

# A POCS approach to Gabor analysis

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## ABSTRACT

Gabor analysis is based on a well-structured composition of a signal or image as a series of building blocks. These are obtained by shifting and modulating a basic signal (atom)  $g$  along a time-frequency (or space-wavenumber) lattice  $\Lambda$ . These families are typically non-orthogonal. Nevertheless suitable expansion coefficients of a signal  $f$  can be computed efficiently via the short time Fourier transform of  $f$ , using a canonically related window, the  $\Lambda$ -dual atom  $\tilde{g}$ . A new iterative method to compute  $\tilde{g}$  is introduced, which is based on the Wexler-Raz principle and an appropriate version of POCS (projection onto convex sets).

**Keywords:** Gabor expansions, digital image processing, space-wavenumber analysis of images, texture analysis, projection onto convex sets, numerical methods in harmonic analysis

## 1. GABOR ANALYSIS

### 1.1. Introduction

A standard method of texture analysis is the use of Gabor expansions of an image.<sup>1-5</sup> Gabor analysis is concerned with the expansion of a signal or image  $f$  by versions of a building block  $g$ , the Gabor atom, shifted and modulated along a sufficiently dense space-wavenumber lattice  $\Lambda$ . (Whereas in one-dimensional problems, analyzing signals, we speak of a time-frequency decomposition, in two-dimensional cases, analyzing images, it would be more appropriate to refer to the Gabor expansion as space-wavenumber analysis.)

It turns out that for the application of Gabor analysis to digital images it is much better to make use of a discrete model in order to achieve perfect reconstruction instead of dealing with pixel images as collections of samples of a continuous image. Thus for a suitable mathematical treatment of Gabor analysis it makes sense to think in terms of *discrete Gabor analysis*, i. e. to interpret images as functions on a finite group  $\mathcal{G} = Z_{n_1} \times Z_{n_2}$  (where  $(n_1, n_2)$  is the image format).

For any such finite Abelian group it is possible to find a complete orthonormal system (of complex-valued functions on  $\mathcal{G}$  of absolute value 1), which are eigenvalues with respect to translation. These so-called *characters* can be viewed as the *pure frequencies* in this context (or *plane waves* in the context of images). It is one of the basic facts from abstract harmonic analysis that this system of functions, consisting of the same number of elements as  $\mathcal{G}$  is closed with respect to conjugation and pointwise multiplication. In other words they form a group of unitary (multiplication) operators on the finite dimensional Hilbert space  $\mathcal{H} = \ell^2(\mathcal{G})$  isomorphic to  $\mathbb{C}^{n_1 \cdot n_2}$ . In the literature this group is called the dual group  $\hat{\mathcal{G}}$  to the finite Abelian group  $\mathcal{G}$ .

Starting from these two groups one may define the group

$$\mathcal{G} \times \hat{\mathcal{G}} = (Z_{n_1} \times Z_{n_2}) \times (Z_{n_1} \times Z_{n_2}) \quad (1)$$

which may be called time-frequency (or space-wavenumber) space (TF-space). A subgroup  $\Lambda$  of TF-space  $\mathcal{G} \times \hat{\mathcal{G}}$  is called a *time-frequency lattice* (TF-lattice). The signal space  $\mathcal{H} = \ell^2(\mathcal{G})$  is the  $n_1 \cdot n_2$  dimensional vector space

$$\mathcal{H} = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \cong \mathbb{C}^{n_1 \cdot n_2} \quad (2)$$

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## 1.2. Notation

For any  $u \in \mathcal{G}$  and  $v \in \widehat{\mathcal{G}}$  we define the *shift* (or *translation*)  $T_u$  and the *modulation*  $M_v$  as unitary operators on  $\mathcal{H}$  as

$$T_u f(x) := f(x - u), \quad (3)$$

$$M_v f(x) := v(x)f(x), \quad (4)$$

where  $v(x)$  is a pure frequency. For the finite 2D-case described above pure frequencies are often identified with integers mod  $n_1$  and mod  $n_2$  resp. and appear thus in a concrete context as the mappings of the form

$$v : (x_1, x_2) \mapsto e^{2\pi i(x_1 v_1/n_1 + x_2 v_2/n_2)}. \quad (5)$$

The translation and the modulation operators are related via the Fourier transform  $\mathcal{F}$  as follows

$$M_v = \mathcal{F}^{-1} T_v \mathcal{F}. \quad (6)$$

To simplify our notation we introduce the *time-frequency shift operator*  $U_\lambda$  which is given as composition of translation and modulation:

$$U_\lambda = U_{(u,v)} = M_v T_u, \quad \lambda = (u, v) \in \mathcal{G} \times \widehat{\mathcal{G}}. \quad (7)$$

Shifts and modulations do (in general) not commute. The following commutation property will be used later:

$$M_v T_u = v(u) T_u M_v. \quad (8)$$

## 1.3. Gabor analysis and synthesis

**Definition 1** For any given  $g \in \mathcal{H}$  and TF-lattice  $\Lambda \triangleleft \mathcal{G} \times \widehat{\mathcal{G}}$  the set

$$\{U_\lambda g\}_{\lambda \in \Lambda} \quad \leftrightarrow \quad (g, \Lambda) \quad (9)$$

is called a Gabor family, generated by the lattice  $\Lambda$  and the window function  $g$ , or shortly by  $(g, \Lambda)$ . The mapping  $G_g$

$$G_g : \mathcal{H} \mapsto \ell^2(\Lambda) \quad f \mapsto \langle f, U_\lambda g \rangle \quad (10)$$

is called the Gabor analysis mapping, where  $\ell^2(\Lambda)$  is the coefficient space which is the vector space of complex valued square summable sequences over  $\Lambda$  (i. e. indexed by  $\Lambda$ )\*, i. e.

$$c \in \ell^2(\Lambda) \quad \Leftrightarrow \quad c : \Lambda \mapsto \mathbb{C}. \quad (11)$$

If the Gabor analysis mapping is injective (i. e. invertible on its range) then the Gabor family generated by  $(g, \Lambda)$  is called a Gabor frame.

The Gabor synthesis mapping  $G_g^*$  is the adjoint mapping of the Gabor transform  $G_g$ , which is given as

$$G_g^* : \ell^2(\Lambda) \mapsto \mathcal{H} \quad c \mapsto \sum_{\lambda \in \Lambda} c_\lambda U_\lambda g. \quad (12)$$

It is one of the basic results of linear algebra that a linear mapping is injective if and only if its adjoint is surjective (i. e. onto). Thus a Gabor family is a frame if and only if  $G_g^*$  is onto, which is of course just another way to express that it spans the whole signal space  $\mathcal{H}$ .

A necessary condition for  $(g, \Lambda)$  to generate a Gabor frame is therefore that the dimension of the coefficient space may not be smaller than the dimension of  $\mathcal{H}$ , since  $|\Lambda| < n_1 n_2$  implies that  $(g, \Lambda)$  can not generate a frame. The

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\*The use of "abstract" indices is not only natural from a mathematical point of view, but avoids —seen from a more practical side— the use of multiple indices or summation, e. g. quadruple sums in the typical case of image analysis. Since in all such cases the order of summation does not matter (it is finite, hence unconditional) symbols of the form  $\sum_{\lambda \in \Lambda}$ , to be read as "summation taking place over the full time-frequency lattice  $\Lambda$ ", appear to be most appropriate.

sparsest possible lattices for which the corresponding Gabor family may form a frame thus satisfy  $|\Lambda| = n_1 n_2$ . They are called *critically sampled* lattices, whereas TF-lattices satisfying  $|\Lambda| > n_1 n_2$  are called an *oversampled* lattices. Sufficient conditions on  $g$  will be presented later.

If some image  $f$  shows up in the range of the Gabor synthesis mapping we say it has a *Gabor representation*. Any such Gabor representation (or *Gabor expansion*) is thus of the form

$$f = \sum_{\lambda \in \Lambda} c_\lambda U_\lambda g. \quad (13)$$

#### 1.4. The dual Gabor frame

The natural questions in Gabor analysis/synthesis are: How can one obtain suitable coefficients  $c_\lambda$  required to represent an arbitrary image  $f$  in terms of TF-shifted versions of a given window function  $g$ , and secondly, how can one reconstruct an image  $f$  from the given values  $c_\lambda = \langle f, U_\lambda g \rangle$  (which determine  $f$  uniquely according to the assumptions).

If  $(g, \Lambda)$  generates a Gabor frame we know that  $G_g$  is injective and  $G_g^*$  is surjective. Thus one way to handle this questions is to view them as the problem of solving a linear system. As such it may be “solved” as follows:

$$\begin{aligned} c &= G_g f \\ \Rightarrow G_g^* c &= G_g^* G_g f \\ \Rightarrow f &= (G_g^* G_g)^{-1} G_g^* c = G_g^+ c, \end{aligned} \quad (14)$$

where  $G_g^+$  is known as the *pseudoinverse* or *Moore-Penrose inverse* of the operator  $G_g$ . This way of solving the linear problem can even be easily implemented for very small size 1D-problems and delivers the minimal-norm solution to the representation problem.

However, this is not a really practical method for the case of images, and it is of course better to make more use of the overall mathematical structure of the problem. A meanwhile well established way of doing so is given by *frame theory*. Recall that in our context the operator  $S_g = G_g^* G_g$  is called the *frame operator* of the Gabor family. It can be formed for any finite family of vectors in  $\mathcal{H}$ , and it is easy to check that  $(g, \Lambda)$  generates a Gabor frame if and only if the corresponding operator  $S_g$  is invertible. The following observation is central to most results concerning the numerical treatment of  $S_g$ .

**Lemma 2** *The frame operator of a Gabor family generated by  $(g, \Lambda)$  and its inverse (if it exists) commute with all TF-shifts  $U_\lambda$ , where  $\lambda \in \Lambda$ .*

*Proof.* Let  $\mu \in \Lambda$  and  $\lambda = (u, v)$ ,  $\mu = (t, s)$  then

$$\begin{aligned} U_\mu S_g f &= U_\mu \sum_{\lambda \in \Lambda} \langle f, U_\lambda g \rangle U_\lambda g = \sum_{\lambda \in \Lambda} \langle U_\mu f, U_\mu U_\lambda g \rangle U_\mu U_\lambda g \\ &= \sum_{\lambda \in \Lambda} \langle U_\mu f, v_1(t_1) U_{\mu+\lambda} g \rangle v_1(t_1) U_{\mu+\lambda} g = \sum_{\lambda \in \Lambda} \langle U_\mu f, U_{\mu+\lambda} g \rangle U_{\mu+\lambda} g = S_g U_\mu f. \end{aligned}$$

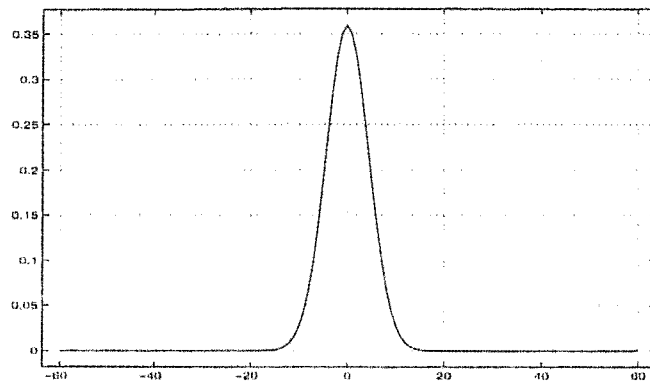
If  $S_g$  is invertible we obtain

$$S_g U_\mu = U_\mu S_g \Leftrightarrow U_\mu S_g^{-1} = S_g^{-1} U_\mu, \quad \mu \in \Lambda$$

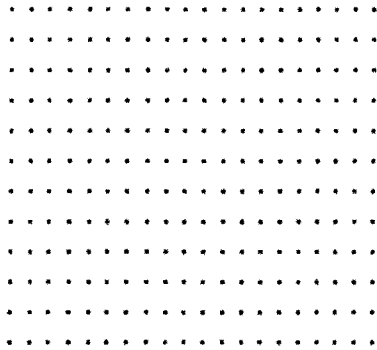
which completes the proof. □

By the use of this lemma, Eq. (14) now becomes

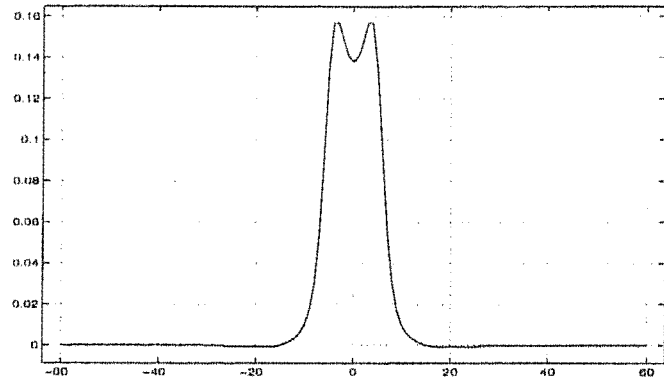
$$f = (G_g G_g^*)^{-1} G_g^* c = S_g^{-1} G_g^* c = \sum_{\lambda \in \Lambda} c_\lambda S_g^{-1} U_\lambda g = \sum_{\lambda \in \Lambda} c_\lambda U_\lambda S_g^{-1} g = \sum_{\lambda \in \Lambda} c_\lambda U_\lambda \tilde{g}, \quad (15)$$



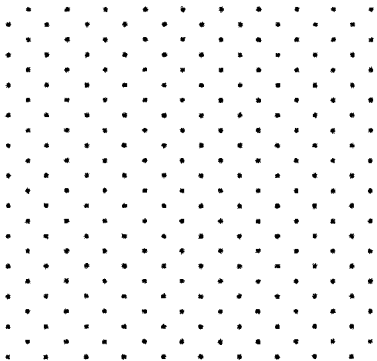
(a) A 1D Gabor atom  $g$ .



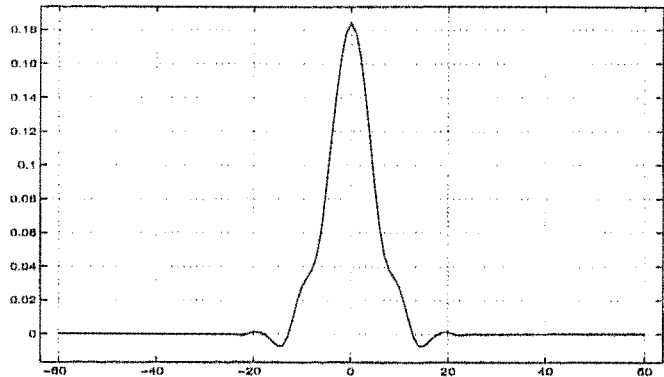
(b) A separable sampling lattice  $\Lambda_1$ .



(c) Dual atom  $\tilde{g}_1$  with respect to TF-lattice  $\Lambda_1$ .



(d) A quincunx sampling lattice  $\Lambda_2$ .



(e) Dual atom  $\tilde{g}_2$  with respect to TF-lattice  $\Lambda_2$ .

Figure 1. A 1D Gabor atom  $g$  (a) and its dual atoms  $\tilde{g}_i$  ( $i = 1, 2$ ) with respect to different TF-lattices  $\Lambda_i$ . Subfigure (c) is the dual with respect to the lattice  $\Lambda_1$  (b) which is separable into T- and F-domain. Subfigure (e) is the dual referring to the quincunx lattice  $\Lambda_2$  (d), which is not separable. In this case it appears as the better choice since the resulting Gabor frame has a better condition number. (Both TF-lattices  $\Lambda_1$  and  $\Lambda_2$  provide the same oversampling redundancy.)

where

$$\tilde{g} := S_g^{-1}g \quad (16)$$

is called *the (canonical) dual Gabor window*. In other words the composition of a surjective Gabor analysis mapping with the Gabor synthesis mapping of its dual window becomes the identity:

$$G_g^* G_{\tilde{g}} = G_{\tilde{g}}^* G_g = Id. \quad (17)$$

**Definition 3** For a given Gabor frame generated by  $(g, \Lambda)$  any function  $\gamma \in \mathcal{H}$  satisfying Eq. (17) is called a dual window of  $g$ , and the set generated by  $(\gamma, \Lambda)$  is called a dual Gabor frame.

Summarizing these observations we obtain the following result.

**Theorem 4** For a Gabor frame generated by  $(g, \Lambda)$  there exists a dual Gabor frame generated by  $(\gamma, \Lambda)$ . If the dimension of the coefficient space  $\ell^2(\Lambda)$  equals the dimension of the signal space, i. e. if  $|\Lambda| = n_1 \cdot n_2$  then the dual Gabor frame is unique. If  $|\Lambda| > n_1 \cdot n_2$  then the dual Gabor frame is not unique.

It remains to show that, in general, there also exist other dual Gabor frames. This will appear as an immediate consequence of Theorem 7.

For a more detailed investigation we have to turn our attention to the space of linear operators acting on  $\mathcal{H}$ , which we denote as  $\mathcal{L}(\mathcal{H})$ . Since all elements of  $\mathcal{L}(\mathcal{H})$  can be represented as matrices there is a canonical inner product:

$$\langle L_1, L_2 \rangle := \frac{1}{n_1 n_2} \sum_{i=1}^{n_1 n_2} \langle L_1 e_i, L_2 e_i \rangle, \quad L_1, L_2 \in \mathcal{L}(\mathcal{H}), \quad (18)$$

where  $\{e_n\}$  is an arbitrary orthonormal basis of  $\mathcal{H}$ .

**Lemma 5** The set of all TF-shift operators  $\{U_\lambda\}_{\lambda \in G \times \hat{G}}$  forms an orthonormal basis of  $\mathcal{L}(\mathcal{H})$ .<sup>6</sup>

*Proof.* The kernel  $k_{u,v}(t, s)$  of  $U_{(u,v)}$  is given by  $k_{u,v}(t, s) = \delta(t - s - u)v(s)$ , hence

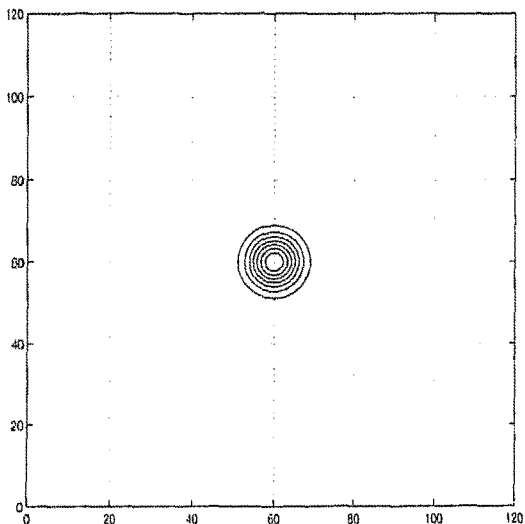
$$\begin{aligned} \langle U_{(u,v)}, U_{(x,y)} \rangle &= \frac{1}{n_1 n_2} \sum_s \sum_t \delta(t - s - u)v(s) \delta(t - s - x)v(s) \\ &= \frac{1}{n_1 n_2} \sum_t \delta(t - t + x - u)(v(s) - y(s)) \\ &= \frac{1}{n_1 n_2} \delta(x - u) \sum_t v(s) - y(s) \\ &= \frac{1}{n_1 n_2} \delta(x - u) N \delta(v - y) = \delta(x - u) \delta(v - y). \end{aligned}$$

Since the number of TF-shifts coincides with the dimension of  $\mathcal{H}$  the theorem is proved.  $\square$

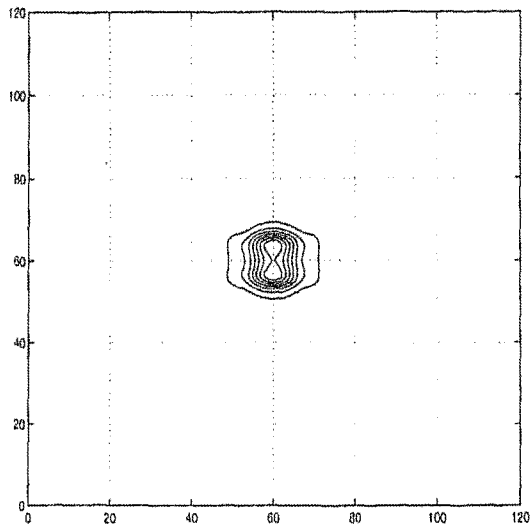
## 1.5. The Wexler-Raz principle

There is a canonical one-to-one correspondence between the elements of the TF-space  $\mathcal{G} \times \hat{\mathcal{G}}$  and the TF-shift operators:  $\lambda \leftrightarrow U_\lambda$ .

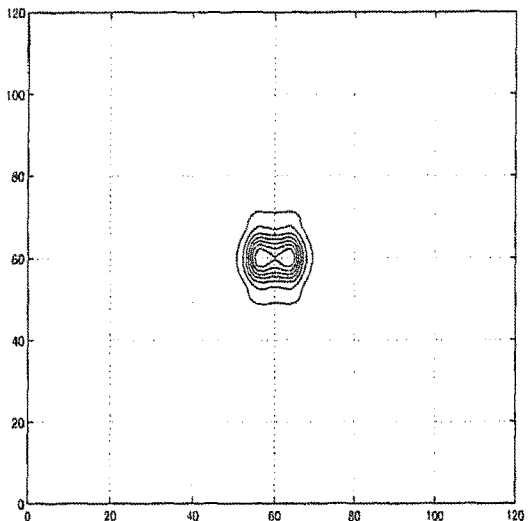
Thus every lattice  $\Lambda \triangleleft \mathcal{G} \times \hat{\mathcal{G}}$  corresponds to a subset of TF-shift operators. Up to scalar factors of modulus 1 these subsets are closed with respect to composition, due to the (additive) lattice structure of  $\Lambda$ . Thus in some sense they are close enough to groups of unitary operators on the signal space  $\mathcal{H}$  and the group theoretical concept of a commutant makes sense. More precisely, for every TF-lattice there exists the so called *adjoint lattice*  $\Lambda^\circ$ , which may be defined via the following commutation relation for TF-shifts.



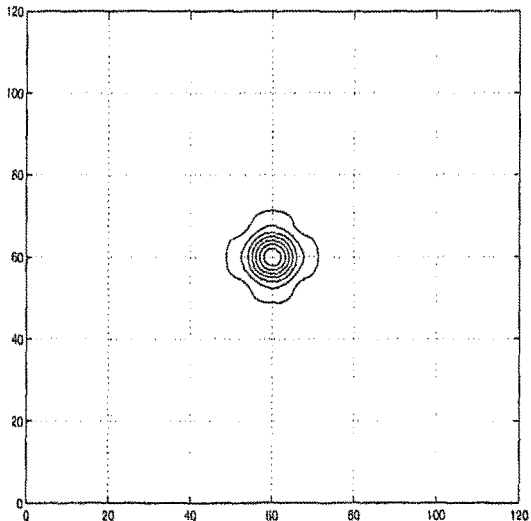
(a) Contour plot of 2D Gabor atom  $g$ .



(b) Dual atom  $\tilde{g}_1$ .



(c) Dual atom  $\tilde{g}_2$ .



(d) Dual atom  $\tilde{g}_3$ .

Figure 2. Contour plots of a 2D Gabor atom  $g$  (a) and its dual atoms  $\tilde{g}_i$  ( $i = 1, 2, 3$ ) with respect to three different TF-lattices. Subfigures (b) and (c) use lattices non-separable regarding one dimension but separable (into T- and F-domain) regarding the other. In the case of a lattice that is non-separable regarding both dimensions, the dual is shown in Subfigure (d), which among the duals is the one best concentrated along the axes. (The three underlying TF-lattices provide all the same oversampling redundancy.)

**Definition 6** The adjoint lattice  $\Lambda^\circ$  of a lattice  $\Lambda$  is defined as

$$\Lambda^\circ := \{\lambda^\circ \in \mathcal{G} \times \widehat{\mathcal{G}} : U_\lambda U_{\lambda^\circ} = U_{\lambda^\circ} U_\lambda, \forall \lambda \in \Lambda\}. \quad (19)$$

It is obvious that large lattices  $\Lambda$  will have small adjoint groups and vice versa, but it is even possible to express this phenomenon in the following quantitative way: The product of the cardinality<sup>†</sup> of a lattice and its adjoint lattice is constant:

$$|\Lambda| \cdot |\Lambda^\circ| = |\mathcal{G} \times \widehat{\mathcal{G}}| = (n_1 n_2)^2, \quad (20)$$

and that the adjoint of  $\Lambda^\circ$  is  $\Lambda$ , i. e.

$$(\Lambda^\circ)^\circ = \Lambda. \quad (21)$$

**Theorem 7 (Wexler-Raz principle)** Let  $(g, \Lambda)$  be a Gabor frame then  $(\gamma, \Lambda)$  is a dual Gabor frame if and only if

$$\langle U_{\lambda^\circ} g, \gamma \rangle = \frac{n_1 n_2}{|\Lambda|} \delta_{0, \lambda^\circ}, \quad \forall \lambda^\circ \in \Lambda^\circ. \quad (22)$$

This theorem is a general version of the *Wexler-Raz principle*<sup>7</sup> as pointed out in Ref. 8, condition (22) as such will be referred to as Wexler-Raz condition.

*Proof.* Let  $f, h \in \mathcal{H}$  be arbitrary functions, and  $G_f^* G_h$  as in (10) and (12). The TF-shifts form an orthonormal basis of the space of linear operators, thus  $G_f^* G_h = \sum_\lambda c_\lambda U_\lambda$  with  $c_\lambda = \langle G_f^* G_h, U_\lambda \rangle$ . According to the proof of Lemma 2 it follows that  $\forall \lambda \in \Lambda$  the operator  $G_f^* G_h$  commutes with  $U_\lambda$ . Thus  $c_\lambda = 0$  if  $\lambda \notin \Lambda^\circ$ .

The functions  $g, \gamma$  are dual windows if and only if  $G_g^* G_\gamma = Id$ . Let  $\{e_n\}$  be an orthonormal basis of  $\mathcal{H}$ , then

$$\begin{aligned} c_{\lambda^\circ} &= \langle G_g^* G_\gamma, U_{\lambda^\circ} \rangle = \frac{1}{n_1 n_2} \sum_{i=1}^{n_1 n_2} \langle G_g^* G_\gamma e_i, U_{\lambda^\circ} e_i \rangle = \frac{1}{n_1 n_2} \sum_{i=1}^{n_1 n_2} \left\langle \sum_{\lambda \in \Lambda} \langle e_i, U_\lambda \gamma \rangle, U_{\lambda^\circ} e_i \right\rangle \\ &= \frac{1}{n_1 n_2} \sum_{i=1}^{n_1 n_2} \sum_{\lambda \in \Lambda} \langle U_\lambda \gamma, e_i \rangle \langle e_i, U_{\lambda^\circ} U_{\lambda^\circ} g \rangle = \frac{|\Lambda|}{n_1 n_2} \langle \gamma, U_{\lambda^\circ} g \rangle. \end{aligned}$$

Since the only nonzero coefficient of the identity is  $c_0$  the theorem is proved.  $\square$

**Corollary 8** A Gabor family  $(g, \Lambda)$  is a Gabor frame if and only if the set  $(g, \Lambda^\circ)$  is linearly independent.

**Remark** In the oversampled case the adjoint group gets sparser:  $|\Lambda^\circ| = n_1 n_2 / |\Lambda| < |\Lambda|$ . Therefore a dual window  $\gamma$  is given by the underdetermined system of equations (22), which proves Theorem 4.

## 1.6. The canonical dual Gabor window

We denote the set of all possible dual windows of a WH-frame  $(g, \Lambda)$  as  $\Gamma_{g, \Lambda}$ , and the set of all “trivial” functions by  $\mathcal{N}_{g, \Lambda}$ , i. e.

$$\Gamma_g := \{\gamma : G_g^* G_\gamma = Id\}, \quad (23)$$

$$\mathcal{N}_g := \{\eta : G_g^* G_\eta = 0\}. \quad (24)$$

Due to Theorem 7 the vector space  $\mathcal{N}_g$  is given as  $\mathcal{N}_g = \ker(G_{g, \Lambda^\circ})$ , where  $G_{g, \Lambda^\circ}$  is the Gabor transform with respect to the TF-lattice  $\Lambda^\circ$ . The affine vector space  $\Gamma_g$  now writes as  $\Gamma_g = \tilde{g} + \mathcal{N}_g$ , where  $\tilde{g} = S^{-1}g$  is the *canonical* dual Gabor window as defined before.

The dual Gabor window  $\tilde{g}$  is special among all dual windows in the way that it is orthogonal to  $\mathcal{N}_g$ . On the other hand the window function  $g$  also is orthogonal to  $\mathcal{N}_g$ , and the trivial spaces of  $g$  and  $\tilde{g}$  coincide, i. e.

$$\mathcal{N}_g = \mathcal{N}_{\tilde{g}}. \quad (25)$$

This characterization of  $\tilde{g}$  is equivalent to the following conventional properties:

<sup>†</sup>The number of elements, written as  $|\Lambda|$ .

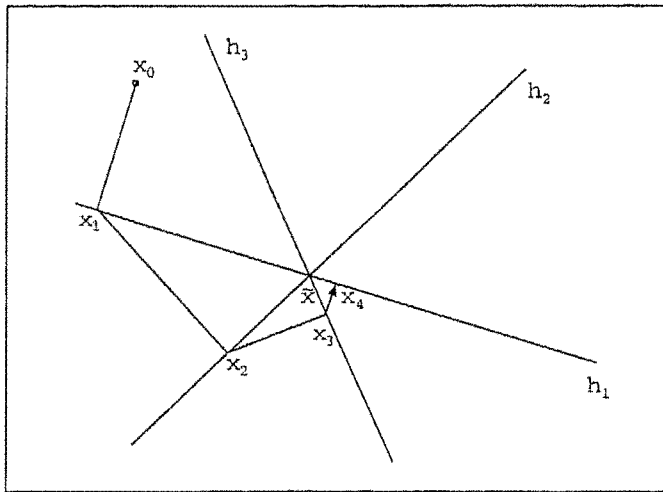


Figure 3. The basic idea of POCS —projection onto convex sets— shown for the trivial but instructive case of three affine subspaces of the Euclidean  $\mathbb{R}^2$  space.

- minimal norm property<sup>7</sup>:  $\|\tilde{g}\| < \|\gamma\| \quad \forall \gamma \in \Gamma_g, \quad \gamma \neq \tilde{g}, \quad \text{and}$
- most likeness<sup>9</sup>:  $\|\frac{\tilde{g}}{\|\tilde{g}\|} - \frac{g}{\|g\|}\| < \|\frac{\gamma}{\|\gamma\|} - \frac{g}{\|g\|}\| \quad \forall \gamma \in \Gamma_g, \quad \gamma \neq \tilde{g}.$

## 2. DUAL GABOR ATOM AND POCS

### 2.1. Computation of a dual atom

The one-dimensional problem of computing the dual of a Gabor atom can be solved efficiently, fast algorithms have been developed by NUHAG,<sup>10-12</sup> see Ref. 13 for a survey.

In the two-dimensional case, if both the atom and the TF-lattice are separable then the computation of a Gabor dual atom can be reduced (by tensorizing) to the one-dimensional case. Under certain format restrictions the non-separable two-dimensional problem again can be solved by the efficient one-dimensional algorithms.<sup>14</sup>

For the general two-dimensional case the involved matrices can reach sizes that let us ask for iterative other than (semi-)direct methods. In this paper the computation of dual Gabor atoms is solved without format or dimensional restrictions by an iterative method, based on the Wexler-Raz principle (Theorem 7) and using a new variant of POCS.<sup>15</sup>

### 2.2. POCS method

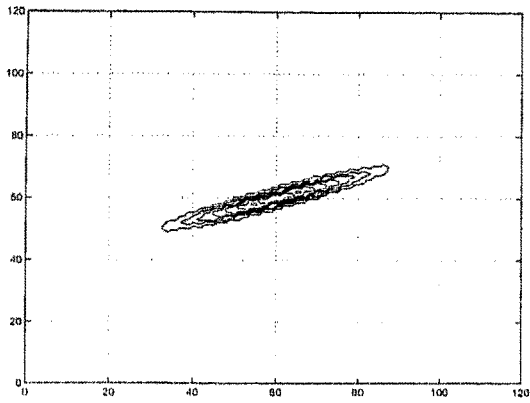
The general POCS<sup>16,17</sup> method is a method to find some element  $\tilde{f}$  in the intersection of closed convex sets, i. e.

$$\tilde{f} \in S := \bigcap_{i \in I} C_i, \quad (26)$$

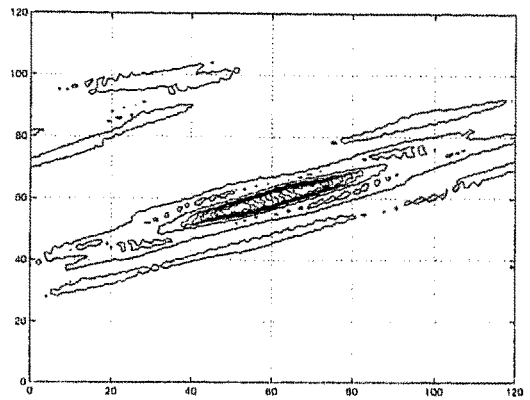
where  $C = (C_i)_{i \in I}$  is a family of convex subsets of a Hilbert space  $\mathcal{H}$ , see Ref. 18. Variants of the POCS method are well known in digital image processing (e. g. in the field of irregular sampling<sup>19,20</sup>). The strategy is to project a starting point  $f_0$  recursively onto each convex subset, one after the other. That is, if  $P_i$  denotes the projection onto  $C_i$  we establish a sequence of approximations by

$$f_{n+1} := P_{i_n} f_n. \quad (27)$$

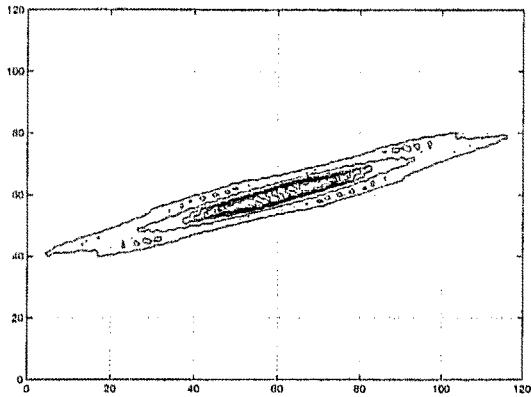
If only a finite number of convex sets is given, which is natural in many relevant problems, the cycle of projections is repeated again after each turn. This cycle of projections is terminated if relevant changes no longer take place.



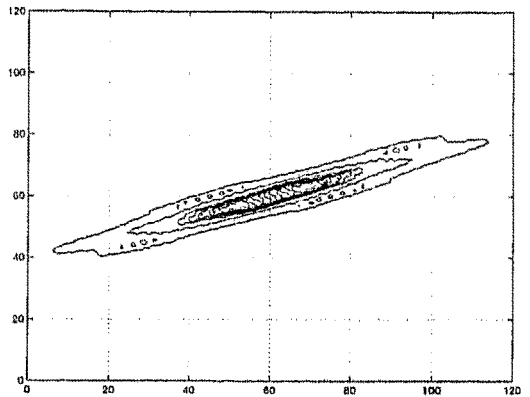
(a) Contour plot of 2D Gabor atom  $g$ .



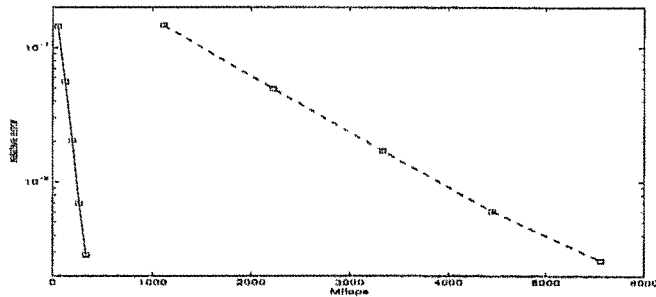
(b) Approximate dual atom after 2 POCS cycles.



(c) Approximate dual atom after 3 POCS cycles.



(d) Dual atom  $\tilde{g}$ .



(e) Performance of the POCS algorithms.

**Figure 4.** Contour plots of a 2D Gabor atom  $g$  (a) and the progression of the POCS based algorithm. The approximate dual atoms (b) and (c) attain the dual  $\tilde{g}$  (d) by a relative (energy-norm) error of 6%, and 2%, respectively. Subfigure (e) is a semi-logarithmic plot of the performance of the POCS variants to the computation of the dual Gabor atom: computational cost (Mflops) versus accuracy (relative energy-norm error, logarithmic scale). The dashed line is related to the new POCS approach to Gabor analysis, the solid line shows the additional reduction of computational cost by using the regularly modulates formula (see Subsection 2.3). Each square indicates one complete POCS cycle.

The use of the resulting element as an approximation to some  $\tilde{f} \in S$  makes POCS an interesting approach to solving Hilbert space problems.

In the concrete situation of a possibly underdetermined system of linear equations there is considerable freedom in choosing the convex sets  $C_i$ , by grouping the equations, either into a large number of small subsystems, or a reduced number of larger systems, as proposed in Ref. 15. As explained there (for the case of unstructured systems) the benefits of a reduced number of iterations are partially lost by the increased cost of calculating a number of (relatively small) pseudo-inverse matrices, as soon as their size went beyond the order of 5-8.

In contrast to the general situation this loss is almost not present if the computational cost of the required projections depends much less on the size of the subsystem, e. g. because of additional structure of the problem allowing the use of FFT-based methods. This is also the case of the variant of POCS proposed below, used to find solutions of the biorthogonality relation with respect to  $\Lambda^\circ$ , because  $\Lambda^\circ$  can be seen as a (finite) union of “horizontal” or “vertical” lattices.

In the present paper, we adapt this POCS approach to the field of Gabor analysis and show how the Wexler-Raz principle (Theorem 7) can be used to establish an efficient variant of POCS for the calculation of some  $\Lambda$ -dual atom  $\gamma$ .

### 2.3. A POCS approach to dual Gabor atoms

We come in this section to the main contribution of the present paper, i. e. the the description of a POCS approach to the computation of dual Gabor atoms making use of the Wexler-Raz condition given by Eq. (22). A direct strategy to obtain a signal  $\gamma$  which satisfies the required biorthogonality relation (over  $\Lambda^\circ$ ) would be the stepwise projection onto each subspace defined by the linear condition

$$\langle U_{\lambda^\circ} g, \gamma \rangle = 0, \quad \forall \lambda^\circ \in \Lambda^\circ, \quad \lambda^\circ \neq 0 = ((0, 0), (0, 0)). \quad (28)$$

and the one defined by the affine condition

$$\langle g, \gamma \rangle = \frac{n_1 n_2}{|\Lambda|}. \quad (29)$$

However, following the idea of Ref. 15 (see Subsection 2.2) a more elaborated approach can be found making use of the special structure of the problem. In the Hilbert space  $\mathcal{H} = \ell^2(\mathcal{G}) = \ell^2(Z_{n_1} \times Z_{n_2}) = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$  the best approximation  $f_T$  of  $f \in \mathcal{H}$  by a linear combination of regularly spaced (arbitrary lattice  $D \triangleleft \mathcal{G}$ ) translates of an atom  $g \in \mathcal{H}$  is given by an easy to handle formula<sup>21,22</sup>:

$$f_T = \mathcal{F}^{-1} \left( \frac{[\hat{f} \cdot \hat{g}]_{D^\perp}}{[|\hat{g}|^2]_{D^\perp}} \hat{g} \right), \quad (30)$$

where  $[h]_{D^\perp}$  denotes periodization of  $h \in \mathcal{H}$  with respect to the orthogonal lattice  $D^\perp \triangleleft \mathcal{G}$ . The fraction (as well as the absolute square operation) is to be taken in a pointwise sense and at positions where to divide by zero the quotient shall evaluate to zero (actually, this “pseudo-division” arises from the regular pseudo-inversion of the involved multiplication operator).

Using this formula, the projection onto the full set of translates corresponding to a fixed modulation parameter can be done efficiently in one step. That is, a series of POCS steps within a cycle is gathered and computed simultaneously. Numerical tests showed that this approach is not only a theoretical improvement but makes the POCS approach a useful tool to find duals of Gabor atoms.

The symmetry between time and frequency (or space and wavenumbers) allows some further improvement, the number of floating point operations of each POCS iteration can be reduced significantly by changing the point of view. The formula for best approximation by translates can simply, but effectively, be modified to a formula for the best approximation  $f_M$  of  $f \in \mathcal{H}$  by regularly spaced (lattice  $D \triangleleft \mathcal{G}$ ) modulates of a function  $g \in \mathcal{H}$ ,

$$f_M = \frac{[f \cdot \bar{g}]_{D^\perp}}{[|g|^2]_{D^\perp}} g, \quad (31)$$

where the fraction is to be taken in the pseudo-division sense as explained above. This new formula no longer acts on the “frequency side” but on the “time side”, that is, the use of the Fourier transform in the formula can be dropped. Thus, not counting rare special cases, the computational effort is reduced, and this fact again is covered by numerical experiments.

## 2.4. The initialization of the POCS algorithm

If the element  $\tilde{f} \in S$  in Eq. (26) is uniquely determined the POCS algorithm converges to  $\tilde{f}$ . If there are more than one element in the intersection  $S = \bigcap_{i \in I} C_i$  then the POCS algorithm converges to some element  $f \in S$ . If the involved convex subsets  $C_i$  are actually linear manifolds (linear or affine subspaces), then a more precise statement holds true. That is, the POCS algorithm then converges to the uniquely determined projection  $P_S f_0$  onto  $S$  of the element  $f_0$  chosen to initialize the POCS algorithm.<sup>23</sup>

This has useful consequences to the Gabor analysis problem where  $S = \Gamma_g$ . If the dual Gabor window  $\gamma \in \Gamma_g$  is uniquely determined the POCS based algorithm will converge to  $\gamma$ . If the Wexler-Raz condition (Eq. (22)) forms an underdetermined system of equations, then  $\Gamma_g$  contains more than one element, that is, the choice of dual windows is not unique. However the solution obtained by means of the POCS algorithm is well defined as follows.

Supposed the POCS method is initialized by the element  $\gamma_0$  and let  $\gamma$  denote the limit element of the algorithm. If  $n = P_{\mathcal{N}_g} \gamma_0$  denotes the projection of  $\gamma_0$  onto the space  $\mathcal{N}_g$  of "trivial" functions as defined in Eq. (24) then  $\gamma_0 = \gamma_0' + n$  where  $n \in \mathcal{N}_g$  and  $\gamma_0' \in \mathcal{N}_g^\perp$ .

Since Eq. (25) implies  $\mathcal{N}_g^\perp = \mathcal{N}_{\tilde{g}}^\perp$  we deduce that the projection of  $\gamma_0'$  onto  $\Gamma_g$  is exactly the canonical dual atom  $\tilde{g}$  as given in Subsection 1.6. Thus we may rewrite the limit element  $\gamma$  as

$$\gamma = P_{\Gamma_g} \gamma_0 = P_{\Gamma_g} (\gamma_0' + n) = \tilde{g} + n. \quad (32)$$

Hence starting the POCS algorithm with the element  $\gamma_0 = \gamma_0' + n$ , the portion  $n \in \mathcal{N}_g$  remains untouched throughout the algorithm whereas the remaining part  $\gamma_0'$  is mapped to  $\tilde{g}$ , its projection onto  $\Gamma_g$ . This property puts us in a position to take influence on the dual atom  $\gamma \in \Gamma_g$  through the choice of the starting element  $\gamma_0$  without explicit computation of the null-space  $\mathcal{N}_g$ . Particularly by taking  $\gamma_0 = g$  we obtain the canonical dual atom  $\tilde{g}$  as defined in Subsection 1.6 and the same result is provided by the choice  $\gamma_0 = 0$  (or, more generally, by any  $\gamma_0 \perp \mathcal{N}_g$ ).

## 3. CONCLUSION

The POCS method is introduced to Gabor analysis as an alternative approach to the determination of the dual Gabor atom using the inverse frame operator. Based on the Wexler-Raz principle, which characterizes dual Gabor windows as the solutions of a system of linear equations (biorthogonality relations) the determination of the canonical dual Gabor atom (for given atom  $g$  and time-frequency lattice  $\Lambda$ ) as the limit of a recursively defined sequence is described. The iterative steps consist of projections onto well-structured affine subspaces which makes the present approach much more efficient than a naive application of the standard POCS-method.

Numerical experiments indicate that the proposed technique is a memory-saving variant of determining dual Gabor atoms, in particular for the higher-dimensional setting, where the fast methods mentioned in Subsection 2.1 face memory limits. Furthermore, it appears to the authors that the POCS approach has potential for further improvement, e. g. due to the fact that the implementation of non-separable time-frequency lattices should not make any serious problems.

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