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A new family of functional spaces
on the Euclidean n-space

It is the purpose of this note to give a first summary of results concerning a new family of functional spaces on the Euclidean n-space, to be called modulation spaces. As will be indicated the family $\{M_{p,q}^s(\mathbb{R}^n) \mid s \in \mathbb{R}, 1 \leq p, q \leq \infty\}$ behaves very much like the family $\{B_{p,q}^s(\mathbb{R}^n)\}$ of (inhomogeneous) Besov spaces. In spite of many formal similarities fairly different methods are to be used in the new setting. In contrast to the study of Besov spaces, where the dilation structure of \mathbb{R}^n plays an important role, the treatment of modulation spaces is based on the properties of \mathbb{R}^n as a locally compact abelian (lca) group. In particular, the concept of smoothness described below does not depend on that of differentiability. A full discussion of the general concept (including Banach spaces of ultra-distributions on \mathbb{R}^n) is given in [7]. In order to avoid technical difficulties in the presentation of the main properties of the new family we shall discuss only the spaces $M_{p,q}^s(\mathbb{R}^n)$ here.

Before giving the formal definition let us give a motivating hint (cf. [1], [11]). If one defines $M_\rho f(x) := \rho^{-n} f(x/\rho)$ for $\rho > 0$, then elements of $B_{p,q}^s(\mathbb{R}^n)$ can be described as those elements $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the function $\rho \mapsto \|M_\rho k * f - f\|_p = \|M_\rho (k - \delta_0) * f\|_p$ satisfies suitable growth conditions for $\rho \rightarrow 0$, for some (any) $k \in \mathcal{S}'(\mathbb{R}^n)$ satisfying $\hat{k}(0) = 1$ (whereas the relation $\|M_\rho (k - \delta_0) * f\|_p \rightarrow 0$ as $\rho \rightarrow 0$ holds true for any $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$). Here $k - \delta_0$ may even be replaced by some bounded measure μ having sufficiently many moments and satisfying $\hat{\mu}(0) = 0$ (cf. [12]).

The family $(M_\rho)_{\rho > 0}$ is a group of isometric automorphisms of the Banach convolution algebra $M(\mathbb{R}^n)$ of all bounded measures on \mathbb{R}^n . It is not only possible to replace the family by another (anisotropic)

group of dilations (yielding anisotropic Besov spaces), but one may also ask what happens if one makes use of the automorphism group of multiplication operators $(M_t)_{t \in \mathbb{R}^n}$, where M_t indicates multiplication by the exponential $x \mapsto \exp 2\pi i (\sum_{i=1}^n x_i t_i)$. Again one has (essentially as a consequence of the Riemann-Lebesgue Lemma) $\|M_t k * f\|_p \rightarrow 0$ for any $k \in L^1(\mathbb{R}^n)$, $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. By way of analogy one is thus lead to the following definition:

Fixing any $k \in \mathcal{Y}(\mathbb{R}^n)$, $k \neq 0$, set for $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$:

$$M_{p,q}^s(\mathbb{R}^n) := \{ \sigma \mid \sigma \in \mathcal{Y}'(\mathbb{R}^n), M_t k * \sigma \in L^p(\mathbb{R}^n) \text{ for all } t \in \mathbb{R}^n, \text{ and } \sigma^{(p)} : t \mapsto \|M_t k * \sigma\|_p \in L_s^q(\mathbb{R}^n) \},$$

the natural norm on $M_{p,q}^s$ being the norm of the control function $\sigma^{(p)}$ of σ (with respect to $L^p(\mathbb{R}^n)$) in $L_s^q(\mathbb{R}^n)$:

$$\|f\|_{M_{p,q}^s(\mathbb{R}^n)} := \left[\int_{\mathbb{R}^n} |\sigma^{(p)}(t)|^q (1+|t|)^{sq} dt \right]^{1/q} \text{ for } 1 \leq q < \infty, \text{ and}$$

$$\|f\|_{M_{p,\infty}^s(\mathbb{R}^n)} := \sup_{t \in \mathbb{R}^n} |\sigma^{(p)}(t)| (1+|t|)^s$$

A summary of basic results is given in the first theorem:

Theorem 1 A) $(M_{p,q}^s(\mathbb{R}^n), \|\cdot\|)$ is a Banach space and the embeddings $\mathcal{Y}(\mathbb{R}^n) \hookrightarrow M_{p,q}^s(\mathbb{R}^n) \hookrightarrow \mathcal{Y}'(\mathbb{R}^n)$ are continuous.

B) Different test functions $k^1, k^2 \in \mathcal{Y}(\mathbb{R}^n)$ define the same space and equivalent norms on $M_{p,q}^s(\mathbb{R}^n)$.

C) For $1 \leq p, q < \infty$ the space $\mathcal{Y}(\mathbb{R}^n)$ is dense in $M_{p,q}^s(\mathbb{R}^n)$, and the following natural duality holds true:

$$(M_{p,q}^s(\mathbb{R}^n))' = M_{p',q'}^{-s}(\mathbb{R}^n), \text{ for } \frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{q} + \frac{1}{q'}. \quad .$$

D) Concerning the complex method of interpolation one has for $1 \leq p_1, q_1 < \infty$, $1 \leq p_2, q_2 \leq \infty$, $s_1, s_2 \in \mathbb{R}^n$ and $\theta \in (0, 1)$.

$$[M_{p_1, q_1}^{s_1}(\mathbb{R}^n), M_{p_2, q_2}^{s_2}(\mathbb{R}^n)]_{[\theta]} = M_{p, q}^s(\mathbb{R}^n),$$

with

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad s = (1-\theta)s_1 + \theta s_2.$$

Concerning the embeddings between different metrics one has:

Proposition 2 Given $1 \leq p_1, p_2, q_1, q_2 \leq \infty$ and $s_1, s_2 \in \mathbb{R}$ the inclusion $M_{p_1, q_1}^{s_1}(\mathbb{R}^n) \subseteq M_{p_2, q_2}^{s_2}(\mathbb{R}^n)$ holds true if and only if $p_1 \leq p_2$ and

$$q_1 \leq q_2 \text{ and } s_1 \geq s_2, \text{ or}$$

$$q_1 > q_2 \text{ and } s_1 > s_2 + n/q_2 - n/q_1 (> s_2).$$

Moreover, equality holds if and only if all corresponding parameters are equal (i.e. $p_1 = p_2$, $q_1 = q_2$, and $s_1 = s_2$).

There is also a variant of Sobolev's embedding theorem. Moreover, using the fact that convolution with the Bessel potentials G_r (or equivalently, pointwise multiplication of the Fourier transforms by $(1+4\pi^2 |t|^2)^{-r/2}$, $r \in \mathbb{R}$) yields an isomorphism between $M_{p,q}^s$ and $M_{p,q}^{s+r}$ one obtains a comparison with the family of Bessel-potential spaces H_p^s (cf. [10]). Altogether one has:

Theorem 3 A) For $s > n/q'$ one has $M_{p,q}^s(\mathbb{R}^n) \hookrightarrow C^0(\mathbb{R}^n)$ (the space of continuous functions vanishing at infinity, with sup-norm).

B) For $q_1 \leq \min(p, p')$ and $q_2 \geq \max(p, p')$ one has $M_{p,q_1}^s(\mathbb{R}^n) \hookrightarrow H_p^s(\mathbb{R}^n) \hookrightarrow M_{p,q}^s(\mathbb{R}^n)$. Consequently $M_{2,2}^s(\mathbb{R}^n) = H_s^2(\mathbb{R}^n)$ for $s \in \mathbb{R}$.

Concerning different dimensions one can prove the following result on traces and extensions, showing the same "loss of smoothness" as in the case of Besov spaces (we use the symbol R_{n-k} for the restriction operator:

$$R_{n-k} f(x) = f(x, 0), (x, 0) \in \mathbb{R}^{n-k} \times \mathbb{R}^k = \mathbb{R}^n, f \in \mathcal{V}(\mathbb{R}^n).$$

Theorem 4 For $1 \leq p, q < \infty$; $k, n \in \mathbb{N}$ satisfying $k < n$ and $s > k/q'$ the restriction operator R_{n-k} extends to a bounded linear mapping on $M_{p,q}^s(\mathbb{R}^n)$ satisfying:

$$R_{n-k}(M_{p,q}^s(\mathbb{R}^n)) = M_{p,q}^{s-k/q'}(\mathbb{R}^{n-k})$$

(the quotient norm on the image being equivalent to the natural norm on the image space).

For those spaces containing $\mathcal{V}(\mathbb{R}^n)$ as a dense subspace one has the usual characterization of the norm-compact subsets:

Proposition 5 For $1 \leq p, q < \infty$ a closed, bounded subset $S \subseteq M_{p,q}^s(\mathbb{R}^n)$ is norm-compact if and only if it is (uniformly) tight and equicontinuous, i.e. if for any $\epsilon > 0$ there exist compact sets $K^1, K^2 \subseteq \mathbb{R}^n$ such that for each $f \in S$ one can find elements $f^1, f^2 \in M_{p,q}^s(\mathbb{R}^n)$ such that $\text{supp } f^1 \subseteq K^1$, $\text{supp } f^2 \subseteq K^2$ and $\|f - f^i\|_{M_{p,q}^s} < \epsilon$ for $i = 1, 2$.

Concerning pointwise multiplication we state the following two results as typical examples.

Proposition 6 A) For $1 \leq p \leq 2$ and $s > n/q'$, or $s \geq 0$ and $q = 1$ the spaces $M_{p,q}^s(\mathbb{R}^n)$ are Banach algebras with respect to pointwise multiplication.

B) For $p \leq 2$ and $s \in \mathbb{R}$ any $f \in M_{2,1}^{|s|}(\mathbb{R}^n)$ defines a compact multiplication operator from $M_{p,q}^s$ into $M_{1,q}^r$ for any $r < s$.

In contrast to the family of Besov spaces $B_{p,q}^s(\mathbb{R}^n)$, which does not contain spaces which are invariant under the Fourier transform one has (cf. [3]):

Theorem 7 A) The Fourier transform \mathcal{F} defines an automorphism on $M_{p,p}^0(\mathbb{R}^n)$, for each $p \geq 1$.

B) For $1 \leq q \leq p \leq \infty$ one has $\mathcal{F}(M_{p,q}^0(\mathbb{R}^n)) \subseteq M_{p,q}^0(\mathbb{R}^n)$.

The perhaps most interesting special cases of modulation spaces are the spaces $S_0 := M_{1,1}^0$ and its dual $S_0' := M_{\infty,\infty}^0$ which are discussed in detail elsewhere (cf. [4],[5],[6]). Besides the invariance of these spaces under the Fourier transform, and the minimality of S_0 within a certain family of functional spaces on \mathbb{R}^n there are various further useful results available for these spaces (even in the setting of lca groups). This fact was also motivating for the introduction of the whole family of modulation spaces. For example, one has in analogy with similar results in the context of tempered distributions the following kernel theorems:

Theorem 8 A) Given any bounded linear operator $T: S_0(\mathbb{R}^n) \rightarrow S'_0(\mathbb{R}^k)$ there exists a uniquely determined kernel $\tau \in S'_0(\mathbb{R}^{n+k})$ such that

$$Tf(g) = \int \tau(f \otimes g) \quad \text{for } f \in S_0(\mathbb{R}^n), g \in S_0(\mathbb{R}^k).$$

B) Given any bounded linear operator $S: S_0(\mathbb{R}^n) \rightarrow S'_0(\mathbb{R}^n)$ which commutes with translations there exists a uniquely determined element $\sigma \in S'_0(\mathbb{R}^n)$ such that $Sf = \sigma * f$ for all $f \in S_0(\mathbb{R}^n)$ (or equivalently: $Sf = \mathcal{F}^{-1}(\hat{\sigma} \cdot \hat{F}f)$).

In view of the fact that one has $S_0(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n) \hookrightarrow S'_0(\mathbb{R}^n)$ for $1 \leq p \leq \infty$ these results are useful even if one is only interested in harmonic analysis involving the spaces $L^p(\mathbb{R}^n)$.

There are various equivalent characterizations of the elements of $M_{p,q}^s(\mathbb{R}^n)$. For example, it is possible to use norms involving maximal functions. For $r \geq 0$, we associate with any continuous function f on \mathbb{R}^n the maximal function $f^{\#}(r)$, given by

$$f^{\#}(r)(x) := \sup_{y \in \mathbb{R}^n} |f(y)| (1 + |x-y|)^{-r}$$

Then one has: For any $r > m$

$$f \mapsto \left[\int_{\mathbb{R}^n} \| (M_t^k * f)^{\#}(r) \|_p^q (1 + |t|)^{sq} \right]^{1/q}$$

defines an equivalent norm on $M_{p,q}^s(\mathbb{R}^n)$, $1 \leq q < \infty$. (An appropriate version of this result is proved for arbitrary lca groups, of course without making use of the Hardy-Littlewood maximal function).

The perhaps most useful characterizations are those involving "uniform" decompositions of the Fourier transforms (in contrast to the "dyadic" decompositions used for a description of Besov spaces, see [11],[13],[15]).

Given any $\psi \in L^1(\mathbb{R}^n)$, $\psi \neq 0$, having compactly supported Fourier transform $\hat{\psi}$ and satisfying $\sum_{k \in \mathbb{Z}^n} \hat{\psi}(t-k) = 1$ (there are many such functions!), another equivalent norm on $M_{p,q}^s(\mathbb{R}^n)$ is the following:

$$\|f\|_{M_{p,q}^s} := \left[\sum_{k \in \mathbb{Z}^n} \|M_k \psi * f\|_p^q (1+|k|)^{sq} \right]^{1/q} \text{ for } 1 \leq q < \infty,$$

with obvious modifications for the case $q = \infty$.

The last mentioned norm (or a reinterpretation of the original norm) shows that modulation spaces can be identified with the inverse Fourier transforms of the so-called Wiener-type spaces $W(\mathcal{G}L^p, L_S^q)$, as introduced by the author in full generality. In fact, many of the results mentioned above can be easily deduced from related results concerning Wiener-type spaces (as given in [2],[3], and [6], for example). Detailed proofs are given in [7], where the setting of ultra-distributions on lca. groups has been chosen in order to reveal the full range of situations to which results of the above kind apply.

The similarity between (inhomogeneous) Besov spaces and modulation spaces as described in this note also suggests to look for "intermediate" spaces, obtained by using partitions of the Fourier transforms which are "between" dyadic and uniform decompositions. That this can be done (i.e. that there exists a four parameter family of spaces $D_{p,q}^{\alpha,1}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, $\alpha \in [0,1]$, such that $D_{p,q}^{0,s} = M_{p,q}^s$ and $D_{p,q}^{1,s} = B_{p,q}^s$) is shown among others in a joint paper with P. Gröbner, where general methods of defining so-called decomposition spaces are described (cf. see [8] for details).

In the Euclidean setting there is some overlap with papers by H. Triebel (see [14]) and M. Goldman (cf. [9]), where certain related results, in particular concerning traces, can be found.

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