Harmonic Analysis of Generalized Stochastic Processes on Locally Compact Abelian Groups

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Introduction

Whereas ordinary functions on a locally compact group map the group elements into the complex numbers, a stochastic process can be understood as a mapping into a Hilbert space. The idea of a generalized function is to reduce the knowledge about the function to that of certain averages. It leads to the concept of generalized functions as continuous linear functionals on spaces of test functions. The combination of both ideas is the basis for generalized stochastic processes on locally compact Abelian groups: Hilbert space valued bounded linear operators on spaces of test functions on G. The properties of the Schwartz space $\mathcal{S}(R^m)$, in particular its invariance under the Fourier transform, make it a very suitable tool for the description of generalized functions, but its generalization, the so-called Schwartz-Bruhat space is very complicated (cf. [19]) and structure theory of lca. groups is required to describe the space.
In contrast, the function space $S_0(G)$ discovered by the first author (cf. [3], [19]) can be defined without the use of structure theory in a simple way for general lca. groups. Moreover it is a Banach space (which greatly simplifies the description of the natural topology on the dual space), and the Fourier transform maps $S_0(G)$ onto $S_0(\hat{G})$, the corresponding space on the dual group $\hat{G}$. Using Pontryagin's duality theorem it is then possible to extend the Fourier transform in order to obtain a generalized Fourier transform from $S_0'(G)$ onto $S_0'(\hat{G})$. Since the space $\mathcal{S}(G)$ of Schwartz-Bruhat is dense in $S_0(G)$ it is clear that the concept of tempered distributions $\sigma \in \mathcal{S}'(G)$ is more general. However, if one is not interested in derivatives the concept of $S_0'(G)$ is general enough and has many practical advantages, mainly due to its simplicity. Let us only mention that for the case $G = T$ the space $S_0(G)$ coincides with $A(G)$, the algebra of absolutely convergent Fourier series, of which it can be seen as a natural generalization for arbitrary lca. groups.
The concept of generalized stochastic processes over Euclidean spaces has been developed in [12], [7] and [8], where the space $\mathcal{D}$ of infinitely often differentiable functions with compact support was used. In [14] the space $\mathcal{K}(G)$ of continuous functions with compact support serves as the space of test functions. The main disadvantage of these function spaces (besides technical questions that may be overcome by developing the appropriate integration or distribution theory) is in our opinion the fact that they are not invariant under Fourier transform and that they are only topological vector spaces. The only work on generalized stochastic processes we know that uses a test function space that is invariant under Fourier transform is [13] for the case $G = R$. There the function space is defined over C or with technical difficulties over C^n, but it seems impossible to extend this definition to locally compact Abelian groups. In addition the test functions are analytic functions which makes it impossible to describe the concept of a support for the corresponding dual space.
The observation that $S_0(G)$ is a conceptually and technically much more convenient space and the observation that most of the relevant concepts arising in the theory of generalized stochastic processes over lca. groups can be proved on the basis of this concept lead to this paper. The interested reader may find more details about generalized stochastic processes, and in particular the definition of the Wigner distribution for a process over R^n in the thesis of the second author [11].

Notations

Let G be a locally compact Abelian group with Haar measure dx; the group operation is written as addition. The dual group is denoted by $\hat{G}$; χ_x denotes a character on $\hat{G}$ which can be identified with $x \in G$. We write $C^b(G), C^0(G)$ and $\mathcal{K}(G)$ for the spaces of continuous complex-valued functions which are bounded, with limit zero at infinity and with compact support respectively. $C^b(G)$ and $C^0(G)$ are endowed with the supremum norm $\|.\|_\infty$, $\mathcal{K}(G)$ with the inductive limit topology. $M(G)$ is considered as the Banach dual of $C^0(G)$ containing $L^1(G)$ as closed subspace. The translation operator L_x and the multiplication (by characters) operator M_t are defined by: $L_x f(y) := f(y-x), x \in G; M_t f(y) := t(y)f(y), t \in \hat{G}$.
The Banach space $S_0(G)$ has been characterized as the minimal Banach space among all Banach spaces which are isometrically invariant under translation and character multiplication and containig all integrable functions $f \in L^1(G)$ with compactly supported Fourier transform (cf. [3] for details). For a summary of properties of this space (such as invariance under the Fourier transform) the reader is referred to [3] or the survey article [4]. We will work mostly with the elements of $S_0'(G)$, which will be called **distributions**. For convenience we shall write for $\sigma \in S_0'(G)$ and $f \in S_0(G) : \langle \sigma, f \rangle := \sigma(f)$.
Let f, g be functions on G_1 and G_2 respectively:
The **tensor product** of f and g is the function $f \otimes g$ on $G_1 \times G_2$ given by

$$f \otimes g(x,y) := f(x) \cdot g(y); \quad x \in G_1, y \in G_2$$

Let B_1 and B_2 be two Banach spaces, which are continuously embedded into $C^b(G_1)$ and $C^b(G_2)$. The **projective tensor product** of B_1 and B_2 is defined as:

$$B_1 \hat{\otimes} B_2 := \left\{ f \mid f = \sum_{n=1}^{\infty} f_n \otimes g_n, \sum_{n=1}^{\infty} \|f_n\|_{B_1} \|g_n\|_{B_2} < \infty \right\}$$

$B_1 \hat{\otimes} B_2$ is a Banach space continuously embedded in $C^b(G_1 \times G_2)$:

$$\|f\|_{\hat{\otimes}} := \inf \left\{ \sum_{n=1}^{\infty} \|f_n\|_{B_1} \|g_n\|_{B_2} \text{ with } f = \sum_{n=1}^{\infty} f_n \otimes g_n \right\}$$

Let $V_0(G_1 \times G_2) := C_0(G_1) \hat{\otimes} C_0(G_2)$ be the projective tensor product. The dual of $V_0(G_1 \times G_2)$, which contains $M(G_1 \times G_2)$ as a proper subspace, is called the space of **bimeasures** $BM(G_1 \times G_2)$. For the properties of bimeasures we refer to [10].
A Radon measure μ (i.e. an element of the dual of $\mathcal{K}(G)$) is called **translation bounded** if for any $k \in \mathcal{K}(G)$ the $\sup_{x \in G} |\mu(L_x(k))|$ is finite.
A detailed discussion of the following concepts is given in [5]:
A bounded set $S \subseteq M(G)$ is called **tight** if for every $\epsilon > 0$ there exists some $k \in \mathcal{K}(G)$ such that $\|k \cdot \mu - \mu\|_M \leq \epsilon$ for all $\mu \in$ S.
A bounded net $(e_\gamma)_{\gamma \in \Gamma}$ in $L^1(G)$ is called a bounded approximate unit for $L^1(G)$ if one has $\lim_\gamma \|e_\gamma * f - f\|_1 \to 0$ for any $f \in L^1(G)$. The limit of e_γ is Dirac's Delta distribution denoted by δ_0.
A net $(\mu_\gamma)_{\gamma \in \Gamma}$ is **vaguely** convergent with limit μ_0 if one has $\lim_\gamma \mu_\gamma(k) = \mu_0(k) \; \forall \, k \in \mathcal{K}$.

1. Definitions

Definition 1: Let $\mathcal{H}$ be an arbitrary Hilbert space:
A measurable mapping $X : G \mapsto \mathcal{H}$ is called a **stochastic process** on G.

Definition 2: A bounded linear mapping $\rho : S_0(G) \mapsto \mathcal{H}$ is called a **generalized stochastic process (GSP)**.

We can think of $\mathcal{H}$ as a Hilbert space of C-valued random variables with zero expectation on an arbitrary probability space (Ω, Σ, P), that is $L_0^2(\Omega, \Sigma, P)$, but we will only use the Hilbert space properties and therefore we write $\mathcal{H}$.
$(.|.)$ will denote the (sesquilinear) inner product and $\|.\|_{\mathcal{H}}$ the norm in $\mathcal{H}$.

The following properties of GSPs will be of importance in this paper. These definitions are of course analogous to the corresponding concepts for classical stochastic processes in the following sense: If a classical is interpreted as a generalized stochastic process (via integration) then the classical terminology is compatible with the one given here.

Definition 3:
a) A GSP is called (wide sense time-) **stationary** if

$$(\rho(f)|\rho(g)) = (\rho(L_x f)|\rho(L_x g)) \; \forall x \in G, \; \forall f, g \in S_0(G)$$

b) A GSP is called (wide sense) **frequency stationary** if

$$(\rho(f)|\rho(g)) = (\rho(M_t f)|\rho(M_t g)) \; \forall t \in \hat{G}, \; \forall f, g \in S_0(G)$$

c) A time-stationary and frequency-stationary process is called **white noise**.

Definition 4:
a) A GSP ρ is called **bounded** if ρ is bounded with respect to $\|.\|_\infty$:

$$\exists\, c > 0 \text{ such that } \|\rho(f)\|_{\mathcal{H}} \leq c\|f\|_\infty \; \forall f \in S_0(G);$$

b) A GSP ρ is called **V-bounded** (variation bounded) if:

$$\exists\, c > 0 \text{ such that } \|\rho(f)\|_{\mathcal{H}} \leq c\|\hat{f}\|_\infty \; \forall f \in S_0(G).$$

Definition 5: A GSP is called **orthogonally scattered** if

$$\text{supp(f)} \cap \text{supp(g)} = \emptyset \text{ implies } \rho(f) \perp \rho(g) \text{ for } f, g \in S_0(G).$$

Due to the tensor product property of $S_0 : S_0(G)\hat{\otimes}S_0(G) = S_0(G \times G)$ (cf. [3] Theorem 7 D) it is justified to hope that the following definition determines an element of $S_0(G\times G)$.

Definition 6: Let ρ be a GSP. The **autocovariance** (or auto-correlation) distribution σ_ρ is defined as: $\langle \sigma_\rho, f \otimes g\rangle := (\rho(f)|\rho(\bar{g})) \; \forall f, g \in S_0(G)$.

A priori the σ_ρ functional is only defined for functions of the form $f \otimes g, f, g \in S_0(G)$, but we want σ_ρ to be a distribution of two variables, that is a linear functional on $S_0(G)\hat{\otimes}S_0(G) = S_0(G \times G)$. Therefore we must extend the definition of σ_ρ to functions $h := \sum_{n=1}^{\infty} f_n \otimes g_n$. This can be done by first defining σ_ρ in the obvious way for finite sums; it is not difficult to show that this definition makes sense and that σ_ρ is bounded; then we can use the fact that the completion of the space of all finite sums is equal to $S_0(G)\hat{\otimes}S_0(G)$ to extend σ_ρ (uniquely) to $S_0(G)\hat{\otimes}S_0(G) = S_0(G \times G)$; hence σ_ρ is an element of $S_0'(G \times G)$.

2. Relations between a GSP and its Covariance

As in the classical case the relations between the properties of a GSP ρ and the covariance σ_ρ associated with ρ are important. The following theorems will deal with this aspect.

Theorem 1: For a GSP ρ the following properties are equivalent:
a) ρ stationary $\iff \sigma_\rho$ diagonally invariant, i.e. $L_{(x,x)}\sigma_\rho = \sigma_\rho \;\forall\, x \in G$;
b) ρ bounded $\iff \sigma_\rho$ extends in a unique way to a bimeasure on $G \times G$;
c) ρ orthogonally scattered
$\iff \sigma_\rho$ is supported by the diagonal, i.e. $\mathrm{supp}(\sigma_\rho) \subseteq \Delta_G := \{(x,x) \mid x \in G\}$;
$\iff$ there exists a positive and translation bounded measure τ_ρ with:

$$\langle \sigma_\rho, f \otimes g\rangle = \langle \tau_\rho, fg\rangle \;\forall\, f, g \in S_0(G).$$

Proof: a) Follows from the definitions.
b) ($\Rightarrow$) By the continuity of σ_ρ it follows from the definition of a bounded GSP that the following holds:

$$|\langle \sigma_\rho, \sum_{n=1}^{\infty} f_n \otimes g_n\rangle| \leq \sum_{n=1}^{\infty} |(\rho(f_n)|\rho(\bar{g}_n))| \leq c^2 \sum_{n=1}^{\infty} \|f_n\|_\infty \|g_n\|_\infty$$

for all admissible representations $\sum_{n=1}^{\infty} f_n \otimes g_n$ of $h \in S_0(G \times G)$;
hence $|\langle \sigma_\rho, h\rangle| \leq c^2 \|h\|_{V_O} \;\forall\, h \in S_0(G \times G)$; the density of S_0 in V_0 now implies that σ_ρ extends to a uniquely determined bimeasure on $G \times G$.
($\Leftarrow$) Boundedness of ρ follows from the estimate

$$\|\rho(f)\|^2_{\mathcal{H}} = (\rho(f)|\rho(f)) = \langle \sigma_\rho, f \otimes \bar{f}\rangle \leq c\|f \otimes \bar{f}\|_{V_O} \leq c\|f\|^2_\infty$$

c) (first equivalence $\Leftarrow$) follows directly from the definitions;
(first equivalence $\Rightarrow$) Assume that $\langle \sigma_\rho, f \otimes g\rangle = 0$ whenever $(\mathrm{supp(f)} \times \mathrm{supp(g)}) \cap \Delta_G = \emptyset$. Applying the formula $S_0(G \times G) = S_0(G)\hat{\otimes} S_0(G)$ and suitably refined partitions of unity (in both factors) one derives therefrom that $\langle \sigma_\rho, h\rangle = 0$ for any $h \in S_0(G \times G)$ having compact support disjoint to Δ_G. This implies $\mathrm{supp}(\sigma_\rho) \subseteq \Delta_G$.
(second equivalence $\Rightarrow$)Δ_G being a set of spectral synthesis (cf. [18] Ch. 7 Theorem 4.1 and Ch. 6 Remark 1.5) a distribution σ_ρ with support on Δ_G satisfies $\sigma_\rho(F) = \sigma_\rho(H)$ if only the two restriction to Δ_G are equal, i.e. if $\mathrm{Restr}_{\Delta_G}(F) = \mathrm{Restr}_{\Delta_G}(H)$ (recall that $\mathrm{Restr}_{\Delta_G}$ maps $S_0(G \times G)$ onto $S_0(\Delta_G)$ by [3], Theorem 7 C). The representation of σ_ρ by τ_ρ follows therefrom using the canonical identification j_G of G and Δ_G and the formula

$$\langle \sigma_\rho, f \otimes g\rangle = \langle \tau_\rho, (\mathrm{Restr}_{\Delta_G}(f \otimes g)) \circ j_G\rangle = \langle \tau_\rho, fg\rangle.$$

To show that τ_ρ is positive we take a net $f_\alpha \in S_0(G)$ with $|f_\alpha|^2 \to \delta_0$ and define:

$$\langle (\tau_\rho)_\alpha, g\rangle = \langle \tau_\rho * |f_\alpha|^2, g\rangle = \langle \tau_\rho, |f_\alpha|^{2\check{}} * g\rangle$$

It is clear that $\langle (\tau_\rho)_\alpha, g\rangle \to \langle \tau_\rho, g\rangle \;\forall\, g \in S_0(G)$. In addition $(\tau_\rho)_\alpha$ can be identified with the bounded function $h_\alpha(x) = \langle \tau_\rho, L_x|f_\alpha|^{2\check{}}\rangle = \langle \tau_\rho, |L_x\check{f}_\alpha|^2\rangle$.
As $\langle \tau_\rho, f\bar{f}\rangle = (\rho(f)|\rho(f)) \geq 0 \;\forall\, f \in S_0$ it is obvious that $h_\alpha(x) \geq 0 \;\forall\, x \in G$; this implies $\langle (\tau_\rho)_\alpha, g\rangle \geq 0 \;\forall\, g \in S_0,\; g \geq 0$ and thus τ_ρ is positive.
The positive elements of $S'_0(G)$ are translation bounded measures (cf. [6] Lemma 3.6 or [11] Appendix: Theorem 2.3). Since the opposite direction is obvious the proof is complete.□

Corollary 2: A GSP ρ is bounded and orthogonally scattered $\iff$
$\exists$ bounded measure μ_ρ on G such that

$$\langle \sigma_\rho, f \otimes g \rangle = \langle \mu_\rho, fg \rangle = \int_G f(x)g(x)d\mu_\rho(x) \; \forall f, g \in S_0(G)$$

Proof: ($\Leftarrow$) Follows by Theorem 1.c and 1.b.
($\Rightarrow$) Theorem 1.c implies the first part of the formula for $\tau_\rho \in S_0(G)$. ρ being bounded τ_ρ is a bimeasure (which is supported by the diagonal). To prove that τ_ρ is a bounded measure we have to show that τ_ρ is bounded with respect to $\|.\|_\infty$.
It is possible to write $f \in C^0(G)$ as $f = f_1 f_2$ with $f_1, f_2 \in C^0(G)$ and
$\|f\|_\infty = \|f_1\|_\infty \|f_2\|_\infty$(for example: $f_1(x) := \arg(f(x))\sqrt{|f(x)|}$, $f_2(x) := \sqrt{|f(x)|}$).
Now the following holds: $|\langle \tau_\rho, f\rangle| = |\langle \tau_\rho, \mathrm{Restr}_{\Delta_G}(f_1 \otimes f_2)\rangle| = |\langle \sigma_\rho, f_1 \otimes f_2 \rangle| \leq$
$\leq \|\sigma_\rho\|_{BM} \|f_1\|_\infty \|f_2\|_\infty = c\|f\|_\infty$ □

3. The Spectral Process

The following definition is analogous to the definition of the Fourier transform for distributions.

Definition 7: Given a GSP ρ on G we define a GSP $\hat{\rho}$ on $\hat{G}$:
$\hat{\rho}(f) := \rho(\hat{f}) \; \forall f \in S_0(\hat{G})$; $\hat{\rho}$ is called the **spectral process** to ρ.

Definition 8: $\check{f}(x) := f(-x)$; $\check{\rho}(f) := \rho(\check{f})$

Since the mappings $f \mapsto \check{f}$ and $f \mapsto \hat{f}$ define isomorphisms of $S_0(G)$ respectively between $S_0(G)$ and $S_0(\hat{G})$ it is clear that $\check{\rho}$ and $\hat{\rho}$ are GSPs. The following lemma contains some simple facts about these operators:

Lemma 3: For a GSP ρ the following properties are equivalent:
a) ρ resp. $\hat{\rho}$ is bounded $\iff \hat{\rho}$ resp. ρ is V-bounded ;
b) ρ resp. $\hat{\rho}$ is stationary $\iff \hat{\rho}$ resp. ρ is frequency-stationary;
c) $\rho = \hat{\tau} \iff \check{\tau} = \hat{\rho}$.
Proof: This follows directly from the definitions.□

From Lemma 3.c it is clear that $\rho \mapsto (\check{\rho})\hat{}$ is the inverse mapping of $\rho \mapsto \hat{\rho}$. This shows that the Fourier transform is a bijective mapping between the GSPs over G and these over $\hat{G}$.

Theorem 4: Let ρ be a GSP:
a) $\langle \hat{\sigma}_\rho, f \otimes g \rangle = \langle \sigma_{\hat{\rho}}, f \otimes \check{g} \rangle$
b) ρ orthogonally scattered $\iff L_{(t,t)}\sigma_{\hat{\rho}} = \sigma_{\hat{\rho}} \; \forall t \in \hat{G}$
Proof: a) $\langle \hat{\sigma}_\rho, f \otimes g \rangle = \langle \sigma_\rho, \hat{f} \otimes \hat{g} \rangle = (\rho(\hat{f})|\rho(\hat{g}^-)) = (\hat{\rho}(f)|\hat{\rho}(\bar{g}^{\check{}\check{}})) = (\hat{\rho}(f)|\hat{\rho}(\bar{g}^{\check{}})) = \langle \sigma_{\hat{\rho}}, f \otimes \bar{g}^{\check{}-} \rangle = \langle \sigma_{\hat{\rho}}, f \otimes \check{g} \rangle$
b) Follows from Theorem 1.c, and the fact that σ is H-invariant $\Leftrightarrow$ supp$(\hat{\sigma}) \subseteq H^\perp$ (cf. [6] Theorem 3.4 A) and part a. □

Corollary 5: Let ρ be a GSP: ρ V-bounded $\iff \hat{\sigma}_\rho$ extends to a bimeasure
Proof: Apply Lemma 3.a, Theorem 1.b and Theorem 4.a. □

4. Characterizations of stationary processes

Corollary 6: Let ρ be a GSP:
a) ρ frequency-stationary $\iff \rho$ orthogonally scattered;

b) ρ stationary $\Longleftrightarrow \exists$ a positive translation bounded measure $\tau_{\hat{\rho}}$ on G with $\langle \sigma_{\hat{\rho}}, f \otimes g \rangle = \langle \tau_{\hat{\rho}}, fg \rangle \; \forall f, g \in S_0(G); \tau_{\hat{\rho}}$ is called the **spectral measure** of ρ.
Proof: a) Follows from Lemma 3.b, Theorem 1.a and Theorem 4.b.
b) Follows from Lemma 3.b, part a and Theorem 1.c.

Remark: Because of Corollary 6.a it is clear that ρ is white noise if and only if ρ is stationary and orthogonally scattered.

Remark: In the classical theory of stochastic processes Corollary 6.a is called the "spectral representation of a stationary process" (cf. [2] p. 527 or [9] p. 244). The spectral measure is called power spectrum or - if it has a continuous density - spectral density of the process.

The following theorem contains a characterization of the covariance of stationary GSPs. The necessary part is a kind of "existence theorem" for stationary GSPs.

Theorem 7: For $\sigma \in S_0'(G \times G)$ the following two properties are equivalent:
σ is covariance of a stationary GSP $\Longleftrightarrow \sigma$ is diagonally invariant and positive definite.
Proof: ($\Rightarrow$) The invariance was already shown in Theorem 1.a. By Corollary 6.b it is clear that the covariance distribution of $\hat{\rho}$ is positive and this implies (by Theorem 4.a) that $\hat{\sigma}$ is positive, which is equivalent to the positive definiteness of σ.
($\Leftarrow$)σ being diagonally invariant and positive definite it follows, that $\hat{\sigma}$ is supported by $\nabla \hat{G} = \left\{ (t| - t), t \in \hat{G} \right\}$ (cf.[6], Theorem 3.4 A), and that $\langle \hat{\sigma}, f \otimes \bar{f}\check{\ } \rangle \geq 0 \; \forall f \in S_0(\hat{G})$(as $f \otimes \bar{f}\check{\ }$ is non-negative on $\nabla \hat{G}$);
this implies $\langle \sigma, \hat{f} \otimes \bar{f}\check{\ }\hat{\ } \rangle \geq 0 \; \forall f \in S_0(\hat{G})$ and this is equivalent to

$$\langle \sigma, f \otimes \bar{f} \rangle \geq 0 \; \forall f \in S_0(G) \quad (\bar{f}\check{\ }\hat{\ } \text{ being equal to } \hat{f}^{-}).$$

We have proved that the form $Q(f, g) := \langle \sigma, f \otimes \bar{g} \rangle$ defines a positive semi-definite sesquilinear form on $S_0(G) \times S_0(G)$. Since $N = \left\{ f \mid \langle \sigma, f \otimes \bar{f} \rangle = 0 \right\}$ is a linear subspace of $S_0(G)$ Q defines a canonical inner product on $\mathcal{H}_1 := S_0(G)/N$. The Hilbert space obtained by completion can be denoted by $\mathcal{H}$. It is then clear that the canonical projection, followed by the embedding of $\mathcal{H}_1$ into $\mathcal{H}$ defines a bounded, linear mapping ρ from $S_0(G)$ into $\mathcal{H}$, i.e. it is a GSP. Of course σ coincides with σ_ρ. By the diagonal invariance of σ the stationarity of ρ follows. □

Corollary 8 : Let $\sigma \in S_0'(G \times G)$:
Then σ is covariance of an orthogonally scattered GSP ρ

$$\Longleftrightarrow \exists \, \tau \text{ positive and translation bounded with: } \langle \sigma, f \otimes g \rangle = \int_G fg d\tau \; \forall f, g \in S_0(G)$$

Proof: ($\Rightarrow$) has been shown in Theorem 1.c.
($\Leftarrow$) Let $\omega \in S_0'(\hat{G} \times \hat{G})$ be defined in the following way:

$$\langle \omega, f \otimes g \rangle := \langle \hat{\sigma}, f \otimes \check{g} \rangle = \langle \sigma, \hat{f} \otimes \check{g}\hat{\ } \rangle = \int_G \hat{f} \check{g}\hat{\ } d\tau \; \forall f, g \in S_0(\hat{G});$$

Then ω is diagonally invariant and positive definite. By Theorem 7 the existence of a stationary GSP$\hat{\rho}$ over $\hat{G}$ with covariance ω follows. Hence by Theorem 4.a ρ is a GSP over G with covariance σ, which is orthogonally scattered in view of Lemma 3.b and Corollary 6.a. □

The following theorem characterizes white noise (cf. Definition 3.c) in three different ways:

Theorem 9: A GSP ρ is white noise if and only if one of the following conditions is satisfied:
$\iff$ a) $\exists\, c \geq 0$ that $(\rho(f)|\rho(\bar{g})) = c\int_G f(t)g(t)dt\ \forall\, f,g \in S_0(G)$
$\iff$ b) $\|\rho(f)\|_{\mathcal{H}} = c\|f\|_2$ for some $c \geq 0$ and $\forall\, f \in S_0(G)$
(i.e. ρ is a scalar multiple of an isometry between $L^2(G)$ and $\mathcal{H}$).
Proof: a) This follows from Theorem 1.a and c and Corollary 6.a together with the uniqueness of the Haar measure as a positive translation invariant measure (cf. [18] Ch. 3,3.1 and references there).
(b $\Rightarrow$ white noise) By means of polar decomposition $(\rho(f)|\rho(\bar{g}))$ can be expressed with the help of $\|\rho(f)\|_{\mathcal{H}} = c\|f\|_2$ and $\|\rho(\bar{g})\|_{\mathcal{H}} = c\|g\|_2$. Thus it is clear, that $(\rho(f)|\rho(\bar{g}))$ is not changed by modulation or translation.
(white noise $\Rightarrow$ b) Follows from part a.$\square$

5. Relations between GSPs and other theories

It is clear that it is not possible to associate arbitrary GSPs with (ordinary) stochastic processes, as there are GSPs with a covariance distribution which cannot be represented by an ordinary function. But we shall prove that any GSP with a covariance distribution induced by some continuous bounded function in $C^b(G \times G)$ can be identified with a uniquely determined stochastic process in the classical sense. On the other hand any mean square continuous stochastic process can be identified with a uniquely determined GSP. The exact formulation of this fact is contained in the following theorem.

Theorem 10: Let ρ be a GSP with covariance σ_ρ:
a) $\sigma_\rho \in S_0'(G \times G)$ is represented by some $h \in C^b(G \times G) \Longrightarrow \rho(f_\alpha)$ is Cauchy net in $\mathcal{H}$ whenever $(f_\alpha)_{\alpha\in I}$ is a vaguely convergent, L^1 - bounded and tight net in $S_0(G)$.
b) In the above situation ρ extends to a bounded linear operator $\tilde{\rho} : M(G) \mapsto \mathcal{H}$, which is σ - norm continuous on tight subsets of $M(G)$. In particular $\lim_{y\to x} \tilde{\rho}(\delta_y) = \tilde{\rho}(\delta_x)$. If $\{\rho(f), f \in S_0(G)\}$ is dense in $\mathcal{H}$ the extension $\tilde{\rho}$ is uniquely determined.
c) The mapping $\rho_G : \rho_G(x) := \tilde{\rho}(\delta_x)$ is a bounded, continuous stochastic process on G, and $h(x,y) = (\tilde{\rho}(\delta_x)|\tilde{\rho}(\delta_y))$ i.e. h is covariance function of ρ_G.
d) For any continuous and bounded stochastic process $\rho_1 : G \mapsto \mathcal{H}$ the covariance function of ρ_1 given by $h(x,y) := (\rho_1(x)|\rho_1(y))$ is bounded and continuous on G×G. By vector-valued integration ρ_1 may be lifted to a bounded linear mapping $\tilde{\rho}_1 : M(G) \mapsto \mathcal{H}$, which is σ-norm continuous on bounded tight subsets. By way of restriction to $S_0(G)$ $\tilde{\rho}_1$ may be considered as a GSP, and h represents the covariance distribution of this GSP.
Proof: a) To prove that the net $(f_\alpha)_{\alpha\in I}$ is a Cauchy net we use:

$$\|\rho(f_\alpha) - \rho(f_\beta)\|^2_{\mathcal{H}} = (\rho(f_\alpha - f_\beta)|\rho(f_\alpha - f_\beta)) = \langle \sigma_\rho, (f_\alpha - f_\beta) \otimes (\bar{f}_\alpha - \bar{f}_\beta)\rangle =$$

$$= \langle \sigma_\rho \cdot k \otimes k, (f_\alpha - f_\beta) \otimes (\bar{f}_\alpha - \bar{f}_\beta)\rangle + \langle \sigma_\rho, k(f_\alpha - f_\beta) \otimes (1-k)(\bar{f}_\alpha - \bar{f}_\beta)\rangle +$$
$$+\langle \sigma_\rho, (1-k)(f_\alpha - f_\beta) \otimes (\bar{f}_\alpha - \bar{f}_\beta)\rangle$$

As f_α is tight and L^1 - bounded there exists $k \in \mathcal{K}(G)$ such that $\|(1-k)(f_\alpha - f_\beta)\|_1 < \epsilon\ \forall\, \alpha \in I$; thus the second of the three terms can be handled in the following way:

$$|\langle \sigma_\rho, k(f_\alpha - f_\beta) \otimes (1-k)(\bar{f}_\alpha - \bar{f}_\beta)\rangle| \qquad =$$

$$\left|\int_G\int_G h_\rho(x,y)k(x)(f_\alpha(x) - f_\beta(x))dx(1-k(y))(\bar{f}_\alpha(y) - \bar{f}_\beta(y))dy\right| \leq$$

$$\leq \int_G \int_G \|h_\rho\|_\infty |k(x)(f_\alpha(x) - f_\beta(x))| dx |(1-k(y))(\bar{f}_\alpha(y) - \bar{f}_\beta(y))| dy \leq$$

$$\leq \int_G \|h_\rho\|_\infty C |(1-k(y))(\bar{f}_\alpha(y) - \bar{f}_\beta(y))| dy$$

and the last integral can be made arbitrarily small by suitable choice of k above independent of α and β. The third term can be treated in the same way as the second. As the first term converges to 0 if f_α is vaguely convergent it follows that $\|\rho(f_\alpha) - \rho(f_\beta)\|_{\mathcal{H}}$ tends to zero and this implies that $(\rho(f_\alpha))_{\alpha\in I}$ is a Cauchy net in the norm topology.

b) As any measure in $M(G)$ can be represented as the w*-limit of a bounded, tight net of functions in S_0, the density of $\{\rho(f), f \in S_0(G)\}$ in $\mathcal{H}$ implies the existence of an uniquely determined element denoted by $\tilde{\rho}(\mu) \in \mathcal{H}$, such that $(\tilde{\rho}(\mu)|\rho(g)) := \lim_\alpha (\rho(f_\alpha)|\rho(g)) \; \forall\, g \in S_0(G)$; the notation is justified because it is independent from the choice of the net (f_α) with $\lim_\alpha f_\alpha = \mu$. This can be seen using the proof of part a with two different L^1-bounded, tight nets with the same vague limit.
Now we show the continuity of $\tilde{\rho}$: Let $(\mu_\beta)_{\beta\in J}$ be a bounded and tight net in $M(G)$, w*-convergent with limit μ; $\forall\, U := U(k_1, k_2, \ldots, k_n, \epsilon) \in \mathcal{U}, k_i \in S_0(G) \;\exists\, f_{(U,\beta)} \in S_0(G)$ that the following holds:

$$|\langle \mu_\beta, k_i\rangle - \langle f_{(U,\beta)}, k_i\rangle| < \epsilon/2 \;\forall\, i = 1 \ldots n, \;\forall\, \beta$$

$$\text{and } |\langle \mu_\beta, k_i\rangle - \langle \mu, k_i\rangle| < \epsilon/2 \;\forall\, \beta > \beta_0$$

$$\text{it follows that } |\langle \mu, k_i\rangle - \langle f_{(U,\beta)}, k_i\rangle| < \epsilon \;\forall\, i = 1 \ldots n \;\forall\, \beta > \beta_0$$

We have shown: $(f_{(U,\beta)})$ is a w*-convergent, L^1 - bounded, tight net with limit μ and $f_{(U,\beta)} \in S_0(G)$, for any $U \in \mathcal{U}$; part a implies that $\rho(f_{(U,\beta)})$ converges in norm and the limit is called $\tilde{\rho}(\mu)$ according to the above definition. Due to the construction of $f_{(U,\beta)}$ it is easy to see that $\tilde{\rho}(\mu_\beta)$ converges to $\tilde{\rho}(\mu)$ as well and this shows the σ - norm continuity of $\tilde{\rho}$ on bounded, tight subsets. Furthermore, this implies the norm-norm continuity and thus the boundedness of the mapping $\tilde{\rho}$ follows.

c) The continuity of ρ_G follows from the σ-norm continuity of $\tilde{\rho}$ on tight subsets.
The boundedness of ρ_G follows from the fact that $\tilde{\rho}$ is bounded with respect to $\|.\|_M$ together with $\|\delta_x\|_M \;=\; 1 \;\forall\; x \in G$.
Let $(f_\alpha)_{\alpha\in I}$ be tight, L^1 - bounded and vaguely convergent with limit δ_0 (i.e. a generalized "Dirac sequence"), then the following completes the proof of part c:

$$h(x,y) = \int_{G\times G} h(t)(\delta_x \otimes \delta_y) dt = \lim_\alpha \langle h, L_x f_\alpha \otimes L_y f_\alpha\rangle =$$

$$= \lim_\alpha \langle \sigma_\rho, L_x f_\alpha \otimes L_y f_\alpha\rangle = (\tilde{\rho}(\delta_x)|\tilde{\rho}(\bar{\delta}_y)) = (\tilde{\rho}(\delta_x)|\tilde{\rho}(\delta_y))$$

d) The estimate $h(x,y) = (\rho_1(x)|\rho_1(y)) = \|\rho_1(x)\|_{\mathcal{H}} \|\rho_1(y)\|_{\mathcal{H}} \leq c^2$ proves that h is bounded. The continuity of h results from the continuity of ρ_1 and of the inner product.
With the help of vector - valued integration we can define:

$$(\tilde{\rho}_1(\mu)|l) := \int_G (\rho_1(x)|l) d\mu \text{ for } l \in \mathcal{H} \text{ and } \mu \in M(G);$$

In view of the Riesz representation theorem $\tilde{\rho}_1(\mu)$ is a well defined element of $\mathcal{H}$, as $|(\tilde{\rho}_1(\mu)|l)| \leq c\|\mu\|_M \|l\|_{\mathcal{H}}$. The last inequality implies $\|\tilde{\rho}_1(\mu)\|_{\mathcal{H}} \leq c\|\mu\|_M$ and thus the boundedness of $\tilde{\rho}_1$ with respect to $\|.\|_M$.

To prove the σ-norm continuity we take a bounded, tight w*-convergent net μ_α in $M(G)$ with limit μ. As the mapping $x \mapsto (\rho_1(x)|l)$ is continuous and bounded for any $l \in \mathcal{H}$ and μ_α is tight we get:

$$\lim_\alpha(\tilde{\rho}_1(\mu_\alpha)|l) = \lim_\alpha \int_G (\rho_1(x)|l)d\mu_\alpha = \int_G (\rho_1(x)|l)d\mu = (\tilde{\rho}_1(\mu)|l)$$

which shows the $\sigma - \sigma$ continuity of $\tilde{\rho}_1$. The convergence of $\|\tilde{\rho}_1(\mu_\alpha)\|$ is shown by the following equality

$$\lim_\alpha(\tilde{\rho}_1(\mu_\alpha)|\tilde{\rho}_1(\mu_\alpha)) = \lim_\alpha \int_G\int_G (\rho_1(x)|\rho_1(y))d\mu_\alpha d\mu_\alpha = \int_G\int_G (\rho_1(x)|\rho_1(y))d\mu d\mu = \|\tilde{\rho}_1(\mu)\|^2$$

which is true as $(\rho_1(x)|\rho_1(y))$ is continuous and bounded and μ_α is w*-convergent, bounded and tight. The last result together with the σ-norm continuity of $\tilde{\rho}_1$ imply the σ-norm continuity we aimed at.
We get the required GSP by restriction to $S_0(G)$: $\rho := \tilde{\rho}_1|_{S_0}$.
It remains to be shown that $h(x,y) = (\rho_1(x)|\rho_1(y))$ represents the covariance distribution σ_ρ of ρ. This follows from the identity

$$\langle\sigma_\rho, f\otimes g\rangle = (\rho(f)|\rho(\bar{g})) = \int_G (\rho_1(x)|\rho(\bar{g}))f(x)dx = \int_G\int_G g(y)(\rho_1(x)|\rho_1(y))dyf(x)dx$$

$$= \int_G\int_G h(x,y)f(x)g(y)dxdy = \langle h, f\otimes g\rangle \text{ for } f,g \in S_0(G). \square$$

Remark: As any measure in $M(G)$ can be represented as the w*-limit of a bounded tight net of functions in S_0 or of discrete measures the σ-norm continuity of the mappings $\tilde{\rho}$ and $\tilde{\rho}_1$ on tight, bounded subsets implies the uniqueness of these extensions from $S_0(G)$ or G to $M(G)$. Therefore Theorem 10 describes a bijective identification between continuous bounded stochastic processes and GSPs with continuous bounded covariance.

Corollary 11: Let ρ be a V-bounded GSP:
a) It follows that ρ can be identified with an uniquely determined stochastic process
b) It follows that ρ extends to $M(G)$ and $\hat{\rho}$ to $\mathcal{F}(M(G))$; therefore

$$\rho(\mu) = \hat{\rho}(h) \text{ if } \hat{\mu} = \check{h}, \text{ and in particular: } \rho(\delta_x) = \hat{\rho}(\chi_x)$$

Proof: Corollary 5 says: ρ V-bounded $\Leftrightarrow$ $\hat{\sigma}_\rho$ extends to a bimeasure; the Fourier transform of a bimeasure being a bounded, continuous function (cf. [10] Theorem 2.4i and Definition 2.1) the corollary follows from Theorem 10. $\square$

Remark: The formula $\rho(\delta_x) = \hat{\rho}(\chi_x)$ in Corollary 11.b can be seen as an alternative formulation of the "representation theorem of V-bounded stochastic processes as Fourier transforms of stochastic measures". That it is actually equivalent to Niemi's formulation in [14], p. 35, will become clear by Proposition 12.

The two preceding theorems show the strong relations between GSPs and stochastic processes. Together with the conditions stated above the concepts are even equivalent. From this point of view it is possible to see aspects of the classical theory in a new light and to apply mathematical methods to stochastic processes in a new way. This new point of view leads of course to new and in many cases short and clear proofs of classical theorems. The theorems stated in the previous chapter were proven for GSPs. If we add the presupposition that the covariance distribution σ_ρ is represented by a bounded, continuous

function all facts are proved for stochastic processes as well, as it is obvious that the definitions of certain properties for GSPs and stochastic processes are the same. We will use this considerations in the sequel to prove some results on V-bounded and harmonizable stochastic processes in a new way.
The following proposition compares GSPs and vector measures as defined by Niemi (cf. [14] p. 15), that are (with respect to the inductive limit topology) continuous and linear mappings $\mu : \mathcal{K}(G) \mapsto \mathcal{H}$. Since $\mathcal{K}(G)$ and $S_0(G)$ are not related by inclusions no general comparison is possible but the following holds:

Proposition 12: Under the assumption of boundedness or V-boundedness the concepts of vector measures and GSPs are equivalent.
Proof: Because of the definition of boundedness and V-boundedness it is obvious, that a bounded GSP extends to a bounded linear mapping from $C^0(G)$. V-bounded ones extend to bounded linear mappings on $\mathcal{F}(C^0(\hat{G}))$. Since $\mathcal{K}(G)$ as well as $S_0(G)$ are dense in these spaces the equivalence of both concepts follows. □

Remark: The concept of stationary GSPs is more general than that of stationary vector measures (cf. [11]).

6. Harmonizable Generalized Stochastic Processes

For GSPs it is also possible to define generalizations of the concept of stationarity. The first of the two different concepts is the concept of V-boundedness (cf. Definition 4.b) which was first introduced by Bochner (cf. [1] p. 18). Using Corollary 11.a it is obvious, that any V-bounded GSP can be identified with a V-bounded stochastic process.

Remark: According to [17] p.315, Theorem 4.2 our definition of V-boundedness for GSPs (with values in a Hilbert space) is equivalent to **weak harmonizability** as defined for the first time in [20].

The following definition corresponds to the definition given in [14] p. 35 for stochastic processes:

Definition 9 : *A* GSP is called (**strongly**) **harmonizable** $\Longleftrightarrow$
$\hat{\sigma}_\rho$ can be identified with a bounded measure.

Proposition 13: Let ρ be a GSP with covariance σ_ρ:
a) ρ harmonizable $\Longleftrightarrow \sigma_\rho$ lies in the Fourier-Stieltjes algebra $B(G \times G) := \mathcal{F}(M(\hat{G} \times \hat{G}))$;

$$\sigma_\rho = h_\rho(x, y) = \langle \nu, \bar{\chi}_x \otimes \bar{\chi}_y \rangle \; \forall\, x, y \in G \text{ and } \nu \in M(\hat{G} \times \hat{G})$$

b) ρ harmonizable $\Longrightarrow \rho$ V-bounded;
c) For stationary GSP ρ:

$$\rho \text{ harmonizable} \Longleftrightarrow \rho \text{ V-bounded}$$

Proof: a) h_ρ element of the Fourier-Stieltjes algebra $\Leftrightarrow$
$\hat{\sigma}_\rho$ bounded measure on $\hat{G} \times \hat{G} \Leftrightarrow \rho$ harmonizable .
b) ρ harmonizable $\Rightarrow \hat{\sigma}_\rho$ extends to a bounded measure (that is a bounded mapping on $C^0(G \times G)) \Rightarrow \hat{\sigma}_\rho$ defines a bimeasure which implies by Corollary 5 that ρ is V-bounded.
c) ($\Rightarrow$) This has been shown already in part b.
($\Leftarrow$) Let ρ be a stationary V-bounded GSP. It follows that $\hat{\rho}$ is bounded and orthogonally scattered. Corollary 2 shows that there exists a bounded measure $\mu_{\hat{\rho}}$ on $\hat{G}$ with:
$\langle \sigma_{\hat{\rho}}, f \otimes g \rangle = \langle \mu_{\hat{\rho}}, fg \rangle$. Therefore we can identify $\mu_{\hat{\rho}}$ (as a measure on the diagonal $\Delta\hat{G}$) with $\sigma_{\hat{\rho}}$ which is therefore a bounded measure. Thus ρ is harmonizable.□

Remark: As the Fourier-Stieltjes algebra is a subset of C^b Proposition 13.a together with Theorem 10 show that any harmonizable GSP can be identified with a (strongly) harmonizable stochastic process, the converse is trivial.

Since there are stationary GSPs having a covariance distribution which cannot be identified with a continuous function it is obvious that a stationary GSP need not be harmonizable or V-bounded. But if we add the continuity of σ_ρ we get:

Proposition 14: Let ρ be a GSP with covariance σ_ρ:

$$\rho \text{ stationary}, \sigma_\rho \in C^b(G \times G) \Longrightarrow \rho \text{ harmonizable} .$$

Proof: ρ stationary and $\sigma_\rho \in C^b$ imply (cf. Theorem 7) that σ_ρ can be identified with a continuous positive definite function; it follows by Bochner's theorem (cf. [21] p.19) that σ_ρ is in the Fourier-Stieltjes algebra $B(G \times G) := \mathcal{F}(M(\hat{G} \times \hat{G}))$. □

Corollary 15: Let X be a continuous stochastic process:

$$X \text{ stationary } \Longrightarrow X \text{ harmonizable } \Longrightarrow X \text{ V-bounded.}$$

Proof: This chain of implications follows from Proposition 13.b and Proposition 14, remembering that any stationary continuous stochastic process can be identified with a stationary GSP with $\sigma_\rho \in C^b(G \times G)$. □

The following theorem (for stochastic processes it was first proved in [14]) states, that any V-bounded GSP can be approximated by harmonizable ones. This is of interest as there are GSPs which are V-bounded but not harmonizable. For the proof we need linear smoothing operators for GSPs which are defined in the same way as for distributions.

Definition 10: Let ρ be a GSP, $f \in S_0(G), k \in L^1(G), h \in A(G) := \mathcal{F}(L^1(\hat{G}))$
a) $h\rho(f) := \rho(hf)$
b) $k * \rho(f) := \rho(\check{k} * f)$

Remark: Since $f \mapsto hf$ and $f \mapsto k * f$ define linear and bounded operaters on $S_0(G)$ it follows that $h\rho$ and $k * \rho$ define GSPs.

Lemma 16:
Let ρ be a GSP, $k \in L^1(G), h \in A(G)$. Then the following equations hold true:
a) $(k * \rho)\hat{} = \hat{k}\hat{\rho}$
b) $\sigma_{h\rho} = (h \otimes \bar{h})\sigma_\rho$
c) $\sigma_{k*\rho} = (k \otimes \bar{k}) * \sigma_\rho$
Proof: The easy calculations are left to the reader.

Theorem 17: For any V-bounded GSP ρ there exists a net $(\rho_\eta)_{\eta \in E}$ of harmonizable GSPs such that: $\sigma_{\rho_\eta}(x, y) \to \sigma_\rho(x, y)$ for $\eta \to \infty$ uniformly on compact sets.
Proof: Let $(e_\alpha)_{\alpha \in I}$ be a net in $S_0(G)$ constituting a tight, L^1-bounded approximate unity for $L^1(G)$, and let $(u_\beta)_{\beta \in J}$ be a bounded approximate unity for the Fourier algebra $A(G)$ in $S_0(G)$. Then we set $\rho_\eta := u_\beta(e_\alpha * \rho), \eta := (\alpha, \beta) \in E := I \times J$; according to Lemma 16.b and c

$$\sigma_{\rho_\eta} = (u_\beta \otimes \bar{u}_\beta)[(e_\alpha \otimes \bar{e}_\alpha) * \sigma_\rho];$$

Furthermore $d_\alpha := e_\alpha \otimes \bar{e}_\alpha$ is a tight, L^1-bounded approximate unit for $L^1(G \times G)$, and $u_\beta \otimes \bar{u}_\beta := v_\beta$ is a bounded approximate unit in $A(G \times G)$.

Let K be a given subset of $G \times G$. We want to show that σ_{ρ_η} converges uniformly on K. Writing $\|f\|_{K,\infty} := \sup_{x\in K} |f(x)|$, we have to verify that for any $\epsilon > 0$

$$\exists\, \eta_0 \text{ such that } \|\sigma_{\rho_\eta} - \sigma_\rho\|_{K,\infty} \leq \epsilon \;\forall\, \eta \geq \eta_0;$$

We can use the following estimate:

$$\|\sigma_{\rho_\eta} - \sigma_\rho\|_{K,\infty} = \|v_\beta(d_\alpha * \sigma_\rho) - \sigma_\rho\|_{K,\infty} \leq$$

$$\leq \|v_\beta(d_\alpha * \sigma_\rho) - d_\alpha * \sigma_\rho\|_{K,\infty} + \|d_\alpha * \sigma_\rho - \sigma_\rho\|_{K,\infty}.$$

It is not difficult to see that the second term of this estimate tends to zero for $\alpha \to \infty$ and that the first term converges to zero for $\beta \to \infty$ for arbitrary fixed α , which proves the uniform convergence of σ_{ρ_η} over K. On the other hand we have $\sigma_{\rho_\eta} \in S_0 * S_0(G \times G) \subseteq S_0(G \times G)$ and this implies that $\hat{\sigma}_{\rho_\eta} \in S_0(\hat{G} \times \hat{G}) \subseteq M(\hat{G} \times \hat{G})$, showing that ρ_η is harmonizable $\forall\, \eta \in$ E. □

Corollary 18: Any continuous V-bounded stochastic process X can be approximated by harmonizable processes uniformly over compact sets.
Proof: Any continuous V-bounded stochastic process can be identified with a uniquely determined V-bounded GSP. Now we apply Theorem 17. The approximating harmonizable GSPs can be identified with harmonizable stochastic processes. The following holds (cf. the proof of Theorem 10): $\tilde{\rho}_\eta(\delta_x) = \tilde{\rho}(u_\beta(e_\alpha * \delta_x)) = \tilde{\rho}(u_\beta(L_x e_\alpha) \to \tilde{\rho}(\delta_x)$
uniformly on compact sets by the vague continuity of $\tilde{\rho}$ □

We conclude this paper with some remarks concerning the dilation theory for GSPs. As will be shown our setting is also suitable to describe the dilation theorem for stochastic processes. More precisely, we want to point out that any V-bounded GSP is the projection of a stationary GSP (that means that there is a stationary dilation). For stochastic processes this was first shown by H. Niemi in [15] using the main result of [16]. We will use the invariance of GSPs under the Fourier transform to obtain this theorem for GSPs as a direct corollary of the main result of [16]. Due to the theory we have developed so far the dilation theorem for stationary stochastic processes is obtained as a corollary as well.

Definition 11: Let $\mathcal{H} \subset \tilde{\mathcal{H}}$ be two Hilbert spaces, ρ a GSP with $\{\rho(f) \mid f \in S_0(G)\}^- = \mathcal{H}$ A GSP $\tilde{\rho}$ into $\tilde{\mathcal{H}}$ is called a **dilation** of ρ if :

$$\rho(f) = P(\tilde{\rho}(f)) \;\forall\, f \in S_0(G),$$

P denoting the orthogonal projection from $\tilde{\mathcal{H}}$ into $\mathcal{H}$.

Theorem 19: For any bounded GSP ρ there exists a dilation $\tilde{\rho}$ into $\tilde{\mathcal{H}}$ which is orthogonally scattered and bounded.
Proof: Due to the equivalence between bounded vector measures and bounded GSPs (cf. Proposition 12) we may refer to the proof for vector measures which is given in [16], Theorem 13. □

Corollary 20: Let ρ be a GSP:
ρ V-bounded $\Longleftrightarrow \exists$ dilation $\tilde{\rho}$ which is V-bounded and stationary .
Proof: $(\Rightarrow)\rho$ V-bounded $\Rightarrow \hat{\rho}$ bounded; Theorem 19 implies $\exists$ dilation $(\hat{\rho})^{\tilde{}}$ of $\hat{\rho}$ which is bounded and orthogonally scattered; it is easy to see that this implies: $\tilde{\rho} := ((\hat{\rho})^{\tilde{}})^{\check{}}$ which is V-bounded and stationary is dilation of ρ.

(⇐) As $\|\rho(f)\|_{\mathcal{H}} \leq \|\tilde{\rho}(f)\|_{\tilde{\mathcal{H}}}\ \forall\, f \in S_0(G)$ it is clear that a GSP with a V-bounded dilation is V-bounded itself. □

As any continuous V-bounded stochastic process can be identified with a V-bounded GSP and vice versa the same result is proved for stochastic processes. Since stationarity implies V-boundedness for stochastic processes (cf. Corollary 15) the assumption of V-boundedness on the right hand side can be omitted.

Corollary 21: Let X be a continuous stochastic process, then

$$X \text{ is V-bounded} \iff \exists \text{ stationary dilation of } X.$$

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