Irregular Sampling of Wavelet and Short-Time Fourier Transforms

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Abstract. We obtain irregular sampling theorems for the wavelet transform and the short-time Fourier transform. These sampling theorems yield irregular weighted frames for wavelets and Gabor functions with explicit estimates for the frame bounds.

1. Introduction

For fixed \( g \in L^2(\mathbb{R}) \) let \( W_g f \) be the continuous wavelet transform of a function \( f \) defined by

\[
W_g f(x, s) = \frac{1}{s} \int_{\mathbb{R}} \hat{g}\left(\frac{t-x}{s}\right)f(t) \, dt \quad \text{for} \quad x, s \in \mathbb{R}, \quad s > 0,
\]

and let the short-time Fourier transform \( S_g f \) be

\[
S_g f(x, y) = \int_{\mathbb{R}} e^{-iyt} \hat{g}(t-x)f(t) \, dt \quad \text{for} \quad x, y \in \mathbb{R}.
\]

In the first case \( g \) is referred to as the analyzing wavelet or just wavelet, in the second case \( g \) is called the window function or also the Gabor wavelet.

Given an admissible function \( g \), a function \( f \in L^2(\mathbb{R}) \) is uniquely determined by its wavelet transform \( W_g f \) or by its short-time Fourier transform \( S_g f \). Moreover, the theory of square-integrable representations provides explicit inversion formulae to recover \( f \) from \( W_g f \) or \( S_g f \) [14].

Both representations are highly redundant, consequently we seek to reduce this redundancy by a sampling of these transforms.

In wavelet theory wavelets \( g \) can be constructed so that the collection \( g_{jk}(x) = 2^{j/2} g(2^j x - k), j, k \in \mathbb{Z} \), is an orthonormal basis for \( L^2(\mathbb{R}) \) [18]. In this case \( W_g f(2^j k, 2^j) = \langle f, g_{jk} \rangle, j, k \in \mathbb{Z} \), and the sampled values of \( W_g f \) are the coefficients of \( f \) with respect to this orthonormal basis. Thus the sampling is free of redundancy.

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However, in general the wavelet or the window function \( g \) can be a fairly arbitrary function, or \( g \) is given by a specific problem. In this case redundancy of the sampling cannot be completely avoided. Then the question arises as to under which conditions on the sampling set a function \( f \) is uniquely and stably determined by the samples of \( W_g f \) or \( S_g f \) and how to reconstruct \( f \).

The case of regular sampling is well understood. For band-limited \( g \) regular sampling has been treated in detail in [4]. More generally, given an almost arbitrary function \( g \), I. Daubechies has found nearly optimal estimates on the lattice density \((\alpha, \beta)\) so that \( f \) can be completely and stably reconstructed from \( W_g f(\alpha k \beta j, \beta k), j, k \in \mathbb{Z} \), or from \( S_g f(\alpha j, \beta k) \) [3]. Regular sampling is also treated in [2].

If the sampling points are distributed randomly in the plane, then our understanding of irregular sampling is less complete and of a qualitative nature only. It is known that in principle \( f \) is uniquely determined and can be reconstructed from the values of \( W_g f \) or \( S_g f \) on a "sufficiently dense," discrete subset of the plane. Moreover, the theory is not restricted to \( L^2(\mathbb{R}) \), but the size of the samples actually characterizes smoothness and decay properties of functions. See [10] and [11].

For other aspects of sampling of wavelet and short-time Fourier transforms we refer to [1], [6], and [9].

In this note we show how recent quantitative results on irregular sampling of band-limited functions also yield irregular sampling theorems for wavelet and short-time Fourier transforms. This extends to the program of [3] and [4] to irregular sampling. The novelty lies

(a) in the explicit constants and error estimates,
(b) in the introduction of adaptive weights to compensate for local variations of the sampling density, and
(c) in an improved reconstruction algorithm.

2. Results

Suppose that the wavelet \( g \) is band-limited, that is, \( \text{supp} \ \tilde{g} \subseteq [-\Omega, \Omega] \) for some \( \Omega > 0 \). With the notation \( \tilde{g}(x) = \overline{g(-x)} \) and \( g_s(x) = s^{-1} g(x/s) \), we see that \( W_g f(x, s) = f * \tilde{g}_s(x) \) is a convolution. Consequently, for \( s > 0 \) fixed, \( W_g f \) is band-limited with \( \text{supp}(f * \tilde{g}_s) \subseteq [-\Omega/s, \Omega/s] \). To make use of this information, we first select a sequence of samples \( (s_j)_{j \in \mathbb{Z}} \) in the scaling parameter \( s \), and then sample \( W_g f(\cdot, s_j) \) irregularly at \( x_{jk}, k \in \mathbb{Z} \). In what follows we consider sampling sets of the form \( \{(x_{jk}, s_j), j, k \in \mathbb{Z} \} \subseteq \mathbb{R} \times \mathbb{R}^+ \) for \( W_g f \) and of the form \( \{(x_{jk}, y_j), j, k \in \mathbb{Z} \} \subseteq \mathbb{R} \times \mathbb{R} \) for \( S_g f \). The sampling sequences will always be arranged by magnitude, so that \( s_j < s_{j+1} \), \( x_{jk} < x_{j,k+1} \), and \( y_j < y_{j+1} \) for all \( j, k \in \mathbb{Z} \). To avoid accumulation points of the sampling set, we also assume that \( \lim_{j \to \pm \infty} s_j = \pm \infty \), \( \lim_{k \to \pm \infty} y_j = \pm \infty \), and \( \lim_{k \to \pm \infty} x_{jk} = \pm \infty \), and call these sequences admissible.

For regular sampling of \( W_g f \) a necessary condition on the wavelet \( g \) and the sampling density in the scaling parameter is the inequality

\[
0 < A \leq \sum_j |\tilde{g}(\beta^j \eta)|^2 \leq B, \quad \forall \eta \neq 0,
\]
for some constants $\beta > 1$, $A$, $B > 0$. The corresponding condition for irregular sampling is more complicated and stated in the following lemma. Note that the assumptions on $g$ are the same as in [4].

**Lemma 1.**

1. Suppose that $f \in L^\infty(\mathbb{R})$ is supported in $[-\Omega, \Omega]$ and bounded away from 0 on $[-\gamma, \gamma]$ for some $\gamma < 2\Omega$. For any admissible sequence $(y_j)_{j \in \mathbb{Z}}$, define

\[
\begin{aligned}
\begin{bmatrix}
a \\
b
\end{bmatrix} = \begin{cases}
\inf_{\xi \in \mathbb{R}} & \sum_{j \in \mathbb{Z}} \frac{y_{j+1} - y_{j-1}}{2} |f(\xi - y_j)|^2,
\sup_{\xi \in \mathbb{R}} & \end{cases}
\end{aligned}
\]

If $\sup_{j \in \mathbb{Z}}(y_{j+1} - y_j) \leq \gamma < 2\Omega$, then

\[
a \geq \frac{\gamma}{\gamma / 2} \inf_{|\xi| \leq \gamma/2} |f(\xi)|^2 > 0 \quad \text{and} \quad b \leq 4 \|f\|_{\infty}^2 \Omega.
\]

2. Suppose that $f \in L^\infty(\mathbb{R})$, $\text{supp} \ f \subseteq [-\Omega, \omega] \cup [\omega, \Omega]$, and that $f$ is bounded away from 0 on $\{\eta: \sqrt{\omega \Omega / \gamma} < |\eta| < \sqrt{\omega \Omega \gamma}\}$ for some $\gamma, 1 < \gamma < \Omega/\omega$. For any admissible sequence $(s_j)_{j \in \mathbb{Z}}$, $s_j > 0$, define

\[
\begin{aligned}
\begin{bmatrix}
a \\
b
\end{bmatrix} = \begin{cases}
\inf_{\eta \in \mathbb{R}} & \sum_{j \in \mathbb{Z}} \frac{1}{2} \ln \frac{s_{j+1}}{s_{j-1}} |f(s_j \eta)|^2,
\sup_{\eta \in \mathbb{R}} & \end{cases}
\end{aligned}
\]

If $\sup_{j \in \mathbb{Z}}(s_{j+1}/s_j) \leq \gamma < \Omega/\omega$, then

\[
a \geq \frac{\ln \gamma}{2} \inf_{\sqrt{\omega \Omega / \gamma} < |\eta| < \sqrt{\omega \Omega \gamma}} |f(\eta)|^2 > 0 \quad \text{and} \quad b \leq 2 \|f\|_{\infty}^2 \ln \frac{\Omega}{\omega}.
\]

Now we can formulate a quantitative irregular sampling theorem for wavelet transforms.

**Theorem 1.** Suppose that $g \in L^2(\mathbb{R})$ is band-limited with $\text{supp} \ \hat{g} = [-\Omega, -\omega] \cup [\omega, \Omega]$, $\hat{g} \in L^\infty(\mathbb{R})$, and $\inf_{|\theta| > 0} |\hat{g}(\theta)| > 0$ for $\{\eta: \sqrt{\omega \Omega / \gamma} < |\eta| < \sqrt{\omega \Omega \gamma}\}$ and for some $\gamma, 1 < \gamma < \Omega/\omega$. If $(s_j)_{j \in \mathbb{Z}}$ and $(x_j)_{j,k \in \mathbb{Z}}$ are any admissible sequences satisfying

\[
\sup_{j \in \mathbb{Z}} \frac{s_{j+1}}{s_j} \leq \gamma < \frac{\Omega}{\omega}
\]

and

\[
\delta := \sup_{j,k \in \mathbb{Z}} \frac{x_{j,k+1} - x_{jk}}{s_j} < \frac{\pi}{\Omega}
\]
and if $a, b > 0$ are the constants defined in Lemma 1.2, then

$$
(9) \quad a \left(1 - \frac{\delta \Omega}{\pi}\right) \|f\|_2^2 \leq \sum_{j,k \in \mathbb{Z}} \frac{x_{j,k+1} - x_{j,k-1}}{4} \ln \frac{s_{j+1}}{s_{j-1}} \left|W_g f(x_{jk}, s_j)\right|^2 \\
\quad \leq b \left(1 + \frac{\delta \Omega}{\pi}\right) \|f\|_2^2.
$$

In particular, $f$ is uniquely determined by the irregular samples of the wavelet transform.

The left inequality in (9) expresses that the sampling is stable.

The role of the constants in (9) is illustrated in the following corollary. For this we define the constants

$$
(10) \quad C = \frac{2\pi^2}{a(\pi - \delta \Omega)^2 + b(\pi + \delta \Omega)^2}
$$

and

$$
(11) \quad \rho = \frac{b(\pi + \delta \Omega)^2 - a(\pi - \delta \Omega)^2}{b(\pi + \delta \Omega)^2 + a(\pi - \delta \Omega)^2}, \quad \sigma = \frac{\rho}{1 + \sqrt{1 - \rho^2}},
$$

and the operator

$$
(12) \quad Sf(x) = C \sum_{j,k \in \mathbb{Z}} \frac{x_{j,k+1} - x_{j,k-1}}{4} \ln \frac{s_{j+1}}{s_{j-1}} W_g f(x_{jk}, s_j)s_j^{-1} g\left(\frac{x - x_{jk}}{s_j}\right).
$$

Then $S$ is bounded on $L^2(\mathbb{R})$ by the right inequality of (9) and we have the following reconstruction algorithm for $f$.

**Corollary 1.** Under the hypothesis of Theorem 1 any $f \in L^2(\mathbb{R})$ can be reconstructed from $W_g f(x_{jk}, s_j)$ as follows: set $f_0 = 0, f_1 = Sf, \lambda_1 = 2$, and

$$
(13) \quad f_n = \lambda_n(f_{n-1} - f_{n-2} + S(f - f_{n-1})) + f_{n-2}, \quad \lambda_n = \left(1 - \frac{\rho^2}{4} \lambda_{n-1}\right)^{-1} \text{ for } n \geq 2.
$$

Then $f_n$ converges to $f$ in $L^2(\mathbb{R})$ and

$$
(14) \quad \|f - f_n\|_2 \leq \frac{2\sigma^n}{1 + \sigma^{2n}} \|f\|_2.
$$

Since $f_1 = Sf$ contains only the samples of $W_g f$, this is indeed a reconstruction of $f$ from its irregular samples.

A slight modification of the arguments shows that irregular sampling of $W_g f$ can also be used to characterize Hilbert spaces of smooth functions or of
distributions. For this let \( \omega > 0 \) be a weight function that satisfies

\[
\sup_{s_0 \leq \xi \leq s_\Omega} w(\xi) \leq D \inf_{s_0 \leq \xi \leq s_\Omega} w(\xi) \quad \text{for all} \quad s > 0
\]

for some constant \( D > 0 \). Let \( \mathcal{L}^2_w \) be the Hilbert space defined by the (quasi-) norm \( \| f \|_w = (\int_{\mathbb{R}} |f(\xi)|^2 w(\xi) \, d\xi)^{1/2} \). Interesting examples are the weights \( w(\xi) = |\xi|^\alpha \) and \( w(\xi) = (1 + |\xi|^2)^{\alpha/2} \), \( \alpha \in \mathbb{R} \), which lead to the (homogeneous) Sobolev spaces \( W^2_\alpha \) with the equivalent norm \( \| f \|_{W^2_\alpha} = \| (-\Delta)^{\alpha/2} f \|_2 \) and the Bessel potential spaces \( H^\alpha \) with equivalent norm \( \| f \|_{H^\alpha} = \| (I - \Delta)^{\alpha/2} f \|_2 \). In the first example \( \| \cdot \|_{W^2_\alpha} \) is only a quasi-norm and \( f \in W^2_\alpha \) is only defined up to polynomials. Since this does not affect the arguments, we leave the remedies of those technical subtleties to the reader and refer to \([18]\) and \([19]\) for a thorough discussion.

**Theorem 2.** Assume that the wavelet \( g \in L^2(\mathbb{R}) \) and the sampling set \( \{(x_{jk}, s_j), j, k \in \mathbb{Z}\} \) satisfy the assumptions of Theorem 1 and that \( w \) is a weight function as in (15). Then

\[
\frac{a}{D} \left(1 - \frac{\delta\Omega}{\pi}\right)^2 \| f \|_w^2 \leq \sum_{j, k \in \mathbb{Z}} \frac{x_{j,k+1} - x_{j,k-1}}{4} \ln \frac{s_{j+1}}{s_{j-1}} \frac{\sqrt{\omega\Omega}}{s_j} |W_g f(x_{jk}, s_j)|^2
\]

\[
\leq bD \left(1 + \frac{\delta\Omega}{\pi}\right)^2 \| f \|_w^2.
\]

In particular, \( f \in W^2_\alpha \) if and only if

\[
d_\alpha a \left(1 - \frac{\delta\Omega}{\pi}\right)^2 \| f \|_{W^2_\alpha} \leq \sum_{j, k \in \mathbb{Z}} \frac{x_{j,k+1} - x_{j,k-1}}{4} \ln \frac{s_{j+1}}{s_{j-1}} s_j^{-\alpha} |W_g f(x_{jk}, s_j)|^2
\]

\[
\leq D_\alpha b \left(1 + \frac{\delta\Omega}{\pi}\right)^2 \| f \|_{W^2_\alpha},
\]

where \( d_\alpha = \min(\omega^{-\alpha}, \Omega^{-\alpha}) \) and \( D_\alpha = \max(\omega^{-\alpha}, \Omega^{-\alpha}) \).

If the constants \( C \) and \( \rho \) in (10) and (11) are defined appropriately (see Proposition 5), then the algorithm of Corollary 1 also converges in the norm of \( \mathcal{L}^2_w \).

The analog of Theorem 1 for the short-time Fourier transform reads as follows.

**Theorem 3.** Suppose that \( g \in L^2(\mathbb{R}) \) is band-limited with \( \text{supp} \, \hat{g} \subseteq [-\Omega, \Omega] \), \( \hat{g} \in L^\infty(\mathbb{R}) \), and \( \inf_{\xi} |\hat{g}(\xi)| > 0 \) for \( \{\xi : |\xi| \leq \gamma/2\} \) and some \( \gamma < 2\Omega \). Let \( (y_{jk})_{j \in \mathbb{Z}} \) and \( (x_{jk})_{k \in \mathbb{Z}}, f \in \mathbb{Z} \), be admissible sequences so that

\[
\sup_{j \in \mathbb{Z}} (y_{j+1} - y_j) \leq \gamma < 2\Omega \quad \text{and} \quad \delta := \sup_{j, k \in \mathbb{Z}} (x_{j,k+1} - x_{jk}) < \frac{\pi}{\Omega}
\]
and let \( a, b > 0 \) be defined as in Lemma 1.1. Then
\[
(18) \quad a \left( 1 - \frac{\delta \Omega}{\pi} \right)^2 \|f\|_2^2 \leq \sum_{j,k \in \mathbb{Z}} \frac{1}{4} (x_{j,k+1} - x_{j,k-1}) (y_{j+1} - y_{j-1}) |S_g f(x_{j,k}, y_j)|^2
\]
\[
\leq b \left( 1 + \frac{\delta \Omega}{\pi} \right)^2 \|f\|_2^2.
\]

This means that \( f \) is uniquely and stably determined by the samples \( S_g f(x_{j,k}, y_j) \). An iterative algorithm can be formulated as in Corollary 1 using the operator
\[
Sf(x) = C \sum_{j,k \in \mathbb{Z}} \frac{1}{4} (x_{j,k+1} - x_{j,k-1}) (y_{j+1} - y_{j-1}) S_g f(x_{j,k}, y_j)e^{iyy}g(x - x_{j,k}).
\]

By means of a symmetry property of \( S_g f \) we also obtain a version of irregular sampling for compactly supported window functions \( g \).

**Corollary 2.** Under the hypothesis of Theorem 2 the following inequalities hold for all \( f \in L^2(\mathbb{R}) \):
\[
(19) \quad a \left( 1 - \frac{\delta \Omega}{\pi} \right)^2 \|f\|_2^2 \leq \sum_{j,k \in \mathbb{Z}} \frac{1}{4} (x_{j,k+1} - x_{j,k-1}) (y_{j+1} - y_{j-1}) |S_g f(y_j, x_j)|^2
\]
\[
\leq b \left( 1 + \frac{\delta \Omega}{\pi} \right)^2 \|f\|_2^2.
\]

This is a quantitative irregular sampling theorem for the short-time Fourier transform with the compactly supported window function \( \hat{g} \).

In the remainder of this article we provide the required background on irregular sampling of band-limited functions and on frames. The theorems then follow by a combination of this information.

3. Irregular Sampling of Band-Limited Functions

We need a quantitative version of irregular sampling for band-limited functions. The following theorem is a consequence of a sampling result in [12], and was explicitly stated in Corollary 1 of [13]. We include a self-contained direct proof which is much shorter than the original one. For more information on irregular sampling of band-limited functions and detailed references see [1], [7], and [12].

**Theorem 4.** Suppose that \( f \in L^2(\mathbb{R}) \) and \( \text{supp } \hat{f} \subseteq [-\Omega, \Omega] \). If \( (x_k)_{k \in \mathbb{Z}} \) is any sequence such that the maximal gap length \( \delta \) satisfies
\[
(20) \quad \delta := \sup_{k \in \mathbb{Z}} (x_{k+1} - x_k) \frac{\pi}{\Omega},
\]
then
\[
(21) \quad \left( 1 - \frac{\delta \Omega}{\pi} \right)^2 \|f\|_2^2 \leq \sum_{k \in \mathbb{Z}} \frac{x_{k+1} - x_{k-1}}{2} |f(x_k)|^2 \leq \left( 1 + \frac{\delta \Omega}{\pi} \right)^2 \|f\|_2^2.
\]
Proof. Let \( y_j = (x_j + x_{j+1})/2 \) and let \( \chi_j \) be the characteristic function of the interval \([y_{j-1}, y_j]\). Then

\[
\left\| \sum_j f(x_j)\chi_j \right\|_2^2 = \sum_j \frac{x_{j+1} - x_{j-1}}{2} |f(x_j)|^2.
\]

Since \( \|f\|_2 - \|\sum_j f(x_j)\chi_j\|_2 \leq \|f - \sum_j f(x_j)\chi_j\|_2 \), the estimates in (21) will follow from \( R(f) := \|f - \sum_j f(x_j)\chi_j\|_2 < (\delta \Omega/\pi)\|f\|_2 \). For an estimate of \( R(f) \) we use \( \chi_j \cdot \chi_k = 0 \) for \( j \neq k \) and Wirtinger's inequality [17] in the form

\[
\int_a^b |f(x)|^2 \, dx \leq \frac{4}{\pi^2} \max\{(b - c)^2, (c - a)^2\} \int_a^b |f'(x)|^2 \, dx
\]

for \( f, f' \in L^2(a, b) \) and \( f(c) = 0, a < c < b \). Then we obtain

\[
R(f)^2 = \sum_j \int_{y_{j-1}}^{y_j} |f(x) - f(x_j)|^2 \, dx
\]

\[
\leq \frac{4}{\pi^2} \sum_j \max\{(y_j - x_j)^2, (x_j - y_{j-1})^2\} \int_{y_{j-1}}^{y_j} |f'(x)|^2 \, dx
\]

\[
\leq \frac{4}{\pi^2} \left( \frac{\delta}{2} \right)^2 \int_{y_{j-1}}^{y_j} |f'(x)|^2 \, dx
\]

\[
= \frac{\delta^2}{\pi^2} \|f'\|_2^2.
\]

Finally, Bernstein's inequality yields

\[
R(f)^2 = \sum_j \int_{y_{j-1}}^{y_j} |f(x) - f(x_j)|^2 \, dx \leq \left( \frac{\delta \Omega}{\pi} \right)^2 \|f\|_2^2
\]

as desired. \( \blacksquare \)

4. Proofs of the Theorems

We first show that the constants \( a, b \) defined in (3) and (5) are well defined. The characteristic function of a set \( A \) is denoted by \( \chi_A \).

Proof of Lemma 1. (a) The upper bound is finite. Since \( \text{supp } f \subseteq [-\Omega, \Omega] \) and since \( f \) is bounded, we obtain

\[
(22) \quad \sum_{j \in \mathbb{Z}} \frac{y_{j+1} - y_{j-1}}{2} |f(\xi - y_j)|^2 \leq \frac{1}{2} \|f\|_2^2 \sum_{j \in \mathbb{Z}} (y_{j+1} - y_{j-1})\chi_{[-\Omega+\gamma, \Omega+\gamma]}(\xi).
\]

For fixed \( \xi \in \mathbb{R} \) the latter sum runs over the subset \( I_{\xi} = \{ j \in \mathbb{Z} | \xi - \Omega \leq y_j \leq \xi + \Omega \} \).

Let \( m(\xi) \) and \( M(\xi) \) be the integers defined by

\[
y_{m(\xi)} - 1 < \xi - \Omega \leq y_{m(\xi)} \quad \text{and} \quad y_{M(\xi)} \leq \xi + \Omega < y_{M(\xi) + 1}.
\]
Since \( y_j \) is an increasing sequence, \( I_\xi \) is just the segment of integers 
\[
\{ j | m(\xi) \leq j \leq M(\xi) \}.
\]

By writing \( y_{j+1} - y_{j-1} = (y_{j+1} + y_j) - (y_j + y_{j-1}) \), the sum in (22) becomes

\[
\sum_{j \in I_\xi} (y_{j+1} - y_{j-1}) = y_{M(\xi)} + y_{M(\xi)} - (y_{m(\xi)} + y_{m(\xi) - 1}).
\]

By definition we have \( y_{M(\xi)} - y_{m(\xi)} \leq 2\Omega \) independent of \( \xi \in \mathbb{R} \), and by assumption \( y_{M(\xi) + 1} \leq y_{M(\xi)} + \gamma \) and \( y_{m(\xi) - 1} \geq y_{m(\xi)} - \gamma \). Therefore

\[
\sum_{j \in I_\xi} (y_{j+1} - y_{j-1}) \leq 4\Omega + 2\gamma \leq 8\Omega
\]

shows that \( b \) is finite.

(b) For the lower bound in (4) we estimate

\[
\sum_{j \in \mathbb{Z}} \frac{y_{j+1} - y_{j-1}}{2} |f(\xi - y_j)|^2 \geq \frac{1}{2} \inf_{|\xi| \leq \gamma/2} |f(\xi)|^2 \sum_{j \in \mathbb{Z}} (y_{j+1} - y_{j-1}) \chi_{[-\gamma/2+y_j, \gamma/2+y_j]}(\xi).
\]

After defining the integers \( n(\xi) \) and \( N(\xi) \) by \( y_{n(\xi) - 1} < \xi - \gamma/2 \leq y_{n(\xi)} \) and \( y_{N(\xi)} \leq \xi + \gamma/2 < y_{N(\xi) + 1} \), we obtain, as before,

\[
\sum (y_{j+1} - y_{j-1}) \chi_{[-\gamma/2+y_j, \gamma/2+y_j]}(\xi) = y_{N(\xi) + 1} + y_{N(\xi)} - y_{n(\xi)} - y_{n(\xi) - 1}
\]

\[
\geq y_{N(\xi) + 1} - y_{n(\xi) - 1} > \left( y + \xi \right) - \left( \xi - \gamma \right) = \gamma
\]

and thus the desired lower estimate in (4), since \( f \) is bounded below on \([-\gamma/2, \gamma/2]\).

(c) The second part of the lemma follows by a simple substitution. Since \( s_j > 0 \), \( \forall j \), we can treat \( \eta > 0 \) and \( \eta < 0 \) separately. Set \( \xi = \ln \eta, y_j = \ln s_j \), and \( h(\xi) = f(e^{\ln \eta}) \). Then \( \inf_{\xi} |h(\xi)| > 0 \) for (ln \( \omega \Omega - \ln \gamma)/2 \leq \xi \leq (\ln \omega \Omega + \ln \gamma)/2 \) and

\[
\sum_{j \in \mathbb{Z}} \frac{1}{2} \ln \frac{s_{j+1}}{s_{j-1}} |f(s_j \eta)|^2 = \sum_{j \in \mathbb{Z}} \frac{y_{j+1} - y_{j-1}}{2} h(y_j + \xi).
\]

Now we can apply part 1 of the lemma.

**Proof of Theorem 1.** We first reconstruct \( W_g f(x, s_j) \) from \( W_g f(x_{jk}, s_j) \) for all \( j \in \mathbb{Z} \).

Since \( W_g f(x, s_j) = f * \tilde{g}_{s_j} \) is band-limited with \( \text{supp}(f * \tilde{g}_{s_j})^c \subseteq [-\Omega_j, \Omega_j] \) with \( \Omega_j = \Omega/s_j \), by Theorem 4 we need the condition \( \delta_j = \sup_{k \in \mathbb{Z}} (x_{j, k+1} - x_{j, k}) < \pi/\Omega_j = \pi s_j/\Omega \) to reconstruct \( W_g f(x, s_j) \). However, this is precisely condition (8).
Since \( \delta \Omega_j = \delta \Omega \) is independent of \( j \), we infer from (21) that

\[
\left(1 - \frac{\delta \Omega}{\pi}\right)^2 \| f \ast \tilde{g}_j \|_2^2 \leq \sum_{k \in \mathbb{Z}} \frac{\chi_{j,k+1} - \chi_{j,k-1}}{2} \left| W_g(f(x_{jk}, s_j)\right|^2 \leq \left(1 + \frac{\delta \Omega}{\pi}\right)^2 \| f \ast \tilde{g}_j \|_2^2.
\]

Next we multiply each term by \( \frac{1}{2} \ln(s_{j+1}/s_{j-1}) \) and sum over \( j \). This gives

\[
\left(1 - \frac{\delta \Omega}{\pi}\right)^2 \sum_{j \in \mathbb{Z}} \frac{1}{2} \ln \frac{s_{j+1}}{s_{j-1}} \| f \ast \tilde{g}_j \|_2^2 \leq \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \frac{1}{2} \ln(s_{j,k+1} - s_{j,k-1}) \| W_g(f(x_{jk}, s_j)\right|^2 \leq \left(1 + \frac{\delta \Omega}{\pi}\right)^2 \sum_{j \in \mathbb{Z}} \frac{1}{2} \ln \frac{s_{j+1}}{s_{j-1}} \| f \ast \tilde{g}_j \|_2^2.
\]

Using Plancherel's theorem gives

\[
\sum_{j \in \mathbb{Z}} \frac{1}{2} \ln \frac{s_{j+1}}{s_{j-1}} \| f \ast \tilde{g}_j \|_2^2 = \sum_{j \in \mathbb{Z}} \frac{1}{2} \ln \frac{s_{j+1}}{s_{j-1}} \| f \ast \tilde{g}_j \|_2^2 = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \left(\sum_{j \in \mathbb{Z}} \frac{1}{2} \ln \frac{s_{j+1}}{s_{j-1}} |\hat{g}(s_j \xi)|^2\right) d\xi.
\]

By Lemma 1 applied to \( \hat{g} \) this expression is bounded below by \( a\|f\|_2^2 \) and from above by \( b\|f\|_2^2 \). Inserting this into (27) gives the desired norm equivalence.

\section*{Proof of Theorem 2.}

Since \( \hat{f} \) is locally in \( L^2 \) by definition, we have \( f \ast \tilde{g}_e \in L^2(\mathbb{R}) \). Then Theorem 4 is applicable and (26) carries over unchanged. Next we observe that (15) immediately implies

\[
\frac{1}{D} w(\xi) \leq w(\eta) \leq Dw(\xi)
\]

for all \( \xi, \eta \in \{\xi: \omega/s_j \leq |\xi| \leq \Omega/s_j\} = \text{supp } \tilde{g}_s, \) in particular for \( \eta = \sqrt{\omega \Omega}/s_j \). Therefore

\[
\frac{1}{D} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \left|\hat{g}(s_j \xi)\right|^2 w(\xi) d\xi \leq w(\sqrt{\omega \Omega}/s_j) \| f \ast \tilde{g}_s \|_2^2 \leq D \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \left|\hat{g}(s_j \xi)\right|^2 w(\xi) d\xi.
\]
Combining the estimates (26), (30), and Lemma 1, we obtain

\[
\sum_{j, k \in \mathbb{Z}} \frac{x_{j, k+1} - x_{j, k-1}}{4} \ln \frac{S_{j+1}}{s_{j-1}} W_j f(x_{jk}, s_j)^2 \leq \frac{1}{D} \left( 1 - \frac{\delta \Omega}{\pi} \right)^2 \sum_{j \in \mathbb{Z}} \frac{1}{2} \ln \frac{S_{j+1}}{s_{j-1}} W_j f^* \tilde{g}_j^2 \geq a \left( 1 - \frac{\delta \Omega}{\pi} \right)^2 \| f \|_w^2.
\]

The upper estimate in (16) is similar.

**Proof of Theorem 3.** Inequality (18) is proved similarly and we just mention the necessary modifications. Let \( M_y g(t) = e^{ixy} g(t) \). Then \( S_y f(x, y) = M_{-y} f \ast \hat{g}(x) \) is band-limited with \( \text{supp}(M_{-y} f \ast \hat{g})^\wedge \subseteq \text{supp} \hat{g} = [-\Omega, \Omega] \). Since

\[
\sup_{k \in \mathbb{Z}} (x_{j, k+1} - x_{j, k}) = \delta < \pi/\Omega,
\]

the short-time Fourier transform \( S_y f(x, y) \) is uniquely determined by \( S_y f(x_{jk}, y_j) \), \( k \in \mathbb{Z} \), for each \( j \in \mathbb{Z} \), and by Theorem 4 we have

\[
\left( 1 - \frac{\delta \Omega}{\pi} \right)^2 \| M_{-y} f \ast \hat{g} \|_2^2 \leq \sum_{k \in \mathbb{Z}} \frac{x_{j, k+1} - x_{j, k-1}}{2} |S_y f(x_{jk}, y_j)|^2 \leq \left( 1 + \frac{\delta \Omega}{\pi} \right)^2 \| M_{-y} f \ast \hat{g} \|_2^2.
\]

Multiplying by the weights \( (y_{j+1} - y_{j-1})/2 \) and summing the right- and left-hand sides over \( j \) yields

\[
\sum_{j \in \mathbb{Z}} \frac{y_{j+1} - y_{j-1}}{2} \| M_{-y} f \ast \hat{g} \|_2^2 = \sum_{j} \frac{y_{j+1} - y_{j-1}}{2} \int_{\mathbb{R}} |\hat{f}(\xi + y_j)|^2 |\hat{g}(\xi)|^2 d\xi = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \left( \sum_{j \in \mathbb{Z}} \frac{y_{j+1} - y_{j-1}}{2} |\hat{g}(\xi - y_j)|^2 \right) d\xi.
\]

Then the inequalities (18) follow as before.

Corollary 2 follows from the relation \( S_y f(x, y) = e^{-ixy} S_y \hat{f}(y, -x) \).
5. Frames [5], [21]

A sequence \(\{e_n, n \in \mathbb{Z}\}\) in a (separable) Hilbert space \(\mathcal{H}\) is called a frame if there are two constants \(A, B > 0\), so that, for all \(f \in \mathcal{H}\),

\[
A \|f\|_\mathcal{H}^2 \leq \sum_n |\langle e_n, f \rangle|^2 \leq B \|f\|_\mathcal{H}^2.
\]

The constants \(A\) and \(B\) are called the frame bounds; they determine the quality of the reconstruction of \(f\). The usual argument for the reconstruction of \(f\) from \(\langle f, e_n \rangle\) goes as follows [5], [21]: Define the frame operator \(S\) as

\[
Sf = \frac{2}{A + B} \sum_{n \in \mathbb{Z}} \langle f, e_n \rangle e_n.
\]

Then \(S\) is a positive, invertible operator and

\[
\|Id - S\| \leq \frac{B - A}{B + A} = \rho < 1
\]

by a simple computation. Denoting \(f_n = \sum_{k=0}^n (Id - S)^k Sf\), we see easily that

\[
f_{n+1} = Sf + \sum_{k=1}^{n+1} (Id - S)^k Sf = Sf + (Id - S)f_n = f_n + S(f - f_n)
\]

and

\[
\|f - f_n\|_\mathcal{H} \leq \rho^{n+1}\|f\|_\mathcal{H}.
\]

This simple iteration scheme suffices when \(B/A \approx 1\). However, as the estimates in [3] or in Theorem 1 show, \(B/A\) is large in many cases. Then the basic iteration (35) converges extremely slowly and is not suitable. Instead, an appropriate acceleration method must be used for a more efficient reconstruction. As an example we present the method of Chebyshev acceleration, see, e.g., [20] and [16], applied to frames. It is chosen here for its simplicity and because it requires explicit estimates for the frame bounds to perform the algorithm. A thorough discussion of improved reconstruction methods for "ill-conditioned" frames must include conjugate gradient methods and other adaptive procedures. For this we may refer to the standard references in numerical analysis, e.g., [20], [16], and [15]. Our point here is to establish a connection between frame reconstruction and numerical acceleration methods and to bring some well-developed numerical tools to the attention of wavelet analysts.

**Proposition 5.** Let \(\{e_n, n \in \mathbb{Z}\} \subseteq \mathcal{H}\) be a frame with frame bounds \(A, B > 0\) and \(\rho = (B - A)/(B + A)\), and let \(S\) be the associated frame operator. Then the sequence \(f_n\), defined by \(f_0 = 0, f_1 = Sf, \lambda_1 = 2, \) and

\[
f_n = \lambda_n(f_{n-1} - f_{n-2} + S(f - f_{n-1})) + f_{n-2},
\]


\[ \lambda_n = \left( 1 - \frac{\rho^2}{4} \lambda_{n-1} \right)^{-1} \quad \text{for} \quad n \geq 2 \]

converges to \( f \) in \( \mathcal{H} \) and satisfies the error estimate

\[ \| f - f_n \|_\mathcal{H} \leq \frac{2\sigma^n}{1 + \sigma^{2n}} \| f \|_\mathcal{H}, \]

where

\[ \sigma = \frac{\rho}{1 + \sqrt{1 - \rho^2}} = \frac{B - A}{B + A + 2\sqrt{AB}} = \frac{\sqrt{B} - \sqrt{A}}{\sqrt{B} + \sqrt{A}}. \]

**Proof.** Let \( C_n(x) \) be the sequence of Chebyshev polynomials defined recursively by \( C_0(x) = 1, C_1(x) = x, \) and

\[ C_n(x) = 2xC_{n-1}(x) - C_{n-2}(x), \]

and let \( P_n(x) = C_n(x/\rho)/C_n(1/\rho) \) be the polynomials normalized on the interval \([-\rho, \rho]\). Using (39) it is not difficult to see the recursion

\[ P_n(x) = \frac{2C_{n-1}(1/\rho)}{\rho C_n(1/\rho)} xP_{n-1}(x) - \frac{C_{n-2}(1/\rho)}{C_n(1/\rho)} P_{n-2}(x) \]

and

\[ \max_{|x| \leq \rho} |P_n(x)| = \frac{1}{C_n(1/\rho)} = \frac{2\sigma^n}{1 + \sigma^{2n}}. \]

Next we verify by induction that \( \lambda_n = (2/\rho)C_{n-1}(1/\rho)/C_n(1/\rho) \) and by (39) that \(-C_{n-2}(1/\rho)/C_n(1/\rho) = 1 - \lambda_n\).

**Claim.**

\[ f - f_n = P_n(Id - S)f. \]

This is true for \( n = 0 \) and \( n = 1 \), and follows for \( n \geq 2 \) by induction, using (40) and the \( \lambda_n \)'s.

\[ P_n(Id - S)f = \lambda_n(Id - S)P_{n-1}(Id - S)f + (1 - \lambda_n)P_{n-2}(Id - S)f \]

\[ = \lambda_n(Id - S)(f - f_{n-1}) + (1 - \lambda_n)(f - f_{n-2}) \]

\[ = f - \lambda_n(f_{n-1} - f_{n-2} + S(f - f_{n-1})) - f_{n-2} \]

\[ = f - f_n. \]
Since $Id - S$ is a self-adjoint operator with spectrum in $[-\rho, \rho]$ by (34), the error estimate follows from the spectral theorem, (42), and (41):

$$
\| f - f_n \|_{\mathcal{H}} = \| P_n (Id - S) f \|_{\mathcal{H}} \leq \max_{|x| \leq \rho} |P_n(x)| \cdot \| f \|_{\mathcal{H}} = \frac{2\sigma^n}{1 + \sigma^2n} \| f \|_{\mathcal{H}}.
$$

We now see that Theorems 1 and 3 express the fact that the collections

$$
\left\{ \frac{1}{2} \left( (x_{j,k+1} - x_{j,k-1}) \ln \frac{s_{j+1}}{s_{j-1}} \right)^{1/2} s_{j-1}^{-1} g\left( \frac{x - x_{jk}}{s_j} \right), j, k \in \mathbb{Z} \right\}
$$

and

$$
\left\{ \frac{1}{2} ((x_{j,k+1} - x_{j,k-1})(y_{j+1} - y_{j-1}))^{1/2} e^{by_j} g(x - x_{jk}), j, k \in \mathbb{Z} \right\}
$$

are frames for $L^2(\mathbb{R})$ with frame bounds $A = a(1 - \delta\Omega/\pi)^2$ and $B = b(1 + \delta\Omega/\pi)^2$. The reconstruction algorithm and the error estimate of Corollary 1 are thus immediate consequences of the proposition.

### 6. Remarks

1. Once $W_g f(x, s_j)$ has been reconstructed from the samples $W_g f(x_{jk}, s_j)$, $j, k \in \mathbb{Z}$, the reconstruction of $f$ can also be written in a closed form as a deconvolution. For this define a sequence of band-limited functions $h_j$ by

$$
\hat{h}_j(\xi) = \hat{g}(s_j \xi) \left( \sum_{\xi \in \mathbb{Z}} |\hat{g}(s_j \xi)|^2 \right)^{-1}.
$$

Then $\sum_{\xi \in \mathbb{Z}} \hat{g}_s(\xi) \hat{h}_j(\xi) \equiv 1$ almost everywhere, and consequently

$$
f = \sum_{j \in \mathbb{Z}} W_g f(\cdot, s_j) * h_j.
$$

Since there are efficient algorithms for the recovery of band-limiting functions from their irregular samples [7], [8], and [12], this deconvolution may be a preferable reconstruction method. A similar remark applies to the sampling of the short-time Fourier transform.

2. Usually a minimal distance between consecutive sampling points $x_{jk}, x_{j,k+1}$ or $s_j, s_{j+1}$ is assumed in order to guarantee that the sequence $W_g f(x_{jk}, s_j)$ is in $l^2(\mathbb{Z}^2)$. It is an important feature of the theorems that such a restriction is unnecessary. A high density of sampling points in some regions is compensated by the adaptive weights $\frac{1}{2} (x_{j,k+1} - x_{j,k-1})$ and $\frac{1}{2} \ln(s_{j+1}/s_{j-1})$. In this way stable sampling and (weighted) frames of the form

$$
\frac{1}{2} \left( (x_{j,k+1} - x_{j,k-1}) \ln \frac{s_{j+1}}{s_{j-1}} \right)^{1/2} s_{j-1}^{-1} g\left( \frac{x - x_{jk}}{s_j} \right), j, k \in \mathbb{Z},
$$
are obtained in situations where the collection $s_j^{-1/2}g((x - x_{jk})/s_j)\), j, k ∈ Z does not constitute a frame. We have learned from the numerical experiments with irregular sampling of band-limited functions [8] that the use of adaptive weights increases the speed of convergence drastically when working with very irregular sampling sets. This feature of irregular sampling has been overlooked so far, and explains perhaps the absence of quantitative results for this problem.

(3) Theorem 3 is instrumental for quantitative results on irregular sampling of wavelet transforms. It is clear that for any sampling theorem for band-limited functions a corresponding sampling theorem for $W_a f$ or $S_a f$ can be stated. For an example of a qualitative sampling theorem of this type we refer to [1]. However, even with the famous irregular sampling theorem of Duffin and Schaeffer [5] and its variations it is difficult to handle local density variations of the sampling set, and no appropriate estimates for the frame bounds are known. On the other hand, it uses a less restrictive definition of density that allows for gaps or missing samples.

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References


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