

Multipliers of Banach Spaces of Functions on Groups

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0. Introduction

It is the main result of this paper that for a wide class of translation invariant Banach spaces of functions on locally compact groups the Banach algebra $M_r(B)$ of all continuous linear operators on B which commute with left translations can be identified with the algebra $m_r(B')$ of all right multipliers of the algebra B' of its tempered elements. For the proof we shall use a general identification theorem.

The results of this paper represent a far reaching generalization of the results of McKennon and Griffin concerning multipliers of type (p, p) .

This paper consists of two parts, the first one being of interest independently from the second one. The main result of the first part is an identification theorem for the space $H_A(B)$ of all (left) module homomorphisms of a Banach module B that can be established under certain general conditions. In the second section the results of the first section will be applied to translation invariant Banach spaces of (classes of) measurable functions on locally compact groups in order to obtain the result mentioned above.

1. A General Identification Theorem

In this section we derive some results concerning certain Banach modules and their multiplier algebras.

According to [9] we call a normed space B a normed left module over a normed algebra A if B is a left module over A in the algebraic sense and satisfies $\|ab\|_B \leq k \|a\|_A \|b\|_B$ for all $a \in A$ and $b \in B$ (cf. [9], § 32). Without loss of generality we may assume $k=1$ (by renorming A : $\|a\|_{A'} = k \|a\|_A$). If further A and B are Banach spaces B is called a left Banach A -module. Right modules are defined in an obvious way. B is called a two-sided A -module, if it is at the same time a left and a right A -module and $(ab)a_1 = a(ba_1)$ holds for all $a, a_1 \in A$ and $b \in B$. It is convenient to use the term Banach ideal in A for a dense ideal of A which is at

the same time a Banach A -module with respect to the multiplication in the algebra A . A normed left A -module B is called essential, if the span of $AB = \{ab \mid a \in A, b \in B\}$ is dense in B . Accordingly B is an essential two-sided A -module if the span of $ABA = \{aba_1 \mid a, a_1 \in A, b \in B\}$ is a dense subset of B .

Throughout this paper we shall freely make use of the factorization theorem for Banach modules ([9], 32.22). It will be used in its full strength. As a consequence of this theorem B coincides with AB if B is an essential left Banach module over a Banach algebra A with bounded, left approximate units (i.e. there is a net (u_i) in A with $\|u_i\|_A \leq C_0$ such that $\lim_i u_i a = a$ for all $a \in A$).

$H_A(B)$ denotes the algebra of all module homomorphisms of the normed left A -module B , i.e. the space of all bounded linear operators on B commuting with multiplication by elements of A on the left. For a normed algebra the set $H_B(B)$ is called the space of right multipliers on B and will be written $m_r(B)$. The corresponding right versions are denoted by $H^A(B)$ and $m_l(B)$.

In this section we shall show that $H_A(B)$ can be identified with various multiplier algebras, if the following situation is given:

A) $(B, \|\cdot\|)$ is an essential, two-sided Banach module over a Banach algebra $(A, \|\cdot\|_A)$ with bounded, two-sided approximate units $(\|u_i\|_A \leq C_0, i \in I)$;

B) A and B are both continuously embedded into a topological vector space and $A \cap B$ is dense in A as well as in B ;

C) On $A \cap B$ the multiplication, inherited from A , and the two module operations coincide, i.e. given $c_1, c_2 \in A \cap B$, the three possible "products" $c_1 c_2$ given by A) are the same.

It is a consequence of C) that for example ac is uniquely determined for all $a \in A$ and $c \in A \cap B$. Therefore, and by the various laws of associativity any possible interpretation of a finite "product" gives the same value. In particular, we need not indicate what kind of "multiplication" we are using to evaluate a certain "product".

For the rest of this section we fix two spaces A and B satisfying the Conditions A), B), and C). We emphasize that we are mainly interested in the case where B is not a Banach algebra itself!

Lemma 1.1. i) $(A \cap B, \|\cdot\|_A + \|\cdot\|)$ is a two-sided Banach ideal of A ;

ii) Let us denote the closure of $(A \cap B)$ $(A \cap B) = \{c_1 c_2 \mid c_1, c_2 \in A \cap B\}$ in $A \cap B$ by D . Then D is a closed two-sided ideal in $A \cap B$ as well as an essential two-sided Banach ideal of A which is dense in B , too. In particular we have $ADA = DA = AD = D$.

Proof. i) is a direct consequence of Condition I) and II). It is now easy to see that the closure of $(A \cap B)$ $(A \cap B)$, $A(A \cap B)$ and $(A \cap B)A$ respectively in $A \cap B$ coincide, but it follows from the factorization theorem that $A(A \cap B)$ and $(A \cap B)A$ are closed submodules of $A \cap B$. We even have $D = DA = AD$ and thus $D \subseteq ADA \subseteq A(A \cap B)A \subseteq DA = D$. Finally, again by the factorization theorem, any $b \in B$ can be written as $b = b_0 a$ with $b_0 \in B$, $a \in A$. Since $b_0 = \lim b_n$ in B with $(b_n) \in A \cap B$ we have $b_n a \in D$ for $n \in \mathbb{N}$ and $b = \lim b_n a$ in B . This establishes the density of D in B . The density in A can be proved in a similar way.

Definition. An element $c \in B$ is called (right) tempered if $\|c\|^t = \sup \{\|dc\|, d \in A \cap B, \|d\| \leq 1\} < \infty$. The space of all such c is denoted by B^t .

Note that c is tempered if and only if the map $d \mapsto dc$ ($d \in A \cap B$) can be extended to a bounded linear operator on B which will be denoted by S_c . $\|c\|'$ is just $\|S_c\|_B$, the norm of this operator. Thus we have $\|c\|' = \sup \{ \|dc\| \mid d \in D, \|d\| \leq 1 \}$.

Theorem 1.1. $(B', \|\cdot\|')$ is a two-sided normed module over A . With the norm $\| \|c\| \| \|' = \|c\|' + \|c\|$ B' is a two-sided Banach module over A and a right normed module over $(B', \|\cdot\|')$. Moreover, $(A \cap B, \|\cdot\|_A + \|\cdot\|)$ is contractively embedded into $(B', \|\cdot\|, \| \|')$ and the multiplication on $A \cap B$ can be extended to B' such that $(B', \|\cdot\|, \| \|')$ is a normed algebra. In particular $(B', \| \| \|')$ is a Banach algebra.

It should be mentioned that in general $A \cap B$ is not a dense subspace of $(B', \|\cdot\|')$.

From now on we shall refer to $(B', \|\cdot\|')$ and $(B', \| \| \|')$ as to the algebra respectively Banach algebra of tempered elements of B .

Proof of Theorem 1.1. Let $c \in B'$, $a \in A$, and $d \in A \cap B$ be given. Then:

$$\|d(ac)\| = \|(da)c\| \leq \|da\| \|c\|' \leq \|d\| \|a\|_A \|c\|'$$

and

$$\|d(ca)\| = \|(dc)a\| \leq \|dc\| \|a\|_A \leq \|d\| \|c\|' \|a\|_A.$$

This shows that $(B', \|\cdot\|')$ is a two-sided A -module. In order to prove the second assertion it will be sufficient to show that $(B', \| \| \|')$ is complete. We only have to show that $\lim c_n = c$ (in B) and $\lim \|c_n\|' = 0$ implies $c = 0$, but this is true because $dc = \lim dc_n = 0$ for all $d \in A \cap B$ implies $ac = 0$ for all $a \in A$ and this implies $c = 0$. Condition I) now implies that $A \cap B$ is contractively embedded in $(B', \| \| \|')$.

It is obvious that multiplication on B' has to be defined by $b \circ c = S_c(b) = \lim b_n c$ (in B) for $b, c \in B'$, $b = \lim b_n$ in B , $(b_n) \subseteq A \cap B$ or even in D . It follows from this definition that $d(S_c(b)) = S_c(db)$ holds for all $d \in A \cap B$, $b, c \in B'$. This, in turn, implies the stability of B' under this extended multiplication. Moreover, we have by $\|d(S_c(b))\| = \|S_c(db)\| \leq \|c\|' \|d\| \|b\|'$ the corresponding norm inequality $\|b \circ c\|' \leq \|c\|' \|b\|'$. It remains to show that this multiplication is associative. It will be sufficient to prove $S_{b \circ c} = S_c S_b$ for all $b, c \in B'$, but this follows from

$$\begin{aligned} S_c S_b(d) &= S_c(db) = S_c \lim (db_n) = \lim S_c(db_n) = \lim (db_n) c = d(\lim b_n c) \\ &= d(S_c b) = S_{b \circ c}(d) \quad \text{for all } d \in A \cap B \end{aligned}$$

(all limits in B). The proof is now complete.

We now write $B'_l(B'_r)$ for the closed linear subspace of $(B', \|\cdot\|')$ generated by $AB'(B'A)$.

Lemma 1.2. $(B'_l, \|\cdot\|')$ is a left A -module containing D . Moreover, B'_l is a closed right ideal in $(B', \|\cdot\|')$ with bounded left approximate units. Symmetric assertions hold for B'_r .

Proof. The only thing that has to be proved explicitly is the existence of left approximate units. Given any $b \in B'_l$ and any $\varepsilon > 0$, there are $a_i \in A, c_i \in B', i = 1, 2, \dots, n$ such that $\|b - \sum a_i c_i\|' < \varepsilon$. Take now any $d \in D \subseteq B'_l, \|d\|' \leq \|d\|_A \leq C_0$ with

$\|da_i - a_i\| \leq \varepsilon/\Sigma \|c_i\|^t$ for $i=1, 2, \dots, n$. Then we have

$$\begin{aligned} \|db - b\|^t &\leq \|db - d(\Sigma a_i c_i)\|^t + \Sigma \|da_i c_i - a_i c_i\|^t \\ &\leq \|d\|_A \varepsilon + \Sigma \|da_i - a_i\|_A \|c\|^t \leq (C_0 + 1) \varepsilon. \end{aligned}$$

We denote the closure of $B'_i(B'_i)$ in $\mathcal{B}(B)$ by $\mathcal{B}_i(\mathcal{B}_i)$. In fact, $\mathcal{B}_i(\mathcal{B}_i)$ is a closed subspace of $H_A(B)$.

Theorem 1.2. *The mapping $c \rightarrow S_c$ defines an isometric embedding of $(B'_i, \|\cdot\|^t)$ into \mathcal{B}_i which satisfies $S_{boc} = S_c S_b$ for all $b, c \in B'_i$. Thus \mathcal{B}_i is a Banach algebra with bounded right approximate units. Moreover, A can be contractively embedded into \mathcal{B}_i (again the embedding is a anti-morphism of the algebra). Therefore, $(B'_i, \|\cdot\|^t)$ can be regarded as essential normed ideal of \mathcal{B}_i . In particular, $(B'_i, \|\cdot\|^t)$ is a (left) Banach ideal of \mathcal{B}_i .*

Proof. The first assertion has been proved in 1.1, the second follows from Lemma 1.2 and the observation that the closure of an algebra with bounded approximate units has again (bounded) approximate units. That A can be embedded into \mathcal{B}_i follows from the fact that B'_i contains $D=AD$ and that this space is dense in A , which, in turn, is contractively embedded into $\mathcal{B}(B)$ by multiplication on the right. Thus B can be regarded as essential normed left module over \mathcal{B}_i . The other assertions follow therefrom.

We now present the main theorem of this section.

Theorem 1.3. *The following spaces of operators are isomorphic as normed spaces (and hence Banach spaces):*

$$M_r(B) \cong m_r(B', \|\cdot\|^t) \cong m_r(B'_i, \|\cdot\|^t) \cong m_l(\mathcal{B}_i).$$

Furthermore, the first two isomorphisms are algebra isomorphisms as well, and the third one is also an anti-isomorphism. Moreover, all three isomorphisms are isometric if A has bounded approximate units of norm 1 ($C_0=1$).

Proof. For $T \in H_A(B)$ we have $dT(c) = T(dc)$ for all $c \in B, d \in D$. Thus $\|T(c)\|^t = \sup\{\|T(dc)\| \|d\| \leq 1\} \leq \|T\|_B \|c\|^t < \infty$ for all $c \in B'$, i.e. T is in $H_A(B', \|\cdot\|^t)$ with $\|T\|^t \leq \|T\|_B$. Moreover, it is easy to see that $T \in H_A(B')$ leaves B'_i invariant and that T vanishes on B'_i iff it vanishes on B' . Thus we have the following contractive embeddings: $H_A(B) \subseteq H_A(B') \subseteq H_A(B'_i)$.

Let now $T \in H_A(B'_i)$ with norm $\|T\|^t$ be given. It follows from the factorization theorem that $d = d_0 a$ for some $d_0 \in D$ and $a \in A$ with $\|d - d_0\| < \delta \|d\|, \|a\|_A \leq C_0$. Furthermore, $a = \lim d_n$ in $A, (d_n) \subseteq D, \|d_n\|_A \leq C_0$ for all $n \in \mathbb{N}$. This implies that (d_n) is a Cauchy sequence in $(B', \|\cdot\|^t)$ and hence $(d_0 T(d_n))$ is a Cauchy sequence in $(B', \|\cdot\|^t)$, by Theorem 1.1. It is clear that the limit equals $T(d)$ since

$$T(d) = \lim T(d_0 d_n) = \lim d_0 T(d_n)$$

in $(B', \|\cdot\|^t)$. In particular, we have $T(d) = \lim d_0 T(d_n)$ in B . Thus

$$\begin{aligned} \|T(d)\| &\leq \overline{\lim} \|d_0 T(d_n)\| \leq \|d_0\| \overline{\lim} \|T(d_n)\|^t \\ &\leq \|d_0\| \|T\|^t C_0 \leq \|d\| (1 + \delta) \|T\|^t C_0. \end{aligned}$$

Since $\delta > 0$ did not depend on d we have $\|T(d)\| \leq C_0 \|T\|' \|d\|$ for all $d \in D$. According to Lemma 1.1. ii) T can be extended to a linear operator on B with $\|T\|_B \leq C_0 \|T\|'$ which is clearly in $H_A(B)$. It remains to show that $H_A(B', \|\cdot\|')$ and $m_r(B', \|\cdot\|')$ as well as $H_A(B'_t, \|\cdot\|')$ and $m_r(B'_t, \|\cdot\|')$ coincide. Since $A \cap B'_t$ is dense in A , one implication is obvious. Let now $T \in H_A(B')$ be given. Then we have $T(b \circ c) = \lim T(b_n c) = \lim b_n T(c) = b \circ T(c)$, the limits taken in B , since we know already that T can be considered as a continuous operator on B as well. Thus T is in $m_r(B')$ and we have proved that $H_A(B)$, $m_r(B'_t, \|\cdot\|')$ and $m_r(B', \|\cdot\|')$ are isomorphic as normed algebras.

It now follows from the proof above that any $T \in H_A(B)$ is in $m_r(B'_t)$ and thus can be extended to an operator in $m_l(\mathcal{B}_t)$ (Theorem 1.2). Let now $d \in D$, $T \in m_l(\mathcal{B}_t)$ be given. Then we may choose (as above) $d_0 \in D$, $a \in A$ with $d = d_0 a$, $\|d - d_0\| \leq \delta \|d\|$, $\|a\|_A \leq C_0$. Consequently $S_a = S_a S_{d_0}$ and further $T(S_a) = T(S_a S_{d_0}) = T(S_a) S_{d_0} = S_c$ for some $c \in B'_t$, since by Theorem 1.2 B'_t is a left ideal in \mathcal{B}_t . We denote c by $T(d)$. It is clear that $T(d) = T(S_a)(d_0)$ holds. This implies $\|T(d)\| \leq \|T\|_{\mathcal{B}_t} \|a\|_A \|d_0\| \leq \|T\|_{\mathcal{B}_t} \|d\| (1 + \delta) C_0$. Thus T can be considered as a bounded linear operator on B with $\|T\|_B \leq C_0 \|T\|_{\mathcal{B}_t}$, which is of course in $H_A(B)$. The proof is now complete.

Remarks. 1) As a consequence of the calculations above we have a continuous embedding of $H_A(B)$ into $H_A(B'_t, \|\cdot\|')$. It is not known whether these spaces must coincide.

2) Finally we note that in a symmetric way one can define tB , the space of all left tempered elements of B , and tB as the closure of tBA in $({}^tB, \|\cdot\|)$. Then $H^t(B)$, the space of all modular homomorphisms of B (now considered as a right A -module) can be identified with $m_l({}^tB)$ and $m_l({}^tB)$. The proof follows from Theorem 1.3 by interchanging left and right in the above proofs.

2. Applications

In this section we apply the results of the first section to a wide class of translation invariant Banach spaces B of (equivalence classes of) measurable functions on locally compact groups. We shall be able to identify the spaces $M_r(B)$ [$M_l(B)$] of all linear operators on B commuting with left [right] translation with various multiplier algebras of normed algebras which are associated with B in a natural way.

Throughout this section G will be a fixed, but arbitrary locally compact group with left Haar measure dx . Write $K(G)$ for the family of all continuous, complexvalued functions on G with compact support (supp). $L^1_{loc}(G)$ denotes the space of all locally integrable functions on G . As usual measurable functions coinciding l.a.e. (locally almost everywhere) shall be identified. The left (right) translation operators $L_y, (R_y)$ are defined by $L_y f(x) = f(y^{-1}x)$, $R_y f(x) = f(xy^{-1})$ $A(y^{-1})$, with A being the modular function of G ; $f^j(x) = f(x^{-1})$. The following properties of a normed space B of measurable functions on G will be of interest:

L1) (R1) B is left (right) invariant, i.e. $L_y, (R_y)$ defines a continuous linear operator on B for every $y \in G$.

L2) (R2) $y \mapsto L_y f(R_y, f)$ is a continuous function from G into B for every $f \in B$. Sometimes the spaces under consideration satisfy

L3) (R3) $L_y(R_y)$ is an isometry on B for all $y \in G$.

Spaces satisfying L1)–L3) on $G = \mathbb{R}$ (so called homogeneous Banach spaces) have been treated in Katznelson’s book ([11], Chap. VI, 1.14.) and may serve as a model.

We shall only be concerned with BF -spaces on G , i.e. Banach spaces of measurable functions such that norm convergence implies convergence in measure. This will be the case for instance if B is a solid Banach space, i.e. if B is a $L^\infty(G)$ -module with respect to pointwise multiplication. Such spaces are often called Banach function spaces (cf. [18]) or Banach lattices ([4], in particular 13.2). For any left invariant, normed space B on G the normed algebra of all bounded linear operators on B commuting with left (right) translation is denoted by $M_r(B)$ ($M_l(B)$).

Given a strictly positive, measurable function w on G , $(L_w^1(G), \| \cdot \|_{1,w})$ denotes the space of all measurable functions on G with $f w$ in $L^1(G)$ with $\| f \|_{1,w} = \| f w \|_1$. This space is a Banach convolution algebra with bounded approximate units, called Beurling algebra ([16], Chap. 1, §6 and Chap. 3, §7), if w satisfies

$$1 \leq w(x) < \infty \quad \text{and} \quad w(xy) \leq w(x)w(y) \quad \text{for all } x, y \in G.$$

Any such function is locally bounded ([1], Theorem A7).

For the rest of this section we shall consider BF -spaces $B \subseteq L_{loc}^1(G)$ satisfying the following conditions:

I) $(B, \| \cdot \|)$ is an essential two-sided Banach module over a suitable Beurling algebra $L_w^1(G)$ with respect to convolution;

II) $B \cap L_w^1(G)$ is dense in B as well as in $L_w^1(G)$.

A discussion of the conditions stated above, showing that they arise quite naturally, will be given in the sequel of this paper. For later use we want to state a further property which will enable us to describe the “extended multiplication” on B' more explicitly.

III) $B * B \subseteq L_{loc}^1(G)$, in the sense that for all $f, g \in B$, $f * g(x) = \int f(y) g(y^{-1}x) dy$ exists i.a.e. and $f * g$ is in $L_{loc}^1(G)$.

Remarks. 1) As already mentioned, we are mainly interested in spaces B which are not Banach convolution algebras.

2) If $B \subseteq L_{loc}^1(G)$ is a BF -space the embedding of B into $L_{loc}^1(G)$ (endowed with the topology, given by the family of seminorms $\{f \rightarrow \int_K |f(x)| dx, K \subseteq G \text{ compact}\}$) must be continuous (by the closed graph theorem).

3) As a consequence of 2) any $k \in K(G)$ defines a continuous linear functional on B by $f \rightarrow \langle f, k \rangle = \int f(y) k(y) dy$.

These functionals form a separating family of functionals on B . In particular $k * f = 0$ for all $k \in K(G)$ implies $f = 0$ in B , $k * f$ being a continuous function on G .

The following results will be important in the sequel of this paper. Detailed proofs can be found in [6].

Theorem 2.1. *Let $B \subseteq L_{loc}^1(G)$ be a BF -space on G . Then B satisfies L1) and L2) [and; or R1) and R2)] iff B is an essential left [two-sided; right] Banach con-*

olution module over a suitable Beurling algebra $L_w^1(G)$ (e.g. $w(y) = \max(1, \|L_y\|_B, \|R_y\|_B)$).

The proof of Theorem 2.1 is essentially a modification of the arguments used in [11] (Chap. VI, 1.14 and exercises 11–14) and is based on vector valued integration.

Theorem 2.2. *Let $B \subseteq L_{loc}^1(G)$ be a solid BF -space on G which is left and right invariant. Then B satisfies I) and II) if $K(G)$ is dense in B , in particular if B is reflexive. Moreover $K(G)$ is dense in $(L_w^1(G) \cap B, \|\cdot\|_{1,w} + \|\cdot\|)$ in this case.*

Proof. It follows from [5], Lemma 1.4 that a solid, translation invariant BF -space containing $K(G)$ as a dense subspace satisfies L2) and R2). Thus I) follows from Theorem 2.1. Moreover $K(G) \subseteq B \cap L_w^1(G)$ is dense in any Beurling algebra ([11], Chap. 3, §7.1). This implies II). That a reflexive, solid left invariant BF -space contains $K(G)$ as a dense subspace has been proved in [6]. Finally, we have to show the density of $K(G)$ in $B \cap L_w^1(G)$. Let $f \in L_w^1(G) \cap B$ and $\varepsilon > 0$ be given. By the assumption, there are $k_1, k_2 \in K(G)$ with $\|f - k_1\| < \varepsilon$ and $\|f - k_2\|_{1,w} < \varepsilon$. Since $K_1 = \text{supp } k_1 \cup \text{supp } k_2$ is compact, there is some $h \in K(G)$, $0 \leq h(x) \leq 1$ with $h(x) = 1$ on K_1 . Hence, $fh \in B \cap L_w^1(G)$, $\|f - fh\| + \|f - fh\|_{1,w} \leq \|f - k_1\| + \|f + k_2\|_{1,w} < 2\varepsilon$, and fh vanishes on the complement of a compact set $K_2 \subseteq G$. The assumption implies $fh = \lim k_j$ in B for a sequence $(k_j) \subseteq K(G)$. Of course we may suppose that all their supports are contained in a compact set $K_3 \supseteq K_2$. By remark 2) this implies $fh = \lim k_j$ in $L^1(G)$ and by the local boundedness of w , $fh = \lim k_j$ in $L_w^1(G)$. The proof is now complete.

We need one more Lemma.

Lemma 2.1. *Let T be a bounded linear operator on B . Then T commutes with left translation if and only if it commutes with convolution on the left by elements of $K(G)$ or $L_w^1(G)$.*

Proof. Let T commute with left translation. Then for $f \in B$ and $h \in L_w^1(G)$ $h * f$ can be represented as the limit of a net of finite sums $(\sum a_n L_{y_n} f)_{i \in I}$ with $(a_n)_{n=1}^i \subseteq \mathbb{C}$ and $(y_n)_{n=1}^i \subseteq G$, only depending on h but not on f (since B satisfies L2). Thus,

$$T(h * f) = T(\lim \sum a_n L_{y_n} f) = \lim \sum a_n L_{y_n} T f = h * T f.$$

Let now T be given with $T(k * f) = k * T(f)$ for all $k \in K(G), f \in B$. Since $f = \lim k_n * f$ and $T(f) = \lim k_n * T(f)$ in B for a suitable sequence $(k_n) \subseteq K(G)$ we have $L_y f = \lim L_y k_n * f$. Therefore $TL_y(f) = \lim T(L_y k_n * f) = \lim L_y k_n * T(f) = L_y T(f)$ for all $y \in G$.

We now return to Section 1. To this end take B as above and $A = L_w^1(G)$. Conditions I) and II) make sure that conditions A)–C) of Section 1 are satisfied. In particular, the theorems derived below will be applicable to all solid, translation invariant Banach spaces of measurable functions on G containing $K(G)$ as a dense subspace (Theorem 2.2). The spaces $B', B'_1, {}^1B, {}^1B, \mathcal{B}_1$ and ${}^r\mathcal{B}$ are assumed to be defined as in Section 1 (multiplication now being replaced by convolution). Using Lemma 2.1 we come to our main result:

Theorem 2.3. *The following spaces of operators are isomorphic as normed algebras (and hence Banach algebras):*

$$M_r(B) \cong m_r(B', \|\cdot\|) \cong m_r(B'_1, \|\cdot\|').$$

Furthermore they are anti-isomorphic to the Banach algebra $m_l(\mathcal{B})$.

We now come to the identification of several spaces appearing in this context. For simplicity of notation let us write $(B'_{l_c}, \|\cdot\|')$ for the space

$$\{f \mid f \in B', y \rightarrow L_y f \text{ is continuous from } G \text{ into } (B', \|\cdot\|')\}.$$

$(B'_{l_c}, \|\cdot\|, \|\cdot\|')$ is defined with $\|\cdot\|'$ being replaced by $\|\cdot\|$.

Theorem 2.4. i) $(B \cap L_w^1(G), \|\cdot\| + \|\cdot\|_{1,w})$ is an essential two-sided Banach ideal in $L_w^1(G)$.

ii) $B'_l = (B'_{l_c}, \|\cdot\|') = (B'_{l_c}, \|\cdot\|, \|\cdot\|') = L_w^1(G) * B'_l = L_w^1(G) * B' = \overline{K(G) * B'}$ (the closure being taken in $(B', \|\cdot\|')$ or in $(B', \|\cdot\|, \|\cdot\|')$).

Proof. i) is a consequence of condition I) and Theorem 2.1. In order to prove ii) we observe that the assumptions imply that B' is left invariant and that $\|L_y\|' \leq \|L_y\|_B$ holds. Thus $y \mapsto \|L_y\|'$ is locally bounded on G . This implies that $(B'_{l_c}, \|\cdot\|')$ is closed in $(B', \|\cdot\|')$ ([5], Lemma 1.2). Since B satisfies L2) we may replace $\|\cdot\|'$ by $\|\cdot\|$. By Theorem 2.1 $(B'_{l_c}, \|\cdot\|)$ is an essential left $L_w^1(G)$ -module. The factorization theorem now implies $L_w^1(G) * B'_{l_c} = B'_{l_c}$. In particular, $K(G) * B'_{l_c}$ is dense in B'_{l_c} with respect to both norms. On the other hand we have

$$K(G) * B'_{l_c} \subseteq K(G) * B' \subseteq L_w^1(G) * B' \subseteq B'_{l_c}$$

since

$$\|L_y(h * f) - h * f\|' \leq \|L_y h - h\|_{1,w} \|f\|' \quad \text{for all } f \in B' \text{ and } h \in L_w^1(G)$$

and since $L_w^1(G)$ satisfies L2). This shows that all these spaces coincide and are equal to $B'_l = L_w^1(G) * B'$.

As a consequence of i) we see that in the present case $A \cap B = L_w^1(G) \cap B$ coincides with D as defined in Section 1. In a similar way one can consider the space

$$B'_r = B' * L_w^1(G) = \overline{B' * K(G)} = \{f \mid f \in B', y \mapsto R_y f \text{ is continuous from } G \text{ into } B'\}.$$

This space has been used in [8].

If G is an Abelian, discrete or compact group, or more general if G is a SIN-group, i.e. if G has (arbitrary) small invariant neighborhoods of the identity ($y U y^{-1} = U$ for all $y \in G$), the distinction between B'_r and B'_l is superfluous.

Lemma 2.2. For $G \in [\text{SIN}]$ the spaces B'_r and B'_l coincide.

Proof. Let $G \in [\text{SIN}]$ be given with a base $\mathfrak{U} = \{U\}_{U \in \mathfrak{U}}$ of invariant neighborhoods of the identity. This directly implies that any Beurling algebra $L_w^1(G)$ on G has bounded central approximate units (take for example the family $\{u/\|u\|_1\}$, where u runs through the family of characteristic functions of the members of \mathfrak{U}). Since $B \cap L_w^1(G)$ is dense in B we have $h * f = f * h$ for all $f \in B$ and h in the center $Z(L_w^1(G))$ of $L_w^1(G)$. The factorization theorem now implies $B'_l = L_w^1(G) * B' = Z(L_w^1(G)) * B' = B' * Z(L_w^1(G)) = B'_r$.

It is now at the time to invoke condition III). The following results are based on this condition:

Theorem 2.5. Let B be a solid BF-space satisfying III). Then $f \in B$ is tempered iff $g * f$ is in B for all $g \in B$. Moreover the extended multiplication on B' is given by

ordinary convolution, i.e. $g \circ f$ coincides in B with the element given by $g * f$ for all $g \in B, f \in B'$.

Proof. Let $f \in B$ be given with $g * f \in B$ for all $g \in B$. We have to show that the linear operator $S_f: g \mapsto g * f$ has a closed graph in $B \times B$. Let $(g_n)_{n \geq 1} \subseteq B$ be given with $g_n \rightarrow g$ and $S_f(g_n) \rightarrow h$ in B . It follows $\langle g_n, k \rangle \rightarrow \langle g, k \rangle$ and $\langle g_n * f, k \rangle \rightarrow \langle h, k \rangle$ for all $k \in K(G)$. Fubini's theorem is now applicable, since B is solid and III) implies $|f| * |g| \in L^1_{loc}(G)$:

$$\begin{aligned} \langle g * f, k \rangle &= \iint k(x) g(y) f(y^{-1}x) dy dx \\ &= \iint g(y) k(x) \check{f}(x^{-1}y) dx dy = \langle g, k * \check{f} \rangle < \infty \end{aligned}$$

for all $g \in B$. It now follows from [18], §71, Theorem 5 and §69, Theorem 2 that $k * \check{f}$ defines a continuous linear functional on B . Hence $\langle g * f, k \rangle = \langle g, k * \check{f} \rangle = \lim \langle g_n, k * \check{f} \rangle = \lim \langle g_n * f, k \rangle = \langle h, k \rangle$ for all $k \in K(G)$. This implies $g * f = h$ (cf. Remark 3). Thus S_f is a bounded linear operator on B , i.e. f is in B' .

Let now $g \in B$ be given. Then there is a sequence $(g_n)_{n \geq 1} \subseteq B \cap L^1_{loc}(G)$ converging to g in B . Without loss of generality we may suppose that $|g_n(x)| \leq |g(x)|$ holds for all x , B being solid. Thus condition III) and Lebesgue's theorem on dominated convergence imply

$$g * f(x) = \int g(y) f(y^{-1}x) dy = \lim \int g_n(y) f(y^{-1}x) dy \text{ l.a.e.}$$

Therefore $g * f$ coincides with $g \circ f = \lim g_n * f$ in B .

It is evident that for solid BF -spaces containing $K(G)$ as a dense subspace, f is tempered if and only if $\sup \{ \|k * f\|, k \in K(G), \|k\| \leq 1 \} < \infty$.

Remark. Condition III), although of technical nature, seems to be indispensable in order to identify the extended multiplication with the ordinary convolution for all $f, g \in B'$. There are at least three sufficient conditions which imply III).

III') $B \subseteq L^1(G)$;

III'') $B \subseteq B_1 \cap B_1^\times$, where B_1 is any solid BF -space on G and B_1^\times is its Köthe dual (e.g. $B_1 = L^p(G)$, $B_1^\times = L^q(G)$ with $1/p + 1/q = 1$, or $B \subseteq L^2(G) = L^2(G)^\times$; in [18] B_1^\times is called associate space of B_1);

III''') $B \subseteq L^1(G) + L^2(G)$ and G is unimodular.

Proof. It is trivial that III') implies III). In the case III'') $f * g$ is defined everywhere for all $f, g \in B$ and moreover is a continuous function on G , since B satisfies L2) and inclusions of BF -spaces are automatically continuous. Finally III''') implies III) by the above calculations and the fact that $L^2(G)$ is a two-sided Banach convolution module over $L^1(G)$ if G is unimodular. Condition III''') seems to be the most interesting in this context. In particular $L^p(G)$ on a unimodular group G satisfies III''') for $1 \leq p \leq 2$.

3. Examples

The most natural examples, to which the above results are applicable, are of course the spaces $L^p(G)$, $1 < p < \infty$ (in particular on non-compact groups). It is clear that we may take $A = L^1(G)$, if G is a unimodular group. On a non-uni-

modular group we have to take $L_w^1(G)$ with $w(x) = \max(1, A^{-1+1/p}(x))$ (cf. [9], 20.14). Essentially the same is true for rearrangement invariant Banach function spaces on G which contain $K(G)$ as a dense subspace, for example, Birnbaum-Orlicz spaces ([2, 4], 13.3, and [19], Chap. 5, § 5, 6, and examples 15, 16), and Lorentz spaces (e.g. [4], 13.4).

Further examples are amalgams of these spaces with shift invariant Banach spaces of sequences X , e.g. $X = l^q$, $1 \leq q < \infty$, as considered in [10], or the spaces $A(B, X)$ defined in [5]. These spaces (with few exceptions) are solid BF -spaces containing $K(G)$ as a dense subspace. Thus, they satisfy I) and II) (Theorem 2.2). Moreover, for any such space B the space $B_w := \{f \mid f \text{ measurable, } fw \in B\}$ with the norm $\|f\|_w = \|fw\|$ satisfies again I) and II), if w is a moderate function on G , i.e. if w is a strictly positive, continuous function on G , such that for all $y \in G$ there is some $M(y) < \infty$ with

$$\max(w(yx), w(xy)) \leq M(y)w(x) \quad \text{for all } x \in G.$$

The most important spaces of this type are the spaces $L_{w_a}^p(\mathbb{R}^n)$ with $1 < p < \infty$, $w_a(x) = (1 + |x|)^a$, $a \geq 0$. They are not convolution algebras for small values of a .

There are of course examples of BF -spaces satisfying I) and II) which are not solid. The best known among them are the so called Sobolev and Besov spaces on $G = \mathbb{R}^n$. For a detailed treatment of these spaces the reader is referred to Chapter V, § 2 and § 5 of [17] and Chapter IV, § 3 of [3]. Except a few special cases, these spaces contain $C_{00}^\infty(\mathbb{R}^n)$, the space of infinitely differentiable functions with compact support on \mathbb{R}^n as a dense subspace. Since these spaces satisfy L 3) property L 2) is a consequence of the continuity of $y \rightarrow L_y f$ for $f \in C_{00}^\infty(\mathbb{R}^n)$. Thus, by Theorem 2.1, condition I) is satisfied for $A = L^1(G)$, II) is now immediate.

The potential spaces $\mathcal{L}_\alpha^p(\mathbb{R}^n)$ ([17], Chap. V, § 3) provide further examples. These spaces are defined by $\mathcal{L}_\alpha^p(\mathbb{R}^n) = \{f \mid f = G_\alpha * g, g \in L^p(\mathbb{R}^n)\}$, $1 \leq p < \infty$ for a suitable function $G_\alpha \in L^1(\mathbb{R}^n)$ with $\hat{G}_\alpha(t) \neq 0$ for all t . $\mathcal{L}_\alpha^p(\mathbb{R}^n)$ is endowed with the norm $\|f\|_{p,\alpha} = \|g\|_p$. It is clear that $\mathcal{L}_\alpha^p(\mathbb{R}^n)$ is an essential Banach module over $L^1(G)$. Moreover, by Wiener's approximation theorem ([16], Chap. 1, § 4.1), we see that $\{h \mid h = G_\alpha * g, g \in L^1 \cap L^p(\mathbb{R}^n)\}$ is a dense subspace of $L^1(\mathbb{R}^n)$ (and, of course, of $\mathcal{L}_\alpha^p(\mathbb{R}^n)$ as well). Thus $\mathcal{L}_\alpha^p(\mathbb{R}^n)$ satisfies II), too.

Finally further remarks concerning the special case $B = L^p(G)$ are appropriate. Above all we have to mention that our results generalize the results of McKennon and Griffin, and Milnes on multipliers of type (p, p) . In fact, our results comprise those given in [12-15] and [8] if G is a SIN-group and p satisfies $1 \leq p \leq 2$. In this case we have for $B = L^p(G)$: $B^l = L_p^l(G)$, $L_p^{vt} = B_r^l = B_l^l$ (cf. Theorem 2.4 and Lemma 2.2; observe furthermore that our definition of tempered elements is not identical with the original one, but coincides with it by Theorem 2.5). On the other hand, these restrictions are necessary, since most of the results of [12, 13] and [8] are based on a false lemma ([12], p. 433, first line, cf. [14] for a detailed explanation. Probably this lemma can only be avoided for the case $G \in [\text{SIN}]$. Then one has approximate units (h_i) in the center of $L^1(G)$ which satisfy $W_{h_i} \circ T = T \circ W_{h_i}$ ([7], p. 433) and the lemma is not needed. Any SIN-group is unimodular. Thus condition III) is fulfilled for $1 \leq p \leq 2$. On the other hand the proof of Theorem 1 of [12] contains an error. We have no other substitute for it if not Theorem 2.5, which again only works for $1 \leq p \leq 2$.

Most of the problems that arise if one tries to restore Theorem 1 of [12] stem from the fact that one must not suppose $f \in L_p^1$ to be positive. In fact, we have the following results:

Lemma 2.3. *Let G be a unimodular, amenable group, $f \geq 0$, $f \in L_p^1$, then f is in $L^1 \cap L^p(G)$.*

Proof. This result follows from the main result of [7].

For unimodular groups G we have the following result:

Theorem 2.6. *Let G be a unimodular group. Then for $1 \leq p < \infty$ the following spaces are isometrically, isomorphic as Banach algebras:*

- i) $M_r(L^p) \cong m_r(L_p, \|\cdot\|) \cong m_r(L_{p,1}, \|\cdot\|) \cong m_r(\mathcal{B})$;
- ii) $M_l(L^p) \cong m_l(L_p, \|\cdot\|) \cong m_l(L_{p,1}, \|\cdot\|) \cong m_l(\mathcal{B}_1)$;

Moreover there is an isometric anti-isomorphism between any space of the first chain and any other space of the second chain.

Proof. Using Theorem 2.3 and the remark at the end of Theorem 1.3 it will be sufficient to establish an isometric anti-isomorphism between $M_r(L^p)$ and $M_l(L^p)$. Such an anti-isomorphism can easily be derived from the involution $f \mapsto \check{f}$ which is an isometry on $L^p(G)$ for unimodular groups. Moreover, it satisfies $(L_y f)^\sim = R_y \check{f}$ and $(k * f)^\sim = \check{f} * \check{k}$ for $f \in L^p(G)$ and $k \in L^1(G)$.

Remark. Theorem 2.6, as it stands, extends to arbitrary rearrangement invariant BF -spaces B on unimodular groups which contain $K(G)$ as a dense subspace.

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