} =: \gamma < 1$ , then

$$(4.10) \quad \|\phi_{k+1}\|_B \leq \gamma \cdot \|\phi_k\|_B \leq \gamma^{k+1} \|\phi\|_B \quad \text{for } k \geq 0.$$

It follows that the operator  $\mathbf{B}$  given as

$$(4.11) \quad \mathbf{B}(\phi) = \left( \sum_{n=0}^{\infty} \phi_n \right) * g$$

is well defined on  $B$ , with values in  $B^\Xi$ , and bounded due to the estimate

$$\begin{aligned}
 \|\mathbf{B}(\phi)\|_B &\leq C_B \cdot \left\| \sum_{n=0}^{\infty} \phi_n \right\|_B \|g\|_{1,a} \\
 (4.12) \qquad &\leq C_B \cdot \sum_{n=0}^{\infty} \|\phi_n\|_B \|g\|_{1,a} \leq (1-\gamma)^{-1} C_B \|\phi\|_B \|g\|_{1,a}.
 \end{aligned}$$

In order to verify that  $\mathbf{B}$  inverts the spline operator  $Sp_\Psi$  over  $B^\Omega$  we have to consider the sequence  $(\phi_k)_{k=0}^\infty$ , now starting with  $\phi_0 := Sp_\Psi f$ , for  $f \in B^\Omega$ . We also use the sequence  $(f_k)_{k=0}^\infty$ , given by recursion (4.8), and starting with  $f_0 := f$ , because it satisfies the following identity:

$$(4.13) \qquad f = f_{k+1} * g + \left( \sum_{n=0}^k Sp_\Psi(f_n * h) \right) * g.$$

Equation (4.13) is clear for  $k=0$  by (4.8) and follows immediately by induction for  $k > 0$ . Moreover, since  $\|f_{k+1} * g\|_B \rightarrow 0$  at a geometric rate, (4.13) yields the following representation of  $f$  as an absolutely convergent series in  $B$ :

$$(4.14) \qquad f = \left( \sum_{n=0}^{\infty} Sp_\Psi(f_n * h) \right) * g.$$

It will now be sufficient to verify that

$$(4.15) \qquad Sp_\Psi(f_k * h) * g = \phi_k * g,$$

because then (4.14) implies the required identity

$$(4.16) \qquad \mathbf{B}(Sp_\Psi f) = \left( \sum_{k=0}^{\infty} \phi_k \right) * g = \sum_{k=0}^{\infty} Sp_\Psi(f_k * h) * g = f \quad \text{for } f \in B^\Omega.$$

In order to verify (4.15) we show first that

$$(4.17) \qquad f_{k+1} = f_k - \phi_k \quad \text{for } k \geq 0.$$

This is clear for  $k=0$  by (4.8) and follows for general  $k \geq 1$  by induction:

$$\begin{aligned}
 f_k - \phi_k &= f_{k-1} * h - Sp_\Psi(f_{k-1} * h) - \phi_{k-1} * h + Sp_\Psi(\phi_{k-1} * h) \\
 &= (f_{k-1} - \phi_{k-1}) * h - Sp_\Psi((f_{k-1} - \phi_{k-1}) * h) \\
 &= f_k * h - Sp_\Psi(f_k * h) = f_{k+1}.
 \end{aligned}$$

Equation (4.15) is true for  $k=0$  and follows from (4.17) by induction for  $k \geq 1$ , using (4.8):

$$\begin{aligned}
 Sp_\Psi(f_{k+1} * h) * g &= Sp_\Psi(f_k * h) * g - Sp_\Psi(\phi_k * h) * g \\
 &= \phi_k * h * g - Sp_\Psi(\phi_k * h) * g = \phi_{k+1} * g.
 \end{aligned}$$

The proof of Theorem 3.1 is thus complete.  $\square$

*Remark 4.2.* Note that both Lemma 4.1 and the proof of Theorem 3.1 work simultaneously for all function spaces  $B$  that are convolution modules over the same Beurling algebra  $L_a^1$ . Thus the same constants arise for all such spaces having the same constant  $C_B$  in (B3). This will be important for Theorem 3.4.

*Proof of Corollary A.* We define the reconstruction operator  $\mathbf{R}$  from  $l_v^p$  into  $L_v^p$  as follows: For  $\Lambda = (\lambda_i)_{i \in I} \in l_v^p$  we set  $\mathbf{R}(\Lambda) := \mathbf{B}(Sp_\Psi(\Lambda))$ , i.e., we form iteratively

$$(4.18) \qquad \phi_0 := \sum_{i \in I} \lambda_i \psi_i \quad \text{and} \quad \phi_{k+1} := \phi_k * h - Sp_\Psi(\phi_k * h) \quad \text{for } k \geq 1.$$

Then  $\mathbf{R}(\Lambda) = (\sum_{k=0}^{\infty} \phi_k) * g$ , the series is well defined in  $L_v^p$  and  $\text{spec } \mathbf{R}(\Lambda) \subseteq \text{spec } g$ . A combination of Theorem 3.1, Remark 2.4, and Proposition 2.2(iv) then yields the boundedness of  $\mathbf{R}$  as an operator from  $L_v^p$  into  $L_v^p$ . Evidently,  $\mathbf{R}$  applied to the sequence  $\Lambda = (f(x_i))_{i \in I}$  yields  $\mathbf{R}(\Lambda) = \mathbf{B}(Sp_{\Psi}f) = f$ . Thus  $\mathbf{R}$  is indeed a reconstruction operator.  $\square$

The following result will play the same role in the proof of Theorem 3.2 as Lemma 4.1 in the proof of Theorem 3.1. The operator  $D_{\Phi}$  is as in Proposition 2.3.

LEMMA 4.2 (Discretization of convolution). *For any compact subset  $\Xi \subseteq \mathbb{R}^m$  there exists some constant  $C_{\Xi}^3 = C(\Xi, a) > 0$  such that uniformly for all  $\eta$ -PUs  $\Phi$  and all spaces  $B$  satisfying (B1)-(B3'),*

$$(4.19) \quad \|(f - D_{\Phi}f) * h\|_{CB} \leq \eta \cdot C_{\Xi}^3 \cdot \|f\|_B \cdot \|h\|_{1,a}$$

for any band-limited  $h \in L_a^1$  with  $\text{spec } h \subseteq \Xi$  and  $f \in B$ .

*Proof.* Given an  $\eta$ -PU  $\Phi = (\varphi_j)_{j \in J}$  we consider for fixed  $j \in J$

$$(4.20) \quad |(f\varphi_j - \langle f, \varphi_j \rangle \delta_{y_j}) * h| \leq \int_{K_{\eta}(y_j)} |h(x-y) - h(x-y_j)| |f\varphi_j|(y) dy =: I.$$

Next we observe that the mean value theorem implies for fixed  $x, y, y_j$ :

$$|h(x-y) - h(x-y_j)| \leq \eta \cdot |\text{grad } h(\xi)|,$$

with  $\xi$  between  $x-y$  and  $x-y_j$ ; hence  $\xi \in K_{\eta}(x-y)$ , and we may continue the estimate by

$$I \leq \int_{K_{\eta}(y_j)} \eta \cdot |\text{grad } h|^*(x-y) |f\varphi_j|(y) dy \leq \eta \cdot |f\varphi_j|^* (\text{grad } h)^*(x).$$

By summation over  $j \in J$  we obtain from this the pointwise estimate

$$(4.21) \quad |(f - D_{\Phi}f) * h| \leq \eta \cdot (|f|^* (\text{grad } h)^*).$$

Since  $\text{spec } (f - D_{\Phi}f) * h \subseteq \text{spec } h = \Xi$ , the  $CB$ -norm is equivalent to its  $B$ -norm by Theorem 2.1(iii) for these functions. Using (B3') we obtain as in (4.4) (cf. Remark 4.1):

$$(4.22) \quad \begin{aligned} \|(f - D_{\Phi}f) * h\|_{CB} &\leq \|(f - D_{\Phi}f) * h\|_B \leq \eta \| |f|^* |\text{grad } h|^* \|_B \\ &\leq \eta \cdot C_B \|f\|_B \| |\text{grad } h|^* \|_{1,a} \leq \eta \cdot C_B C_{\Xi}^2 \cdot \|h\|_{1,a} \cdot \|f\|_B. \end{aligned}$$

Thus  $C_{\Xi}^3 := C_B \cdot C_{\Xi}^2$  is an appropriate choice.  $\square$

*Proof of Theorem 3.2.* By the theorem of Wiener-Levy (cf. [Rei, Chaps. 1, 6.5]) there exists a band-limited function  $g_1 \in L_a^1$  such that  $\hat{g}_1(t) = 1/\hat{g}(t)$  on  $\Omega$ . Next we choose a band-limited function  $h \in \mathcal{S}'(\mathbb{R}^m) \subseteq L_a^1$  with  $\hat{h}(t) = 1$  on  $\text{spec } g \cup \text{spec } g_1$ . Then

$$(4.23) \quad \begin{aligned} g * h &= g, & g_1 * h &= g_1, \\ f * h &= f * g_1 * g = f \quad \text{for all } f \in B^{\Omega}. \end{aligned}$$

We want to show that in the identity  $f = (f * g_1) * g$  the factor  $f * g_1$  can be replaced by a discrete measure. To this end we start an iteration procedure similar to that of Theorem 3.1, but with a different approximation of the convolution  $f \mapsto f * h$ . We define

$$(4.24) \quad f_0 := f * g_1, \quad f_{k+1} := (f_k - D_{\Phi}f_k) * h,$$

with  $D_{\Phi}: B \rightarrow MB$  as in Lemma 4.2. It follows by induction that

$$(4.25) \quad f = f_{k+1} * g + \left( \sum_{n=0}^k D_{\Phi}f_n \right) * g.$$

Substituting (4.24) and (4.23) into (4.25), we obtain for  $k = 0$

$$f_1 * g + (D_\Phi f_0) * g = f * g_1 * h * g - D_\Phi(f * g_1) * h * g + D_\Phi(f * g_1) * g = f.$$

The step from  $k$  to  $k + 1$  is also clear, if we again use  $g * h = g$ . If we now choose  $\eta = \eta(\Omega, g) < 1 / (C_\Xi^3 \cdot \|h\|_{1,a})$ , or  $\gamma := \eta \cdot C_\Xi^3 \|h\|_{1,a} < 1$ , then Lemma 4.2 implies for any  $\eta$ -PU  $\Phi$

$$(4.26) \quad \|f_{k+1}\|_B \leq \gamma \cdot \|f_k\|_B \leq \gamma^{k+1} \|f_0\|_B \quad \text{for } k \geq 0.$$

Taking the limit as  $k \rightarrow \infty$  in (4.25) the series representation for  $f$  follows:

$$(4.27) \quad f = \left( \sum_{n=0}^{\infty} D_\Phi(f_n) \right) * g = D_\Phi \left( \sum_{n=0}^{\infty} f_n \right) * g.$$

The interchange of brackets is justified by the fact that the series  $F := \sum_{n=0}^{\infty} f_n$  is absolutely convergent in  $B$ , by the continuity of  $D_\Phi$  from  $B$  into  $MB$ , and by the continuity of convolution by  $g \in C_a^1$  (Theorem 2.1(ii)). Thus

$$(4.28) \quad f = (D_\Phi F) * g = \sum_{j \in J} \langle \varphi_j, F \rangle L_{y_j} g$$

yields the desired expansion of  $f$  in terms of translates of  $g$ . The continuous dependence of the measure  $D_\Phi F$  follows from Proposition 2.3(i) and (4.26):

$$(4.29) \quad \begin{aligned} \|D_\Phi F\|_{MB} &\leq C_D \|F\|_{MB} \leq C_D \|F\|_B \leq C_D \cdot \sum_{k=0}^{\infty} \|f_k\|_B \\ &\leq C_D \left( \sum_{k=0}^{\infty} \gamma^k \right) \|f_0\|_B \leq C_D C_B (1 - \gamma)^{-1} \|g_1\|_{1,a} \|f\|_B. \end{aligned}$$

Thus the operator  $\mathbf{D}$  is given as  $\mathbf{D}(f) := D_\Phi F$ .  $\square$

*Remark 4.3.* If  $\hat{g}(t) \equiv 1$  on  $\Omega$ , the auxiliary function  $g_1$  is not necessary and the reconstruction of  $f$  is much simpler. The iteration is then  $f_0 = f$

$$(4.30) \quad f_{k+1} = f_k * h - (D_\Phi f_k) * h$$

where  $\hat{h}(t) \equiv 1$  on  $\text{spec } g$ . The rest of the argument is the same.

*Proof of Corollary B.* It is no loss of generality to assume that  $X$  is well spread (by selecting a  $\delta$ -dense and separated subfamily, and setting the coefficient equal to zero for the omitted points). Thus all we have to show is that in (4.29)  $\|D_\Psi F\|_{ML_a^p}$  is equivalent to  $(\sum_{i \in J} |\langle \psi_i, F \rangle|^p \nu(x_i))^{1/p}$ . This follows from Remark 2.4.  $\square$

For the proof of Theorem 3.3 we have to use a different discretization operator (only valid on  $B^\Omega$  but not defined on all of  $B$ ), which will take the same role as  $D_\Phi$  in the proof of Theorem 3.2. For  $f \in CB$  we set

$$(4.31) \quad D_\Psi^+(f) := \sum_{i \in I} \left( \int \psi_i(y) dy \right) f(x_i) \delta_{x_i}.$$

These operators  $D_\Psi^+$  combines the features of  $Sp_\Psi$  and  $D_\Psi$ . Since  $D_\Psi^+$  maps  $CB$  into  $MB_X$  and uses sampling values of  $f \in CB$  it can be used as an approximation operator in both Theorems 3.1 and 3.2. Lemma 4.3 provides the necessary estimate.

**LEMMA 4.3.** *Let  $(B, \|\cdot\|_B)$  satisfy (B1)–(B3) (for some  $a > 0$ ). Then there exists  $C_{D^+} > 0$  (depending only on  $a$ ) such that for any  $\delta$ -PU  $\Psi$ ,*

$$(4.32) \quad \|D_\Psi(f) - D_\Psi^+(f)\|_{MB} \leq \delta \cdot C_{D^+} \|f\|_B \quad \text{for all } f \in B^\Omega.$$

*In particular, for any  $\varepsilon > 0$  there exists  $\delta_0 > 0$  such that*

$$(4.33) \quad \|(D_\Psi^+ f) * h - f * h\|_{CB} \leq \varepsilon \cdot \|f\|_B$$

*for all  $f \in B^\Omega$  and all  $\delta$ -PU  $\Psi$  with  $\delta \leq \delta_0$ .*

In the proof we shall use the following a priori estimate.

*Remark 4.4.* Given complex-valued sequences  $(c_i)_{i \in I}$  and  $(d_i)_{i \in I}$  satisfying  $|c_i| \leq d_i$  for all  $i \in I$  we have

$$(4.34) \quad \mu = \sum_{i \in I} d_i \delta_{x_i} \in MB \text{ implies } \nu := \sum_{i \in I} c_i \delta_{x_i} \in MB \text{ and } \|\nu\|_{MB} \leq \|\mu\|_{MB}.$$

*Proof of Lemma 4.3.* The mean-value theorem implies for  $i \in I$ :

$$(4.35) \quad \int |f(y) - f(x_i)| \psi_i(y) dy \leq \delta \cdot \int |\text{grad } f|^*(y) \psi_i(y) dy.$$

After summation over  $i \in I$  Remark 4.4 implies

$$(4.36) \quad \begin{aligned} \|D_\Psi f - D_\Psi^+ f\|_{MB} &\leq \left\| \sum_{i \in I} \left( \int |f(y) - f(x_i)| \psi_i(y) dy \right) \delta_{x_i} \right\|_{MB} \\ &\leq \delta \cdot \|D_\Psi |\text{grad } f|^*\|_{MB}. \end{aligned}$$

Applying Proposition 2.3(i) and Remark 4.1 we obtain

$$(4.37) \quad \begin{aligned} \|D_\Psi |\text{grad } f|^*\|_{MB} &\leq C_D \cdot \| |\text{grad } f|^* \|_{MB} \leq C_D \| |\text{grad } f|^* \|_B \\ &= C_D \| \text{grad } f \|_{CB} \leq (C_D C_0 \|u\|_{C_a^1}) \|f\|_B. \end{aligned}$$

Thus  $\|D_\Psi f - D_\Psi^+ f\|_{MB} \leq \delta \cdot C_3 \|f\|_B$  for all  $f \in B^\Omega$ . Combining this fact with Lemma 4.2, the proof is complete if we choose  $\delta_0 \leq \varepsilon \cdot \min(\frac{1}{2}, \frac{1}{2} C_3^2)$ .  $\square$

*Proof of Theorem 3.3.* We refer to the proof of Theorem 3.2 for the iterative procedure, indicating only that  $D_\Phi$  has to be replaced by  $D_\Psi^+$ . Furthermore,  $g_1$  may be chosen to be identical to  $g$  (cf. Remark 4.3). By using Lemma 4.3 instead of Lemma 4.2 geometric convergence of the sequence  $(f_n)_{n=0}^\infty$  can be verified for sufficiently small  $\delta$ . In analogy to (4.27) we obtain

$$(4.38) \quad f = D_\Psi^+ \left( \sum_{n=0}^\infty f_n \right) * g.$$

*Proof of Theorem 3.4.* We use the operator  $\mathbf{B}$  from Theorem 3.1 in order to define

$$(4.39) \quad e_i := \mathbf{B}(\psi_i).$$

Since  $\psi_i \in L_a^1$ , Theorem 3.1 implies  $e_i \in L_a^1(\mathbb{R}^n)$  and  $\text{spec } e_i \subseteq \Xi$ , thus  $e_i \in C_a^1$  for  $i \in I$  by Theorem 2.1(iii). If  $\mathcal{H}(\mathbb{R}^m)$  is dense in  $B$ , then obviously

$$\lim_{n \rightarrow \infty} \sum_{i \in F_n} f(x_i) \psi_i = \sum_{i \in I} f(x_i) \psi_i$$

in  $B$  for any increasing sequence of finite subsets  $F_n \subseteq I$ , exhausting  $I$ . Therefore by the boundedness of  $\mathbf{B}$

$$(4.40) \quad \begin{aligned} f &= \mathbf{B}(Sp_\Psi f) = \mathbf{B} \left( \lim_{n \rightarrow \infty} \sum_{i \in F_n} f(x_i) \psi_i \right) \\ &= \lim_{n \rightarrow \infty} \sum_{i \in F_n} f(x_i) e_i = \sum_{i \in I} f(x_i) e_i, \end{aligned}$$

i.e., the series converges in  $B$ . Since the family  $(e_i)_{i \in I}$  has joint spectrum we may apply Theorem 2.1(iii) to derive convergence in  $CB$ , hence uniform convergence over compact sets in  $\mathbb{R}^m$ .  $\square$

The direct method of obtaining suitable coefficients from the sampling values as described in Theorem 3.5 requires the following technical lemma.

**LEMMA 4.4** (From sampling values to coefficients). *Given  $\Xi \subseteq \mathbb{R}^m$ , compact and  $\rho > 0$  there exist  $\delta = \delta(\Xi)$  and  $\gamma = \gamma(\Xi, h) > 0$  such that*

$$(4.41) \quad \|f * h - (D_\Phi Sp_\Psi f) * h\|_B \leq \rho \cdot \|f\|_B \text{ for all } f \in B^\Xi$$

and uniformly for all  $\delta$ -PUS  $\Phi$  and all  $\gamma$ -PUS  $\Phi$ .

*Proof.* We combine Lemmas 4.1 and 4.2 and obtain for  $f \in B^{\Xi}$

$$\begin{aligned} & \|f * h - (D_{\Phi} S p_{\Psi} f) * h\|_B \\ & \leq \| (f - D_{\Phi} f) * h \|_B + \| D_{\Phi} (f - S p_{\Psi} f) * h \|_B \quad (\text{by Lemma 4.2}) \\ & \leq \gamma C_{\Xi}^3 \|h\|_{1,a} \|f\|_B + C_0 \|D_{\Phi} (f - S p_{\Psi} f)\|_{MB} \cdot \|h\|_{C_a^1} \quad (\text{by Theorem 2.1(ii)}) \\ & \leq \gamma C_{\Xi}^3 \|h\|_{1,a} \|f\|_B + C_D C_0 \|f - S p_{\Psi} f\|_B \|h\|_{C_a^1} \\ & \hspace{15em} (\text{by Proposition 2.3 and Lemma 4.1}) \\ & \leq (\gamma C_{\Xi}^3 \|h\|_{1,a} + C_D C_0 C_{\Xi}^1 \delta \|h\|_{C_a^1}) \cdot \|f\|_B. \end{aligned}$$

Now choose  $\gamma, \delta$  small enough so that the coefficient of  $\|f\|_B$  is less than  $\rho$ .  $\square$

*Proof of Theorem 3.5.* The proof is similar to those of Theorems 3.1. and 3.2. We choose  $h \in L_a^1$  band-limited with  $\hat{h}(t) = 1$  on  $\text{spec } g$  and define

$$(4.42) \quad f_0 := f, \quad f_{k+1} := f_k * h - (D_{\Phi} S p_{\Psi} f_k) * h.$$

Then  $f_k \in B^{\Xi}$  for all  $k$ , and if  $\delta$  and  $\gamma$  are small enough, by Lemma 4.4

$$(4.43) \quad \|f_{k+1}\|_B \leq \rho \cdot \|f_k\|_B$$

with  $\rho < 1$ . Since  $f = f * g = f * h * g$  and  $g = g * h$  we have for  $n \geq 0$

$$(4.44) \quad f = f * g = f_{n+1} * g + \sum_{k=0}^n D_{\Phi} (S p_{\Psi} f_k) * g.$$

Since  $\rho < 1$  it follows

$$(4.45) \quad f = \left( \sum_{k=0}^{\infty} D_{\Phi} (S p_{\Psi} f_k) \right) * g.$$

Since the series of discrete measures  $\sum_{k=0}^{\infty} D_{\Phi} (S p_{\Psi} f_k) = D_{\Phi} (\sum_{k=0}^{\infty} S p_{\Psi} f_k)$  is supported on  $Y = (y_j)_{j \in J}$  and absolutely convergent in  $MB$ , the result can be written as  $\sum_{j \in J} c_j \delta_{y_j}$ , with  $c_j := \langle \varphi_j, \sum_{k=0}^{\infty} S p_{\Psi} f_k \rangle$ . This sum is unconditionally convergent in  $MB$ , and norm convergent, if  $\mathcal{H}(\mathbb{R}^m)$  is dense in  $B$ . It follows therefore that  $f = \sum_{j \in J} c_j L_{y_j} g$ , as was required.

As in (4.29) we verify that the coefficient mapping (it can be described by an infinite matrix)  $M: f \mapsto C = (c_j)_{j \in J}$ , is continuous. Finally, we show as in the proof of Theorem 3.1 that the  $f_k$ 's and therefore  $(c_j)_{j \in J}$  depend only on the sampling values  $(f(x_i))_{i \in I}$ .  $\square$

Corollary C follows from Theorem 3.3 in the same way that Corollary B follows from Theorem 3.2.

**Note added in proof.** Since the submission of this manuscript we have obtained further results on the algorithms of Theorems 3.1-3.3: (a) [FG4] contains a detailed error analysis of these algorithms and shows their stability, again in the general function space setting. (b) The results of numerical experiments have been very convincing; see [FGH], [FCH], [FCS]. Particularly, the adaptive weights method described in Theorem 3.5 using the  $D_{\Psi}^+$ -operator has turned to be simple and extremely effective.

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