Non-Orthogonal Wavelet and Gabor Expansions, and Group Representations

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In this article we explain the construction of nonorthogonal series expansions with respect to basis functions which are simple transforms of a single function. In mathematics such expansions are often called atomic decompositions, while in physics they are series expansions with respect to discrete sets of coherent states [22]. This means that all basis functions are derived from a single function by elementary operations such as shifts, modulations, scaling or rotations. An expansion of this type had been proposed very early for applications in the theory of communication and signal analysis [14]. In the last decade, after such series expansions had been obtained for spaces of analytic functions [1] and for Besov spaces [11], it became clear that these atomic decompositions must be part of a more general phenomenon. Inspired by the work of A. Grossmann and the Marseille group [17], such a general theory was found in the series of papers [8, 9, 10, 16] by the authors.

This article is intended to be a counterpart to the abstract theory of [9, 10], which for reasons of length do not contain any examples. Here we will focus on the applications of the general theory, in particular on the

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construction of non-orthogonal wavelet and Gabor-type expansions.

In the excitement and enthusiasm about orthogonal wavelets other representations of functions have been somewhat neglected, but nonorthogonal expansions and frames also have their merits:

(1) Orthogonal wavelets are rather complicated functions, whereas any "nice" function can serve as a basic wavelet for an non-orthogonal expansion.

(2) Instead of translations and dilations other operations can be used to obtain the expanding functions from a "basic wavelet". In some cases of interest, appropriate orthogonal bases of coherent states do not even exist; and then it is natural to look at non-orthogonal expansions of the same structure.

(3) Since no lattice structure is required, one obtains greater stability. As an additional feature one can prove irregular sampling theorems for the continuous wavelet transform and the short time Fourier transform.

The article is organized as follows. Section 1 contains some generalities on non-orthogonal expansions and frames. Section 2 – the main body of the article – presents various examples of non-orthogonal expansions and some applications. All expansions mentioned, in particular wavelet and Gabor-type expansions, are special cases of the general theorem 6.1 in [9]. In Section 3 – 5 we explain the ideas underlying the construction of non-orthogonal expansions with respect to coherent states and give a partial proof for the statements in Section 2. In this part we follow [16], which is devoted to frames for Banach spaces and simplifies the original construction of non-orthogonal expansions. Section 6 treats the computational aspects and states the general theorem.

1. GENERALITIES

1.1. A sequence of functions $\{e_i, i \in I\}$ in a Hilbert space $\mathcal{H}$ gives rise to nonorthogonal expansions, if every function $f \in \mathcal{H}$ has a series expansion

$$f = \sum_{i \in I} c_i e_i \,, \tag{1}$$

such that the coefficient sequence c depends linearly on f and satisfies the norm equivalence

$$A_1 \|f\|_{\mathcal{H}}^2 \leq \sum_i |c_i|^2 \leq A_2 \|f\|_{\mathcal{H}}^2 \tag{2}$$

for some constants $0 < A_1 \leq A_2$ (or in short $\|f\|_{\mathcal{H}} \cong \|c\|_2$) .

Convergence of a sum $\sum_{i\in I}\dots$ over a general index set I is understood as unconditional convergence, i.e., any sequence of partial sums converges and the limit is independent of the summation procedure.

In general the basis functions are neither orthogonal nor linearly independent, therefore f may be representable with other coefficient sequences. This non-uniqueness of the coefficients in the absence of linear independence is not a serious problem, because from (1) and (2) there is a canonical way to construct the coefficients c_i from the inner products $\langle e_i, f\rangle$, cf. 1.3.

1.2. If $g_i \in \mathcal{H}$ denotes the coefficient functional $f \mapsto c_i = \langle g_i, f\rangle$, then from the equality $\langle h, f\rangle = \langle(\sum_i\langle g_i, h\rangle e_i), f\rangle = \langle h, \sum_i\langle e_i, f\rangle g_i\rangle$ for all $h \in \mathcal{H}$, one obtains $f = \sum_i\langle e_i, f\rangle g_i$ with the norm equivalence

$$A_2^{-1}\|f\|_{\mathcal{H}}^2 \leq \sum_i |\langle e_i, f\rangle|^2 \leq A_1^{-1}\|f\|_{\mathcal{H}}^2 \tag{3}$$

If (3) holds, then the set of function $\{e_i, i \in I\}$ is called a *frame* for $\mathcal{H}$. By (2) the coefficient functionals g_i are also a frame.

CONSTRUCTION OF THE COEFFICIENTS c_i OR RECONSTRUCTION OF f FROM $\langle e_i, f\rangle$

Clearly, f is uniquely determined by the frame coefficients $\langle e_i, f\rangle$. It is a remarkable fact that from the norm equivalence (3), one can derive a method for the reconstruction of f from the frame coefficients and also of the coefficients in the series expansion (1), cf. [5]. For this one observes that by (3), the operator

$$Sf = \sum_i \langle e_i, f\rangle e_i, \tag{4}$$

the so-called *frame operator*, is positive, bounded below and above by A_1^{-1} and A_2^{-1}. Consequently, S is invertible on $\mathcal{H}$ and

$$f = S^{-1}Sf = \sum_i\langle e_i, f\rangle S^{-1}e_i = SS^{-1}f = \sum_i\langle S^{-1}e_i, f\rangle e_i. \tag{5}$$

Since S^{-1} has a Neumann series $S^{-1} = \alpha\sum_{n=0}^{\infty}(1-\alpha S)^n, \alpha = 2/(A_1^{-1} + A_2^{-1})$, this can easily be written as an iterative algorithm for the construction of the coefficients c_i or the reconstruction of f, cf. Section 6 or [5, 3].

1.3. In our context $\mathcal{H}$ is a Hilbert space of square-integrable functions $L^2(S,\mu)$ and the basis functions e_i are coherent states, i.e., they are transforms of one function g by simple operations. In many applications it is desirable to recognize finer details of functions from their series expansions. This leads to the questions, which Banach spaces can be characterized by means of their expansions with respect to coherent states or by the size of their frame coefficients. If such a description is possible, how can the coefficients be computed or how can the function be reconstructed from its frame coefficients. The question of frames for Banach spaces is more complicated since the argument of 1.3 breaks down in this case. The general answer to these questions has been obtained in [9, 10, 16].

As a practical consequence of the extension to Banach spaces one obtains statements about the quality of the convergence of the reconstruction in 1.3.

2. SOME EXAMPLES OF NONORTHOGONAL EXPANSIONS

This section is a collection of results on non-orthogonal series expansions from different branches of mathematics. All decomposition theorems can be obtained as special cases of a single theorem. Most statements also have direct, "hard analysis" proofs, but in most examples the main theorem in Section 6 provides more detailed information on the class of admissible wavelets, irregular expansions or stability.

NONORTHOGONAL WAVELET EXPANSIONS

Recall first that the moments of a multivariate function f are defined by $\int_{R^n} x_1^{k_1} x_2^{k_2} \dots x_n^{k_n} f(x)\, dx$, where $x = (x_1, x_2, \dots, x_n) \in R^n$ and the $k_i \geq 0$ are integers.

Let $g \in L^2(R^n)$ be a function (i) with enough smoothness and a sufficient number of vanishing moments, and (ii) such that $|\hat{g}(\xi)| \geq c > 0$ for $a \leq |\xi| \leq b$, some $a, b, c > 0$, e.g. g is a Schwartz function with some vanishing moments , or supp $\hat{g} \subseteq \{0 < r_0 \leq |x| \leq R_0 < \infty\}$, or g has compact support and vanishing moments, etc. (cf. Section 6). Then there exist $\alpha_0 > 0, \beta_0 > 1$, the size depending only on g, with the property that for $0 < \alpha \leq \alpha_0, 1 < \beta \leq \beta_0$, every $f \in L^2(R^n)$ has a nonorthogonal wavelet expansion

$$f = \sum_{j \in Z, k \in Z^n} c_{jk} \beta^{jn/2} g(\beta^j x - \alpha k) \tag{6}$$

with convergence in $L^2(R^n)$. The coefficients c_{jk} can be constructed as in Section 1.3 and satisfy $\|f\|_2 \cong (\sum_{j \in Z, k \in Z^n} |c_{jk}|^2)^{1/2}$.

Condition (i) guarantees the convergence of the wavelet expansion (6), while condition (ii) implies that the basis functions span $L^2(R^n)$.

WAVELET FRAMES

By duality, the functions $g_{jk}(x) = \beta^{jn/2} g(\beta^j x - \alpha k), j \in Z, k \in Z^n$ constitute a frame for $L^2(R^n)$: thus $f \in L^2(R^n)$ can be completely reconstructed from the frame coefficients $\langle g_{jk}, f\rangle = \beta^{jn/2} \int \bar{g}(\beta^j y - \alpha k) f(y) dy$ and

$$A_1 \|f\|_2 \leq \Big(\sum_{j\in Z, k\in Z^n} |\langle g_{jk}, f\rangle|^2\Big)^{1/2} \leq A_2 \|f\|_2 \quad . \tag{7}$$

The frame coefficients $\langle g_{jk}, f\rangle$ are a regular sampling of the *continuous wavelet transform*

$$W_g(f)(x,t) = \int_{R^n} t^{-n/2} \bar{g}\Big(\frac{y-x}{t}\Big) f(y) dy \tag{8}$$

at the points $(x,t) = (\alpha k \beta^{-j}, \beta^{-j})$. Thus (7) can also be interpreted as a sampling theorem for the wavelet transform analogous to the sampling theorem for band-limited functions.

IRREGULAR SAMPLING AND EXPANSIONS

Similar results are true for irregular sampling sequences $X = (x_i, y_i)_{i\in I}$ in $R^n \times R^+$. Let $Q_\alpha(x)$ denote the cube of side length α with center at $x \in R^n$. If g, α, β are given as in (6) and if X is any set in $R^n \times R^+$ that satisfies (i) the density condition $\bigcup_i (Q_{y_i \alpha}(x_i) \times [y_i \beta^{-1/2}, y_i \beta^{1/2}]) = R^n \times R^+$, and (ii) such that this covering is of finite height, then the collection of functions

$$y_i^{-n/2} g(y_i^{-1}(x - x_i)), i \in I, \quad \text{is a frame for } L^2(R^n)$$

Every $f \in L^2(R^n)$ has an irregular non-orthogonal wavelet expansion

$$f(x) = \sum_{i\in I} c_i y_i^{-n/2} g(y_i^{-1}(x - x_i)) \tag{9}$$

with $\|f\|_2 \cong \|c\|_2$.

DIRECTIONAL SENSITIVITY

In multidimensional image processing, it is desirable to obtain information about the directional behavior of the frequencies of an image. This can be achieved by a variation of (6), which also includes rotations.

Given any $g \in L^2(R^n)$ with sufficient smoothness and a sufficient number of vanishing moments, there exist $\alpha > 0, \beta > 1$ and a finite number of orthogonal matrices $O_l \in \mathcal{O}(n), l = 1, \ldots, r$ such that the collection

$$g_{jkl}(x) = \beta^{-jn/2} g(O_l^{-1}(\beta^{-j} x - \alpha k)) \, , j \in Z, k \in Z^n, l = 1, \ldots, r$$

is a frame for $L^2(R^n)$. By duality every $f \in L^2(R^n)$ admits a nonorthogonal expansion

$$f = \sum_{j,k,l} c_{jkl} \beta^{-jn/2} g(O_l^{-1}(\beta^{-j}x - \alpha k)) \tag{10}$$

with $\|f\|_2 \cong \|c\|_2$.

In order to obtain directional sensitivity, one chooses a basic wavelet g with supp $\hat{g} \subseteq C_{v,\alpha}$, where $C_{v,\alpha} = \{x \in R^n : |x|^{-1}\langle x, v\rangle > \cos\alpha\}$ is the cone with its axis along the unit vector $v \in R^n, |v| = 1$, and angle α. Then for each l the function $f_l = \sum_{j\in Z, k\in Z^n} c_{jkl} g_{jkl}$ has spectrum supp$\hat{f}_l \subseteq C_{O_l^{-1}v,\alpha}$, and f_l contains the essential information about the frequency content of f in the direction of $O_l^{-1}v$.

CHARACTERIZATIONS OF THE CLASSICAL FUNCTION SPACES

Under some additional conditions on g, $g \in S_0$, say, functions and distributions outside $L^2(R^n)$ also have nonorthogonal wavelet expansions of the form (6) and (9). As an illustration we consider the homogeneous Besov spaces

$$\dot{B}^s_{pq}(R^n) = \Big\{ f \in S_0' : \|f\|^q_{B^s_{pq}} = \int \left(\int_{R^n} |W_g(f)(x,t)|^p dx \right)^{q/p} t^{-qs-nq/2} dt/t < \infty \Big\} \tag{11}$$

for $1 \leq p, q < \infty, s \in R$. For this family of Banach spaces the decay of the wavelet transform serves as a measure for the smoothness of a function or distribution. In particular, if $p = q = 2$ and if $s \geq 0$ is an integer, then the norm $\|f\|_{B^s_{pq}}$ is equivalent to the Sobolev norm $\sum_{s_1+s_2+\ldots+s_n=s} \|\partial^s f/\partial x_1^{s_1} \partial x_2^{s_2} \partial x_n^{s_n}\|_2$. In other words, the Besov spaces fill in the gaps between the ordinary Sobolev spaces.

Functions in Besov spaces admit wavelet expansions (6) and (9) and are characterized by expansions

$$f \in \dot{B}^s_{pq} \Longleftrightarrow \left(\sum_{j\in Z} \Big(\sum_{k \in Z^n} |c_{jk}|^p \Big)^{q/p} \beta^{-jq(s+n/2-n/p)} \right)^{1/q} < \infty \tag{12}$$

Both norms in (12) are equivalent and (6) converges in $\dot{B}^s_{pq}$.

Similarly $f \in \dot{B}^s_{pq}$ can be recognized by the size of the frame coefficients

$\langle g_{jk}, f\rangle$ as follows: there are two constants $A_1, A_2 > 0$ such that

$$A_1\|f\|_{B^s_{pq}} \leq \left(\sum_{k\in Z^n} |\langle g_{jk}, f\rangle|^p)^{q/p}\beta^{-jq(s+n/2-n/p)}\right)^{1/q} \leq A_2\|f\|_{B^s_{pq}}$$

Writing $Q_{jk} = \prod_{r=1}^n[\beta^j\alpha k_r, \beta^j\alpha(k_r+1)], j \in Z, k = (k_1, \ldots, k_n) \in Z^n$,

$$f \in L^p(R^n) \Longleftrightarrow (\int_{R^n}(\sum_{j,k}|c_{jk}|^2\beta^{-jq(s+n/2)}\chi_{Q_{jk}}(x))^{p/2}dx)^{1/p} < \infty$$

and (6) converges in L^p.

Likewise all Besov-Triebel-Lizorkin (= BTL-) spaces can be characterized by the size of the coefficients in the nonorthogonal wavelet expansions (compare [13, 16]).

All characterizations are of course true for orthogonal wavelet bases as well. These are even unconditional bases for the BTL-spaces [20, 15]. Despite the power and elegance of orthogonal wavelet expansions, in particular for fast numerical computations, non-orthogonal expansions might be preferable in certain applications, because *any* reasonable function may serve as a basic wavelet. On the other hand orthogonal wavelets are rather complicated functions and require a delicate construction, see e.g. [21].

REFERENCES

All results mentioned occur as special cases of [9], Theorem 6.1. and [16], Theorem T. An extensive study of non-orthogonal wavelets and some applications has recently appeared in the impressive paper [13], see also the preceding versions in [11, 12]. The case $L^2(R^n)$ is treated in [2, 4].

GABOR TYPE EXPANSIONS

If $g \in L^2(R^n)$ satisfies $\int_{R^n}|g(x)|^2(1+|x|)^{2+\epsilon}dx < \infty$ and $\int_{R^n}|\hat{g}(\xi)|^2(1+|\xi|)^{2+\epsilon}d\xi < \infty$ for some $\epsilon > 0$, then there are $\alpha_0 > 0, \beta_0 > 0$ with the property that for any $0 < \alpha \leq \alpha_0, 0 < \beta \leq \beta_0$ every function $f \in L^2(R^n)$ has a Gabor-type expansion

$$f = \sum_{k,m\in Z^n} c_{km}e^{2\pi i\alpha kx}g(x-\beta m) \tag{13}$$

The series converges in $L^2(R^n)$ and coefficients satisfy
$\|f\|^2 \cong (\sum_{k,m}|c_{km}|^2)^{1/2}$.

By duality, the set of functions $\{g_{km} = e^{2\pi i\alpha kx}g(x-\beta m), k, m \in Z^n\}$ is a frame for $L^2(R^n)$, i.e.

$$A_1\|f\|^2 \le (\sum_{k,m} |\langle g_{km}, f\rangle|^2)^{1/2} \le A_2\|f\|_2 \tag{14}$$

In this case the frame coefficients $\langle g_{km}, f\rangle$ are a regular sampling of the *short time Fourier transform* with window function g:

$$S_g(f)(x,y) = \int_{R^n} e^{-2\pi i y\xi}\bar{g}(\xi - x)f(\xi)d\xi \tag{15}$$

at the points $(\beta m, \alpha k)$. The equivalence of norms (14) implies that the short time Fourier transform $S_g(f)$ and f can be completely reconstructed from these sampled values by the method of 1.3.

D. Gabor [14] has proposed such an expansion for $g(x) = e^{-\pi x^2}, \alpha\beta = 1$ in order to decompose a signal into components with optimal concentration in the time-frequency plane. In this case, however, (14) is violated, and the expansions (13) are extremely unstable. It can be shown that stable expansions require "oversampling" $\alpha\beta < 1$ [2], but sharp bounds on the required density are still unknown.

IRREGULAR SAMPLINGS AND EXPANSIONS

Let $(x_i, y_i)_{i\in I}$ be any discrete set in $R^n \times R^n$ such that $\bigcup_i (Q_\alpha(x_i) \times Q_\beta(y_i)) = R^n \times R^n$, where $Q_\alpha(x)$ is the cube of side length α centered at $x \in R^n$, and g, α, β are as above. Then the set of functions

$$e^{2\pi i y_i\xi} g(\xi - x_i), i \in I, \quad \text{constitutes a frame for } L^2(R^n)$$

Any $f \in L^2(R^n)$ has a jittered Gabor-type expansion

$$f = \sum_i c_i e^{2\pi i y_i \xi} g(\xi - x_i) \tag{16}$$

with convergence in $L^2(R^n)$ and $\|f\|_2 \cong \|c\|_2$.

GABOR EXPANSIONS AND THE BEHAVIOR OF FUNCTIONS IN THE TIME-FREQUENCY PLANE

The short time Fourier transform can be viewed as a representation of a signal in the time-frequency plane. The behavior of signals simultaneously in time and frequency is best studied in the frame-work of the *modulation spaces* $M^s_{pq}(R^n)$. Let $g \in \mathcal{S}$ be fixed, $1 \le p, q \le \infty$, $s \in R$, then $M^s_{pq}(R^n)$ is defined as

$$M^s_{pq}(R^n) = \left\{ f \in \mathcal{S}' : \|f\|^q_{pqs} \right.$$

$$:= \int \left(\int |S_g(f)(x,y)|^p dx \right)^{q/p} (1+|y|)^{sq} dy < \infty \Big\}$$

with the obvious modifications for $p, q = \infty$. In particular $L^2(R^n) = M^0_{22}$ and the Bessel potential spaces $H^s = \{f \in S' : \|f\|^2_{H^s} = \int |\hat{f}(\xi)|^2(1 + |\xi|^{2s})d\xi < \infty\} = M^s_{22}$.

The space $S_0(R^n) := M^0_{11}$ has some interesting properties: it is an algebra with respect to pointwise multiplication and convolution, invariant under modulation and under Fourier transform. It is a natural domain for Poisson's formula and can often be used as a substitute for the more complicated Schwartz space, e.g. in generalized Fourier analysis [6].

For appropriate g, say $g \in \mathcal{S}$ and $\alpha, \beta > 0$ small enough, depending only on g and $|s|$, every function $f \in M^s_{pq}$ has a Gabor expansion (13) or (16), which converges in the norm of M^s_{pq} for $p, q < \infty$, and

$$f \in M^s_{pq} \iff (\sum_{k \in Z^n} (\sum_{m \in Z^n} |c_{km}|^p)^{q/p}(1+|k|)^{sq})^{1/q} < \infty \tag{18}$$

$$\iff \left(\sum_{k \in Z^n} (\sum_{m \in Z^n} |\langle g_{km}, f\rangle|^p)^{q/p}(1+|k|)^{sq} \right)^{1/q} < \infty \tag{19}$$

Both expressions are equivalent norms for $f \in M^s_{pq}$.

SYSTEM IDENTIFICATION

A linear time-invariant system T on $L^p(R^n)$ can be modeled by a convolution operator $T = T_\sigma$, where $T_\sigma f = \sigma * f$ is the response of T to the input signal f. It can be shown that σ is always a tempered distribution in $S_0'(R^n) = M^0_{\infty\infty}$. A complete identification would require testing T against all "Dirac pulses" δ_x or all pure modulations $e^{iy\xi}$ and is often impossible. Since the modulation of a signal and its sampling are often technically easier to realize than sharp "Dirac pulses", a more realistic approach to the identification of T proceeds as follows:

1. Test the system T against a collection of input signals which are complex modulates $M_{\alpha k}g(t) = e^{2\pi i \alpha k t}g(t)$ of a given smooth envelope g.

2. Then sample the output of T in each case. (The sampling does not have to be equally spaced, but may depend on the modulation αk, if necessary.)

$$M_{k\alpha}g \xrightarrow{T} \sigma * M_{k\alpha}g \xrightarrow{sampling} \sigma * M_{k\alpha}g(m\beta)$$

In other words, the data obtained from this procedure are the numbers $\sigma * M_{k\alpha} g(m\beta) = e^{2\pi i\alpha\beta km} S_g\ (f)(m\beta, k\alpha) = e^{2\pi i\alpha\beta km} \langle g_{km}, f\rangle$, i.e., nothing but the frame coefficients of σ (here $g\ (x) = \overline{g(-x)}$). Therefore, if $\alpha, \beta > 0$ are small enough, σ is completely determined by these data and can be completely reconstructed by a variation of the frame method.

REFERENCES

Modulation spaces and their atomic decompositions are studied in [7], the above results are a special case of [9], Theorem 6.1 and [16]. The L^2-case is treated in [2, 4], [23] contains a Zak transform approach to Gabor-type expansions of modulation spaces.

OTHER NONORTHOGONAL EXPANSIONS

A) THE BARGMANN-FOCK SPACES [9, 16, 19]

Let $\mathcal{F}^p = \{f$ entire on $C^n : \|f\|^p_{\mathcal{F}^p} = \int_{C^n} |f(z)|^p e^{-\pi p|z|^2/2} dz < \infty\}$ for $1 \leq p < \infty$. $\mathcal{F}^2$ is a reproducing kernel Hilbert space with kernel $e_w(z) = e^{\pi \bar{w} z}$, i.e., $f(w) = \langle e_w, f\rangle = \int_{C^n} e^{\pi w \bar{z}} f(z) e^{-\pi|z|^2} dz$.

These spaces are of interest in signal analysis because they arise essentially as spaces of short time Fourier transforms $S_g(f)$ with the fixed window $g(x) = e^{-\pi x^2}$. They also occur in the study of the canonical commutation relations in quantum mechanics.

For arbitrary $g \in \mathcal{F}^1$, there is a lattice density $\alpha, \beta > 0$ depending only on g, such that

$$f(z) = \sum_{k,m \in Z^n} c_{km} e^{\pi w_{km} z - \pi |w_{km}|^2/2} g(z - \bar{w}_{km}) \tag{20}$$

where $w_{km} = \beta m + i\alpha k \in C^n$ and the series converges in $\mathcal{F}^p$. Moreover, $\|f\|_{\mathcal{F}^p} \cong (\sum_{k,m \in Z^n} |c_{km}|^p)^{1/p}$. In particular, if $g(z) \equiv 1 \in \mathcal{F}^1$ is chosen, then (20) yields a stable expansion with respect to the reproducing kernel functions e_w .

The corresponding result for frames states that every $f \in \mathcal{F}^p$ can be completely reconstructed from the coefficients $\lambda_{km} = \int_{C^n} e^{\pi w_{km} \bar{z} - \pi |w_{km}|^2/2} \bar{g}(z - \bar{w}_{km}) f(z) e^{-\pi|z|^2} dz$ and $\|f\|_{\mathcal{F}^p} \cong \|\lambda\|_p$. If in particular $g \equiv 1$, then $\lambda_{km} = f(w_{km}) e^{-\pi|w_{km}|^2/2}$ and one obtains a well-known sampling theorem for entire functions $\|f\|_{\mathcal{F}^p} \cong (\sum_{k,m} |f(w_{km})|^p e^{-\pi p |w_{km}|^2/2})^{1/p}$.

Instead of the w_{km}, irregularly distributed sets as in (16) can be used in (20) and the sampling theorem.

B) GABOR-TYPE EXPANSIONS IN ABELIAN GROUPS

Let $\mathcal{G}$ be a separable locally compact Abelian group, $\hat{\mathcal{G}}$ its dual group, and g be a "nice" function on $\mathcal{G}$. For instance any $g \in L^1(\mathcal{G})$ with a decomposition $g = \sum_n g_n(x - x_n)$, where $x_n \in \mathcal{G}, \operatorname{supp} g_n \subseteq Q$, a fixed compact set in $\mathcal{G}$, and $\sum_n \|\hat{g}_n\|_1 < \infty$, qualifies. If the neighborhood $U \times V \subseteq \mathcal{G} \times \hat{\mathcal{G}}$ of the identity is small enough (depending only on g) and if $(x_m, \chi_k)_{k,m\in N} \subseteq \mathcal{G}\times\hat{\mathcal{G}}$ is a discrete set, such that $\bigcup_{k,m\in N}(x_n + U) \times (\chi_m \cdot V) = \mathcal{G} \times \hat{\mathcal{G}}$ then every $f \in L^2(\mathcal{G})$ has a nonorthogonal expansion

$$f(x) = \sum_{k,m\in Z} c_{km}\chi_m(x)g(x - x_k) \tag{21}$$

and $\|f\|_2 \cong (\sum_{k,m} |c_{km}|^2)^{1/2}$.

Also the collection $g_{km}(x) = \chi_m(x)g(x - x_k), k, m \in Z$ is a frame for $L^2(R^n)$.

For $\mathcal{G} = R^n$ one recovers the Gabor-type expansions of Section 2.2. It is easy to extend the theory of modulation spaces and their Gabor expansions to general locally compact Abelian groups.

In particular, for $\mathcal{G} = Z^n$ or $\mathcal{G} = Z_N$ (the cyclic group of order N) we obtain discrete models of Gabor expansions. They furnish a systematic procedure for the discretization and numerical implementation of the Gabor-type expansions (13).

C) ANISOTROPIC WAVELET EXPANSIONS

Let D be a $n \times n$ matrix such that each eigenvalue has strictly positive real part and set $\Delta(t) = |\det e^{-tD/2}|$. If $g \in L^2(R^n)$ satisfies $|\hat{g}(\xi)| \geq c > 0$ for $a \leq |\xi| \leq b$ and mild decay and moment conditions, then for $\alpha, \beta > 0$ small enough, every $f \in L^2(R^n)$ has an anisotropic nonorthogonal wavelet expansion

$$f(x) = \sum_{j\in Z, k\in Z^n} c_{jk}\Delta(\beta j)g(e^{-\beta jD}x - \alpha k) \tag{22}$$

with convergence in $L^2(R^n)$and $\|f\|_2 \cong \|c\|_2$.

The class of anisotropic Besov-Triebel-Lizorkin spaces can be characterized by this type of nonorthogonal expansions.

If the matrix $A = e^D$ leaves Z^n or some other lattice invariant, it is known that orthogonal wavelet expansions (22) exist. In the other cases non-orthogonal expansions are the only tool available.

D) BERGMAN SPACES ON THE UPPER HALF-PLANE [1]

Let $U = \{z = x + iy, y > 0\} \subseteq C$ by the upper half plane and $A^{p,m}, 1 \leq p < \infty, m \geq 3, m \in N$ be the Bergman space

$$A^{p,m} = \{f \text{ analytic on } U : \|f\|_{p,m}^p = \int\int |f(z)|^p y^{pm/2-2} dx dy < \infty\} \tag{23}$$

Then for $\alpha > 0$ and $\beta > 1$ small enough, every $f \in A^{p,m}$ has an expansion

$$f(z) = \sum_{j,k \in Z} c_{jk} \beta^{-2jm} (z - \alpha k + i\beta^{2j})^{-m} \tag{24}$$

with convergence in $A^{p,m}$ and coefficients satisfying $\|c\|_p \cong \|f\|_{p,m}^p$.

3. THE GENERAL STRUCTURE OF THESE EXPANSIONS

Despite the diversity of these expansions, they are but special cases of a single theorem. To gain some insight into the structure of these expansions, we observe their common structure.

3.1. *The basis functions are derived from a single function g by the action of a (square-)integrable unitary representation π of a (Lie) group $\mathcal{G}$ on a Hilbert space $\mathcal{H}$.*

3.2. *All occurring spaces are defined by the magnitude of representation coefficients $V_g(f)(x) = \langle \pi(x)g, f \rangle$, where g is a fixed test function.*

To any appropriate Banach space Y of functions on $\mathcal{G}$ is associated a Banach space of functions of distributions on the level of $\mathcal{H}$, the *coorbit of Y under the representation π* [9]

$$Co_\pi Y = \{f : \langle \pi(x)g, f \rangle \in Y\} \text{ with norm } \|f\|_{Co\,Y} = \|\langle \pi(x)g, f \rangle\|_Y \tag{25}$$

in particular, $Co_\pi L^2(\mathcal{G}) = \mathcal{H}$.

3.3. The nonorthogonal series expansions are of the form

$$f = \sum_i \lambda_i \pi(x_i) g \tag{26}$$

with convergence in $Co_\pi Y$, where x_i is a discrete, sufficiently dense set in $\mathcal{G}$. A $f \in Co_\pi Y$ can be completely characterized by the size of the set of coefficients λ_i, e.g. $f \in Co_\pi L^p(\mathcal{G}) \iff (c_i) \in l^p$.

3.4. Frames for $Co_\pi Y$: $\{\pi(x_i)g, i \in I\}$ being a frame means essentially that f is uniquely determined by the sequence $\langle \pi(x_i)g, f\rangle_{i\in I}$. In other words, the representation coefficient $x \to \langle \pi(x)g, f\rangle$ is completely determined by and can be reconstructed from its sampled values on the discrete set $\{x_i, i \in I\}$ in $\mathcal{G}$. Thus the construction of coherent frames amounts to proving a sampling theorem for representation coefficients. The analogy with the famous Shannon-Whittacker-Kotel'nikov sampling theorem for band-limited functions and its extensions has been very fruitful for understanding the theory of frames.

The examples of Section 2 are attached to the following groups and representations.

i) The group for wavelet expansions is $\mathcal{G} = R^n \times R^+$ ($ax + b$-group) with multiplication $(x, t) \cdot (y, u) = (x + ty, tu), x, y \in R^n, u, v > 0$, a group of affine transformations on R^n. The representation is $\pi(x, t)f(y) = L_x D_t f(y) = t^{-n/2} f(y/t - x/t)$ on $L^2(R^n)$ [17]. The continuous wavelet transformation $W_g(f)(x, t)$ is identical to the representation coefficient $\langle \pi(x, t)g, f\rangle$.

A comparison of (11) and (25) shows that $Co_\pi L^p = \dot{B}_{pp}^{n/p-n/2}(R^n)$.

The expansions (10) use the larger group $\mathcal{G}' = R^n \times (R^+ \times O(n)$ with multiplication $(x, t, O_1) \cdot (y, u, O_2) = (x + tO_1 y, tu, O_1 O_2), x, y \in R^n, u, v > 0, O_i$ being orthogonal matrices. The representation is $\pi'(x, t, O)f(y) = t^{-n/2} f(O^{-1}(y - x)/t)$ on $L^2(R^n)$. In contrast to π, this representation is irreducible and allows us to pick the wavelets in a larger class of functions.

ii) The group for Gabor expansions is the Heisenberg group $\mathcal{G} = R^n \times R^n \times T$ with multiplication $(x, y, \sigma) \cdot (u, v, \tau) = (x + u, y + v, \sigma\tau e^{2\pi i y u})$, $x, y, u, v \in R^n, \sigma, \tau \in C, |\sigma| = |\tau| = 1$. The occurring representation π is the Schrödinger representation $(x, y, \tau) \mapsto \tau L_x M_y f(\xi) = \tau e^{2\pi i y(\xi - x)} f(\xi - x)$ on $L^2(R^n)$. Here L_x is the shift operator and M_y denotes the modulation operator. The short time Fourier transform is identified as $S_g(f)(x, y) = \tau e^{-2\pi i x y}\langle \pi(s, y, \tau)g, f\rangle$. Since the torus acts trivially by scalar multiplication, this additional phase factor is of no consequence and does not occur in the results.

In this case the coorbits of $L^p(\mathcal{G})$ are the modulation spaces M_{pp}^0.

iii) In Example 2.3a) we find again the Heisenberg group, but acting by the Bargmann-Fock representation $\pi(x, y, \tau)f(z) = \tau e^{-i\pi x y} e^{-\pi|w|^2/2} e^{\pi w z} \times f(z - \bar{w})$ on $\mathcal{F}^2(C^n)$, where $w = x + iy \in C^n$. To see that the Bargmann-Fock spaces can also be defined by the size of a representation coefficient, choose $g(z) \equiv 1$, then $|\langle \pi(x, y, \tau)g, f\rangle| = |f(\bar{w})|e^{-\pi|w|^2/2}$ and the representation coefficient is in $L^p(\mathcal{G})$ if and only if $f \in \mathcal{F}^p$, in other words $Co_\pi L^p = \mathcal{F}^p$.

iv) Example 2.3b) is also based on a group of Heisenberg type, namely $\mathcal{G} \times \hat{\mathcal{G}} \times T$ with multiplication $(x, \chi, \sigma) \cdot (y, \psi, \tau) = (x + y, \chi\psi, \sigma\tau\chi(y))$ for $x, y \in \mathcal{G}, \chi, \psi \in \hat{\mathcal{G}}, |\sigma| = |\tau| = 1$. The representation is $\pi(x, \chi, \tau)f(y) = \tau\chi(y - x)f(y - x)$ acting on $f \in L^2(\mathcal{G})$.

v) For the anisotropic wavelet expansions, the "anisotropic" $ax+b$-group is used: $\mathcal{G} = R^n \times R$ with multiplication $(x, t) \cdot (y, v) = (x + e^{tD}y, t + v)$ for $x, y \in R^n, t, v \in R$. In this case the representation acts on $f \in L^2(R^n)$ by $\pi(x, t)f(y) = \Delta(t)f(e^{-tD}(y - x))$.

vi) The last example uses $\mathcal{G} = SL(2, R)$ and its discrete series representations, see [8] for more details.

4. THE MAIN TOOL: THE ORTHOGONALITY RELATIONS

4.1. Let $\mathcal{G}$ be a locally compact group with (left) Haar measure dx and π be an irreducible, unitary representation of $\mathcal{G}$ on a Hilbert space $\mathcal{H}$ that is square-integrable, i.e. there exists $g \in \mathcal{H}, g \neq 0$, such that

$$\int_{\mathcal{G}} |\langle \pi(x)g, g \rangle|^2 dx < \infty \tag{27}$$

Then there exists a positive, self-adjoint, densely defined operator A on $\mathcal{H}$, such that

$$\int_{\mathcal{G}} \overline{\langle \pi(x)g_1, f_1 \rangle} \langle \pi(x)g_2, f_2 \rangle dx = \langle Ag_2, Ag_1 \rangle \langle f_1, f_2 \rangle \tag{28}$$

for all $g_1, g_2 \in \operatorname{dom} A, f_1, f_2 \in \mathcal{H}$. If $\mathcal{G}$ is unimodular, e.g. for the Heisenberg group, then A is a multiple of the identity operator. See [17, 22, 3] for proofs and details.

Special Case: $g_1 = g_2 = g \in \operatorname{dom} A, \|Ag\| = 1, f_1 = \pi(y)g, f_2 = f \in \mathcal{H}$. Set $V_g(f)(x) = \langle \pi(x)g, f \rangle$, then

$$V_g(f) * V_g(g)(y) = \int \langle \pi(x)g, f \rangle \overline{\langle \pi(x)g, \pi(y)g \rangle} dx = \langle \pi(y)g, f \rangle = V_g(f)(y) \,, \tag{29}$$

where $*$ stands for the usual convolution $F * G(x) = \int_{\mathcal{G}} F(y)G(y^{-1}x)dy$ between two functions F and G on $\mathcal{G}$. Thus the orthogonality relations imply the reproducing formula $V_g(f) * V_g(g) = V_g(f)$ for representation coefficients. In the sequel we assume without mentioning that $g \in \mathcal{H}$ is normalized $\|Ag\| = 1$, and thus the reproducing formula (29) holds true.

In some applications the following modification of (29) is very useful, cf.

e.g. [13]: if $\langle Ah, Ag\rangle = 1$, then by (28)

$$V_g(f) * V_g(h) = \langle Ah, Ag\rangle V_g(f) = V_g(f) \tag{30}$$

holds for all $f \in \mathcal{H}$. This is a general statement of the fact that the analyzing wavelet g and the synthesizing wavelet h can be chosen (almost) independent of each other.

For the $ax+b$-group, (29) is equivalent to Calderòn's reproducing formula. For the Heisenberg group the orthogonality relations are known as Moyal's formulas.

The irreducibility of the representation is only a technical assumption; the reproducing formula holds for a larger class of representations, e.g. for groups of $ax+b$-type. For instance, given the representation $\pi(x,t)f(y) = \Delta(t)f(e^{-tD}(y-x))$ of the anisotropic $ax+b$-group (Example 2.3c) and $g, h \in L^2(R^n)$, the reproducing formula (30) for the wavelet transform $\langle \pi(x,t)g, f\rangle$ holds under the condition

$$\int \overline{\hat{g}(e^{tD}\xi)}\hat{h}(e^{tD}\xi)dt = 1 \tag{31}$$

for all $\xi \in R^n, |\xi| = 1$. This explains the conditions on the support of $\hat{g}$ and the vanishing moments in Examples 2.1 and 2.3c).

EXTENSION TO OTHER SPACES

For the decomposition of general $Co_\pi Y$-spaces, the stronger condition

$$\int_{\mathcal{G}} |\langle \pi(x)g, g\rangle| w(x)\, dx < \infty \tag{32}$$

for some $g \in \mathcal{H}, g \neq 0$ is needed. Given a Banach space Y of functions on $\mathcal{G}$, the weight has to be chosen such that $L^1_w(\mathcal{G}) * Y \subseteq Y$ and $Y * L^1_w(\mathcal{G}) \subseteq Y$, where $L^1_w(\mathcal{G}) = \{f \text{ measurable on } \mathcal{G} : \int_{\mathcal{G}} |f(x)|w(x)\, dx < \infty\}$.

With this assumption:

a) It is possible to define a suitable space of test functions embedded in $\mathcal{H}$, namely $Co\, L^1_w$, and distributions (the dual of $Co\, L^1_w$) from which to select elements. Then definition (25) makes sense even for functions not contained in $\mathcal{H}$.

b) The reproducing formulas (29) and (30) carry over to general coorbit spaces $Co_\pi Y$. Moreover, it follows from the orthogonality relations that the definition of $Co_\pi Y$ is independent of the choice of g, such that different g's yield equivalent norms for $Co_\pi Y$. A direct proof of this statement for the examples of Section 2 is usually more involved.

c) The convolution $V_g(f) * V_g(g) = V_g(f)$ in $L^p(\mathcal{G}) * L^1(\mathcal{G}) \subseteq L^p(\mathcal{G})$ or in $Y * L^1_w(\mathcal{G}) \subseteq Y$ in general, is easy to analyze.

4.2. If $f = g$ in (29), then $V_g(g) * V_g(g) = V_g(g)$, and the convolution operator $F \mapsto F * V_g(g)$ is the orthogonal projection from $L^2(\mathcal{G})$ onto the closed subspace of representation coefficients $\{\langle\pi(x)g, f\rangle, f \in \mathcal{H}\}$. If $\langle\pi(x)g, g\rangle \in L^1_w(\mathcal{G})$, then this statement extends to $Co_\pi Y$ and yields the fundamental Correspondence Principle.

Correspondence Principle [9], Prop. 4.3. *A function $F \in Y$ on $\mathcal{G}$ is a generalized representation coefficient $V_g(f)(x) = \langle\pi(x)g, f\rangle$ for a unique $f \in Co_\pi Y$, if and only if $F * V_g(g) = F$. The translate $L_x F \in Y$ corresponds to $\pi(x)f \in Co_\pi Y$, where $L_x F(y) = F(x^{-1}y), x, y \in \mathcal{G}$.*

The Correspondence Principle is the very reason why all the different examples of Section 2 can be treated by a single method. As a result of this *unification* by representation theory the objects are now nice, smooth functions $V_g(f)(x)$ on the group $\mathcal{G}$, instead of rough functions, measures, or distributions on R^n, no matter what the original space looked like! Instead of a possibly complicated integral formula these representation coefficients satisfy a simple convolution equation, which can be analyzed with methods of abstract harmonic analysis.

5. THE CONSTRUCTION OF NONORTHOGONAL EXPANSIONS

We present the main ideas for the series expansions of $Co_\pi L^p$, which for the various groups contains the Besov spaces and the modulation spaces with indices p, p, the Bargmann-Fock spaces etc. The rigorous proof for the decomposition of general coorbit spaces $Co_\pi Y$ differs only in technical details.

Our goal is to approximate the reproducing formula (29) by a sum of translates of $V_g(g)$ and then iterate on the remainder (see [1], where a similar idea is first used to construct atomic decompositions.)

In the following we abbreviate $F(x) = \langle\pi(x)g, f\rangle, G(x) = \langle\pi(x)g, g\rangle$.

APPROXIMATION OPERATORS

Let $U \subseteq \mathcal{G}$ be a neighborhood of the identity e in $\mathcal{G}$. A subset $(x_i)_{i\in I}$ in $\mathcal{G}$ is called U-dense, if $\cup_{i\in I} x_i \cdot U = \mathcal{G}$. Given (x_i), choose a partition of unity $\Psi = (\psi_i)_{i\in I}$ in $\mathcal{G}$, such that (1) $\sum_i \psi_i \equiv 1$ almost everywhere, (2) $0 \le \psi_i \le 1$, (3) $\operatorname{supp} \psi_i \subseteq x_i \cdot U$.

Example. In the n-dimensional $ax+b$-group $R^n \times R^+$ the set $(\alpha k\beta^j, \beta^j)$, $j \in Z, k \in Z^n$ is dense with respect to the neighborhood $U = [-\alpha/2, \alpha/2]^n \times [1/\sqrt{\beta}, \sqrt{\beta}]$ and the characteristic functions of $[\beta^j \alpha(k-1/2), \beta^j \alpha(k+1/2)] \times [\beta^{j-1/2}, \beta^{j+1/2}]$ form a partition of unity. The U-density for an irregular sampling set (x_i, y_i) in $R^n \times R^+$ has been explicitly stated in Section 2.1.

Definition. Given a partition of unity Ψ, the operator T_Ψ, acting on functions on $\mathcal{G}$, is defined by

$$T_\Psi F = \sum_i \langle \psi_i, F \rangle L_{x_i} G \tag{33}$$

where $\langle \psi_i, F \rangle = \int_{\mathcal{G}} \bar{\psi}_i F(x)\, dx$ is essentially a local average of F near x_i.

Given a neighborhood $U \subseteq \mathcal{G}$ of e, the *U-oscillation* of a function G is

$$\mathrm{osc}_U G(x) = \sup_{u \in U} |G(ux) - G(x)| \tag{34}$$

Facts [9]. Assume that $g \in \mathcal{H}$, such that $G(x) = \langle \pi(x)g, g \rangle \in L^1(\mathcal{G})$ and also $\mathrm{osc}_U G \in L^1(\mathcal{G})$ for some, hence for all, neighborhoods U of e. Then:

1. T_Ψ is bounded from $L^p(\mathcal{G})$ into the closed subspace $L^p * G$ with a bound independent of Ψ.

2. If $f \in Co_\pi L^p$, then $\sum_i \langle \pi(x_i)g, f \rangle \psi_i \in L^p(\mathcal{G})$. In particular, if $(x_i)_{i \in I}$ is a discrete subset in $\mathcal{G}$, then $\langle \pi(x_i)g, f \rangle \in l^p(I)$.

3. If $(x_i)_{i \in I}$ is a discrete subset in $\mathcal{G}$ and $(c_i)_{i \in I} \in l^p(I)$, then $F = \sum_i c_i L_{x_i} G \in L^p(\mathcal{G})$ and thus by the Correspondence Principle $f = \sum_i c_i \pi(x_i) g \in Co_\pi L^p$.

THE MAIN ESTIMATE

We approximate the convolution $F * G = F$ (29) by a Riemann type sum, i.e. by T_Ψ, as follows:

$$F(x) = F * G(x) = \int_{\mathcal{G}} F(y) G(y^{-1}x) dy = \sum_i \int F(y) \psi_i(y) G(y^{-1}x) dy \approx$$

$$\approx \sum_i \left(\int F(y) \psi_i(y) dy \right) G(x_i^{-1} x) = T_\Psi F$$

Since $y \in x_i \cdot U \Leftrightarrow y = x_i u \Leftrightarrow x_i^{-1} = u y^{-1}$ for some $u \in U$, one obtains a pointwise estimate

$$|F(x) - T_\Psi F(x)| = |\int_{\mathcal{G}} F(y) \left(\sum_i \psi_i(y)\right) (G(y^{-1}x) - G(x_i^{-1}x))\, dy| \le$$

$$\le \sum_{i\in I} \int |F(y)|\psi_i(y) \sup_{u\in U} |G(y^{-1}x) - G(uy^{-1}x))|dy = |F| * \mathrm{osc}_U G(x) \quad (35)$$

Upon taking norms

$$\|F - T_\Psi F\|_p \le \| \, |F| * \mathrm{osc}_U G\|_p \le \|F\|_p \| \, \mathrm{osc}_U G\|_1 \quad (36)$$

holds for all $F \in L^p * G$. If the neighborhood U is chosen so small that $\|\mathrm{osc}_U G\|_1 < 1$, then $\mathbf{1} - T_\Psi$ is a contraction on the closed subspace $L^p * G$ and the operator norm satisfies

$$|||\mathbf{1} - T_\Psi||| < 1 \quad (37)$$

THE ITERATION

By (37) T_Ψ is invertible on $L^p * G$ and

$$T_\Psi^{-1} = \sum_{n=0}^{\infty} (\mathbf{1} - T_\Psi)^n \quad \text{on } L^p * G \quad (38)$$

Consequently $F = T_\Psi T_\Psi^{-1} F = \sum_{i\in I} \langle \psi_i, T_\Psi^{-1} F\rangle L_{x_i} G$ and the Correspondence Principle implies

$$f = \sum_{i\in I} c_i \pi(x_i) g \quad (39)$$

with coefficients $c_i = \langle \psi_i, T_\Psi^{-1} F\rangle \in l^p(I)$.

FRAMES

From (39) it is clear that the coefficients c_i are given by functionals e_i in the dual of $Co_\pi L^p$: $c_i = \langle e_i, f\rangle$. Precisely e_i is given by $V_g(e_i) = T_\Psi^{*-1}\psi_i$. Using the duality theory for coorbit spaces [9, 10], the argument of Section 1 carries over and yields that $\{\pi(x_i)g, i \in I\}$ is a (Banach) frame for $Co_\pi L^p, 1 \le p < \infty$, i.e. for some constants $0 < A_1 \le A_2$

$$A_1 \|f\|_{Co_\pi L^p} \le \|\langle \pi(x_i)g, f\rangle_{i\in I}\|_p \le A_2 \|f\|_{Co_\pi L^p} \ , \quad (40)$$

and the reconstruction of f from the frame coefficients is

$$f = \sum_i \langle \pi(x_i)g, f\rangle e_i \ .$$

Note, however, that for Banach spaces the norm equivalence (40) by itself does not imply a reconstruction method.

6. CONCLUSION

NUMERICAL ASPECTS

For numerical calculations it is better to work with the modified approximation operator $S_\Psi F = \sum_i F(x_i)\gamma_i L_{x_i} G$ on $L^p * G$, where the partition of unity Ψ comes in only through the weight factors $\gamma_i = \int \psi_i(x)dx$.

If $(x_i)_{i\in I}$ is dense enough in $\mathcal{G}$, then S_Ψ is invertible on $L^p * G$ [16] and

$$F = S_\Psi^{-1} S_\Psi F = \sum_i F(x_i)\gamma_i S_\Psi^{-1} L_{x_i} G = \sum_{n=0}^{\infty} (1 - S_\Psi)^n S_\Psi F \tag{41}$$

provides a complete reconstruction of F from the samples $F(x_i) = \langle \pi(x_i)g, f\rangle$.

With the Correspondence Principle the latter expression translates into the following reconstruction scheme on $Co_\pi L^p$:

$$\phi_0 = \sum_{i\in I} \langle \pi(x_i)g, f\rangle \gamma_i \pi(x_i) g \tag{42}$$

$$\phi_{n+1} = \phi_n - \sum_{i\in I} \langle \pi(x_i)g, \phi_n\rangle \gamma_i \pi(x_i) g \tag{43}$$

$$f = \sum_{n=0}^{\infty} \phi_n \tag{44}$$

COMPUTATION OF THE COEFFICIENTS c

The identity (41) also implies the series expansion $f = \sum_i c_i \pi(x_i) g$. The reconstruction scheme (42) can be turned into an algorithm to compute the coefficients c_i as follows. Set $\phi_n = \sum_{i\in I} c_i^{(n)} \pi(x_i)g$, where the initial data $c_i^{(0)} = \gamma_i \langle \pi(x_i)g, f\rangle$ is the sampling of the representation coefficient. Let $A = (A_{ij})_{i,j\in I}$ be the matrix with elements $A_{ij} = \gamma_i \langle \pi(x_i)g, \pi(x_j)g\rangle$. Then

$$c^{(n+1)} = c^{(n)} - Ac^{(n)} \quad \text{and} \quad c_i = \sum_{n=0}^{\infty} c_i^{(n)} \ , \tag{45}$$

as follows readily from

$$\phi_{n+1} = \sum_i c_i^{(n+1)} \pi(x_i) g = \phi_n - \sum_{i\in I} \langle \pi(x_i)g, \phi_n\rangle \gamma_i \pi(x_i) g =$$

$$= \sum_i \left(c_i^{(n)} - \sum_j c_j^{(n)} \gamma_i \langle \pi(x_i)g, \pi(x_j)g \rangle \right) \pi(x_i)g \, .$$

This is essentially the same method as in 1.3. At this point the group theoretic background of the reconstruction method disappears and the implementation becomes a problem in linear algebra.

The following theorem summarizes the conclusions of Sections 3–5 and contains the examples discussed in Section 2 as special cases.

Theorem 1. *Given $\mathcal{G}$, $\pi, \mathcal{H}, L^p(\mathcal{G}), 1 \le p < \infty$. Assume that $g \in \mathcal{H}$ is normalized $\|Ag\| = 1$ and satisfies $G(x) = \langle \pi(x)g, g\rangle \in L^1(\mathcal{G})$ and $osc_{U_0} G \in L^1(\mathcal{G})$ for some neighborhood U_0 of e. Then choose U so small that*

$$\|osc_U G\|_1 < 1$$

a) If $(x_i)_{i\in I}$ in $\mathcal{G}$ is U-dense and discrete, then any $f \in Co_\pi L^p$ has a (nonorthogonal) expansion

$$f = \sum_{i\in I} c_i \pi(x_i) g$$

The series converges unconditionally in $Co_\pi L^p$ and the coefficients c satisfy

$$A_1 \|f\|_{Co_\pi L^p} \le \|(c_i)_{i\in I}\|_p \le A_2 \|f\|_{Co_\pi L^p}$$

b) On the other hand, $\{\pi(x_i)g, i \in I\}$ is a frame for $Co_\pi L^p(\mathcal{G})$, i.e., $f \in Co_\pi L^p$ is uniquely determined by the coefficients $\langle \pi(x_i)g, f\rangle_{i\in I}$,

$$\|f\|_{Co_\pi L^p} \cong \Big(\sum_i |\langle \pi(x_i)g, f\rangle|^p\Big)^{1/p}$$

and $f = \sum_i \langle \pi(x_i)g, f\rangle e_i$ for a fixed set $e_i \in Co_\pi L^1$.

Final Remarks. a) The Theorem gives a universal condition on g to be a wavelet and on the sampling density U required for a nonorthogonal expansion. The conditions on the wavelets in the concrete examples are sufficient to ensure the required properties of the representation coefficient $\langle \pi(x)g, g\rangle$. The explicit determination of the size of U is more tricky and is being investigated intensively.

b) It contains an extension of the theory of frames to Banach spaces [16].

c) Since the construction works for *any* U-dense subset (x_i), the Theorem allows for irregular sampling of representation coefficients (which in turn yields sampling theorems for wavelet transforms and short time Fourier transforms) and for "jittered" expansions.

d) The missing technical details and the exact proof for general coorbit spaces can be found in [9, 16].

e) The original papers [9, 16] also contain a stability theory for non-orthogonal expansions and frames of Banach spaces. It is shown that the reconstruction is robust under small changes of the wavelet g or the sampling set $x_i, i \in I$.

f) With Theorem 1 and the identification of the underlying groups and representations it is now a routine task to obtain the explicit expansions which were discussed in Section 2.

Since each of the examples of Section 2 possesses a rich background, it is impossible to include all substantial contributions. The following list of references can therefore serve only as a guide to more detailed collections of references. Especially useful should be I. Daubechies' lecture notes [3], the review article [18], Y. Meyer's book [21] and the original paper [8] on the unified theory.

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