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Multipliers from $L^1(G)$ to a Homogeneous Banach Space

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INTRODUCTION

For the analysis of a left Banach module B over a Banach algebra A, the space $H_A(A, B)$ of all continuous module homomorphisms from A to B plays an important role. It is the purpose of this paper to present several characterizations of $H_{L^1}(L^1, B)$, the space of (right) multipliers from $L^1(G)$ to a homogeneous Banach space B. In any case $H_{L^1}(L^1, B)$ will be identified with a Banach space of Radon measures closely related to B. Incidentally, the methods used in this paper give compactness criteria for certain homogeneous Banach spaces. These results are of independent interest inasmuch as they represent generalizations of the classical Weil criterion for compactness in the $L^p(G)$-spaces to a great class of homogeneous Banach spaces.

The main results of this paper are the characterizations of $H_{L^1}(L^1, B)$ given in Theorems 1.5, 2.5, 2.9, 3.1–3.3, 3.6, and 3.8, and the compactness criteria, Theorems 2.3, 2.4 and 2.8. They represent extensions, in several directions, of results of [4, 9, and 10].

1. NOTATION AND BASIC FACTS

Let G be a locally compact group with left Haar measure dx. Left translation is given by $L_y f(x) = f(y^{-1}x)$ for all $y \in G$. $K(G)$ denotes the topological vector space of all continuous functions on G with compact support, endowed with the usual inductive limit topology. A Radon measure on G is a continuous linear functional on $K(G)$, i.e., a linear functional μ satisfying $|\langle k, \mu\rangle| \leqslant C_K \| k \|_\infty$ for all compact subsets $K \subseteq G$ and all $k \in K(G)$ with $\operatorname{supp} k \subseteq K$. The space $R(G)$ of all Radon measures on G is in a natural way a topological vector space with the vague topology. A net$(\mu_\alpha) \subseteq R(G)$ converges vaguely to $\mu \in R(G)$ iff $\langle k, \mu_\alpha\rangle \rightarrow \langle k, \mu\rangle$ for all $k \in K(G)$. Any vaguely bounded subset of $R(G)$ is vaguely relative compact. The subspace of bounded measures (i.e., with $C_K \leqslant C < \infty$ for all compact $K \subseteq G$) shall be written $M(G)$. $M(G)$ with the variation norm can be considered as Banach dual of $(C^0(G), \| \|_\infty)$, the space of all continuous functions on G vanishing at infinity. On norm bounded subsets of $M(G)$ the w^*-topology and the vague topology coincide.

Throughout this paper B shall denote a *homogeneous Banach space* on G in the sense of Katznelson [11, p. 127]. Thus we shall suppose that $(B, \|\,\|_B)$ is a Banach space of (equivalence classes of) locally integrable functions such that

(H1) *convergence in B implies convergence in measure;*

(H2) *B is left invariant and $y \to L_y f$ is a continuous function from G into B for all $f \in B$;*

(H3) $\|L_y f\|_B = \|f\|_B$ *for all $y \in G$ and $f \in B$.*

Thus B is a *Banach space with continuous shift* in the sense of [13]. If B is furthermore a dense subspace of $L^1(G)$, then B is a *Segal algebra* in the sense of Reiter [15, Sect. 4]. As indicated in [11, Chap. VI, p. 127–128; Sect. 1, Exercises 11–13], any homogeneous Banach space is an essential left $L^1(G)$-Banach module with respect to ordinary convolution (The proof is essentially the same as in [11] and is based on vectorvalued integration.)

The space of all right multipliers, i.e., (bounded) linear operators from $L^1(G)$ into B satisfying $T(f * g) = f * Tg$ for all $f \in L^1(G)$ and $g \in B$ shall be denoted by $H_{L^1}(L^1, B)$, or abbreviated as (L^1, B). In [13], (L^1, B) is written Mult B.

We now summarize some results on translation bounded measures (cf. [1, Theorem 1.1; 2, Proposition 1.12]):

THEOREM 1.1. *Let μ be a Radon measure on G. Then the following properties are equivalent*

(i) $\sup_{y \in G} |\mu| (L_y k) < \infty$ *for all $k \in K(G)$;*

(ii) $\sup_{y \in G} |\mu| (xK) < \infty$ *for all compact sets $K \subseteq G$;*

(iii) *$k * \mu$ is a bounded (continuous) function on G;*

(iv) *the linear operator $T_\mu \colon k \mapsto k * \mu$ defines a continuous multiplier from $K(G)$ into $C^b(G)$ (the space of all bounded continuous functions on G with the supremum norm).*

DEFINITION. A Radon measure μ satisfying one (and hence all) of the conditions of Theorem 1.1 is called *left translation bounded.*

The space of all such measures constitutes a Banach space $T(G)$ with $\|\mu\|_T = \sup_{y \in G} |\mu| (xK)$ as norm (K being any compact set in G with nonvoid interior). It can be shown that $T(G)$ is a left Banach module over $M(G)$ with respect to convolution [1]. In fact, by [1, Theorem 1.2], $T(G)$ is the maximal subspace of $R(G)$ with this property.

As in the case of the real line it can be shown that a homogeneous Banach space is a subspace of $\mathscr{L}(G) = L^1_{\mathrm{loc}}(G) \cap T(G)$ (cf. [11, p. 127]). Note that $T(G)$ is $\mathscr{M}_B(G)$ in the terminology of [1], and in [13], $T(G)$ is denoted by $\mathscr{W}_\infty$ and $\mathscr{L}(G)$ by $\mathscr{V}_\infty$.

LEMMA 1.2. *The unit ball of $T(G)$ is compact in the vague topology.*

Proof. It follows directly from the definition of $T(G)$ that the unit ball of $T(G)$ is vaguely bounded in $R(G)$ and hence vaguely relatively compact in $R(G)$. Thus it remains to prove that the vague limit $\mu = v\text{-}\lim \mu_\alpha$ of a net $(\mu_\alpha) \subseteq T(G)$ with $\| \mu_\alpha \|_T \leqslant 1$ again satisfies $\| \mu \|_T \leqslant 1$, but this is a consequence of the fact that $\sup_\alpha \sup_{y \in G} | \mu_\alpha | (L_y k_0) \leqslant 1$ implies $| \mu | (L_y k_0) \leqslant \limsup_\alpha | \mu_\alpha | (L_y k_0) \leqslant 1$ for all $y \in G$, i.e., $\sup_{y \in G} | \mu | (L_y k_0) \leqslant 1$.

THEOREM 1.3. *(L^1, T) is isometrically isomorphic to $T(G)$.*

Proof. Since $T(G)$ is a left Banach module over $L^1(G)$ with respect to convolution we only have to show that any $T \in H_{L^1}(L^1, T)$ is of the form $Tf = f * \mu$ for some $\mu \in T(G)$ with $\| \mu \|_T \leqslant \| T \|$. Let (u_α) be an approximate unit in $L^1(G)$ with $\| u_\alpha \| \leqslant 1$. Then $\| Tu_\alpha \|$ is bounded in $T(G)$ by $\| T \|$. Therefore (Tu_α) is vaguely relatively compact in $T(G)$. Hence for any cluster point $\mu \in T(G)$ of this net there is a subnet that converges vaguely to μ. Since the corresponding subnet of $(k * Tu_\alpha)$ converges to $k * \mu$ vaguely for every $k \in K(G)$ and $\lim_\alpha k * T(u_\alpha) = \lim_\alpha T(k * u_\alpha) = T(k)$ in $T(G)$ by the continuity of T we have $T(k) = k * \mu$ for all $k \in K(G)$. In particular μ is unique and satisfies $\| \mu \|_T \leqslant \| T \|$. Finally we obtain by approximation $T(f) = f * \mu$ for all $f \in L^1(G)$.

We now turn to the space (L^1, B). Let us recall that B can be considered as an isometric subspace of (L^1, B) since $L^1(G)$ has approximate units (u_α) of norm 1 satisfying $\lim_\alpha u_\alpha * g = g$ for all $g \in B$. Furthermore it is known that with the strong operator topology (L^1, B) is a complete, locally convex topological vectorspace, containing B as a dense subspace [17, Theorem 3.5]. This topology (restricted to B) is often called β-topology or strict topology. It is equally well known that (L^1, B) can be identified with B if B is the dual of an $L^1(G)$-module [13, 16], in particular if B is reflexive.

Before we can prove our first identification theorem we state a lemma.

LEMMA 1.4. *A subset $M \subseteq B$ is totally bounded in (B, β) iff M is bounded in B and $g * M$ is relatively compact in B for all g in a dense subset of $L^1(G)$.*

The proof is based on the uniform boundedness principle and a simple approximation argument.

THEOREM 1.5. *Let B be a homogeneous Banach space on G. Then (L^1, B) is isometrically isomorphic to the space $\tilde{B}_1 = \{\mu, \mu \in R(G),\ \mu = v\text{-}\lim f_\alpha,\ (f_\alpha)$ is totally bounded in $(B, \beta)\}$, with the infimum of $\sup_\alpha \| f_\alpha \|_B$ over all admissible families $(f_\alpha) \subseteq B$, with $\mu = v\text{-}\lim f_\alpha$, as norm $\| \mu \|_1^{\sim}$ in $\tilde{B}_1$.*

Proof. (i) Let $\mu \in \tilde{B}_1$ be given, $(f_\alpha) \subseteq B$ as in the definition of $\tilde{B}_1$ with $\sup \| f_\alpha \|_B \leqslant \| \mu \|_1^{\sim} (1 + \epsilon)$, $\epsilon > 0$. Then $f * \mu = v\text{-}\lim_\alpha f * f_\alpha$ for every $f \in K(G)$. By the assumptions, $(f * f_\alpha)$ has compact closure in B. Therefore $f * \mu$ must lie in B. Furthermore, $\| f * f_\alpha \|_B \leqslant \| f \|_1 \| f_\alpha \|_B \leqslant \| f \|_1 \| \mu \|_1^\sim (1 + \epsilon)$ implies $\| f * \mu \|_B \leqslant \| f \|_1 \| \mu \|_1^\sim (1 + \epsilon)$, since a subset of $(f * f_\alpha)$ converges to $f * \mu$ in B. $\epsilon > 0$ being arbitrary, we have obtained the required inequality. To see that this inequality extends to $f \in L^1(G)$ we note that for $f = \lim f_n$ in $L^1(G)$, $(f_n) \subseteq K(G)$, the sequence $(f_n * \mu)$ is a Cauchy sequence in B. Since $\lim f_n * \mu = f * \mu$ in $T(G)$ the limit (in B) must be $f * \mu$ and satisfies the same inequality. The proof is now complete.

(ii) Now let $T \in (L^1, B)$ be given. Then we know, by Theorem 1.3, that $T = T_\mu$ for some $\mu \in T(G)$, and that $\mu = v\text{-}\lim T(u_\alpha)$. (Without loss of generality, we may suppose that the members of (u_α) have common compact support.) If we define $f_\alpha = T(u_\alpha)$, it is clear that $\mu = v\text{-}\lim f_\alpha$ and $\| f_\alpha \|_B \leqslant \| T \|$ hold. Now let $k \in K(G)$ be given. Then $(k * u_\alpha)$ is a bounded subset of $L^1(G)$ whose elements have a common compact support. Since $y \to L_y k * u_\alpha$ is an equicontinuous family of functions from G into $L^1(G)$ $(k * u_\alpha)$ has compact closure in $L^1(G)$ (cf. Theorem 2.3). Thus, by Lemma 1.4, (f_α) is totally bounded in (B, β). Note that, by Theorem 1.1, (f_α) is contained in $C^b(G)$ if $(u_\alpha) \subseteq K(G)$.

2. COMPACTNESS CRITERIA AND APPLICATIONS

In this and the following section we shall replace the β-compactness in Theorem 1.5 by several other properties. To this aim we need some results on compactness in homogeneous Banach spaces.

DEFINITION. A subset $F \subseteq M(G)$ is called *(uniformly) tight* if for every $\epsilon > 0$ there is some compact set $K_\epsilon \subseteq G$ such that $| \mu | (G \backslash K_\epsilon) < \epsilon$ for all $\mu \in F$.

LEMMA 2.1. *Let F_1, F_2 be two tight, bounded families in $M(G)$. Then $F_1 * F_2 = \{\mu_1 * \mu_2, \mu_1 \in F_1, \mu_2 \in F_2\}$ is a (bounded) tight family in $M(G)$.*

Proof. Let $\mu_i \in F_i$, $\epsilon > 0$ be given. By the assumption, $\sup\{\| \mu \|, \mu \in F_i\} \leqslant C < \infty$ for $i = 1, 2$. Since F_1 and F_2 are tight, there are (symmetric) compact sets $K_\epsilon^1, K_\epsilon^2 \subseteq G$ with $| \mu_i | (G \backslash K_\epsilon^i) < \epsilon/2C$ for all $\mu_i \in F_i$, but this implies for $K_\epsilon = K_\epsilon^1 K_\epsilon^2$

$$| \mu_1 * \mu_2 | (G \backslash K_\epsilon) \leqslant | \mu_1 | * | \mu_2 | (G \backslash K_\epsilon) < \epsilon$$

since it is readily verified that

$$| \mu_1 | * | \mu_2 | (G \backslash K_\epsilon) \leqslant | \mu_1 | (G \backslash K_\epsilon^1) \| \mu_2 \| + | \mu_2 | (G \backslash K_\epsilon^2) \| \mu_1 \|$$

holds.

THEOREM 2.2. *Let B be a homogeneous Banach space on G, and (μ_α) be a bounded, tight net in $M(G)$ with $\mu = v\text{-}\lim \mu_\alpha$ for some $\mu \in M(G)$. Then we have*

(i) $\| \mu_\alpha * f - \mu * f \|_1 \to 0$ *for all* $f \in L^1(G)$;

(ii) $\| \mu_\alpha * g - \mu * g \|_B \to 0$ *for all* $g \in B$;

(iii) *if furthermore B is a symmetric Segal algebra* [15, Sect. 3] *and* $(\mu_\alpha) = (f_\alpha)$ *is bounded in B we have*

$$\| f * f_\alpha - f * \mu \|_B \to 0 \qquad \textit{for all}\, f \in L^1(G).$$

Note that already (i) becomes false if the tightness condition is dropped (take $\mu_n = L_{y_n} k$, $k \in K(G)$, $y_n \to \infty$ on a noncompact group). By [12, Theorem 3.3], the assumption $\| \mu_\alpha \| \to \| \mu \|$ implies "almost tightness" of the net (μ_α), in the sense that for $\epsilon > 0$ there exists $K_\epsilon \subseteq G$ and α_0 with $| \mu_\alpha | (G \backslash K_\epsilon) < \epsilon$ for all $\alpha \geqslant \alpha_\alpha$. This assumption would, of course, be sufficient for the proof of the theorem. For sequences "almost tightness" is the same as tightness. Special cases of Theorem 2.2(ii) are already known [12, Theorem 4.2; 9]. The proof of [12, Theorem 4.2] gives just Theorem 2.2(i).

Proof of the theorem. The proof will be given in several steps.

(i) By Lemma 2.1, for every $k \in K(G)$ there is a compact set $K_\epsilon \subseteq G$ with $\int_{G \backslash K_\epsilon} | (\mu_\alpha - \mu) * k(y)\, dy < \epsilon/2$ for all α. On the other hand by the vague convergence of (μ_α) we have $(\mu_\alpha - \mu) * k(y) = (\mu_\alpha - \mu)(L_y \check{k}) \to 0$ for every $y \in G$. Moreover as a consequence of the compactness of the set $\{L_y \check{k}, y \in K_\epsilon\}$ in $K(G)$, we have uniform convergence on K_ϵ. This implies $\int_{K_\epsilon} |(\mu_\alpha - \mu) * k(y)|\, dy < \epsilon/2$ for all $\alpha \geqslant \alpha_0$. Altogether we have obtained $\|(\mu_\alpha - \mu) * k \|_1 < \epsilon$ for all $\alpha \geqslant \alpha_0$, i.e., $\lim_\alpha \| \mu_\alpha * k - \mu * k \|_1 = 0$. Since $K(G)$ is dense in $L^1(G)$ and (μ_α) is bounded it is easy to see that $k \in K(G)$ can be replaced by $f \in L^1(G)$.

(ii) follows from (i) by means of the factorization theorem: Any $g \in B$ can be written as $g = f * g_1$, $f \in L^1(G)$, $g_1 \in B$. Hence $\|(\mu_\alpha - \mu) * g \|_B \leqslant \|(\mu_\alpha - \mu) * f \|_1 \| g_1 \|_B$ tends to zero. Of course (ii) can also be proved without factorization theorem, using an $\epsilon/3$-argument.

To prove (iii) we note at first that $\| f * f_\alpha - f * \mu \|_1 \to 0$ for all $f \in L^1(G)$ because the involution in $M(G)$ conserves tightness and inverts the order of convolution. As in (ii) it follows thereform that $\| g * f_\alpha - g * \mu \|_B \to 0$ for all $g \in B$, if B is a symmetric Segal algebra. It will now be sufficient to prove that $(f * f_\alpha)$ is a Cauchy net in B for all $f \in L^1(G)$. Let $\epsilon > 0$ be given. Since B is dense in $L^1(G)$, there is some $g \in B$ with $\| f - g \|_1 \leqslant \epsilon / \sup_\alpha \| f_\alpha \|_B$. Thus we have

$$\begin{aligned} \| f * f_\alpha - f * f_\beta \|_B &\leqslant \| f * f_\alpha - g * f_\alpha \|_B + \| g * f_\alpha - g * \mu \|_B \\ &\quad + \| g * \mu - g * f_\beta \|_B + \| g * f_\beta - f * f_\beta \|_B \leqslant 4\epsilon \end{aligned}$$

for all $\alpha, \beta \geqslant \alpha_0$ and the proof is complete.

THEOREM 2.3. *Let B be a symmetric Segal algebra. Then $M \subseteq B$ is compact iff it is closed and satisfies the following three conditions*:

(i) *M is bounded in B*;

(ii) *$\{y \rightarrow L_y f, f \in M\}$ is an equicontinuous family of functions from G into B; and*

(iii) *M is tight (in $L^1(G)$).*

Proof. Let a net $(f_\alpha) \subseteq M$ be given. Since M is bounded in $L^1(G)$ there is some $\mu \in M(G)$ and a subnet (f_β) satisfying $\mu = v\text{-}\lim f_\beta$. We can show that (f_β) is already a Cauchy net in B. The assumptions imply that for $\epsilon > 0$ there is some $f \in L^1(G)$ with $\| f * f_\beta - f_\beta \|_B < \epsilon/4$ for all β. According to Theorem 2.2(i), there is some β_0 such that $\| f * f_\beta - f * \mu \|_B < \epsilon/4$ for $\beta \geqslant \beta_0$. These together imply $\| f_{\beta_1} - f_{\beta_2} \|_B < \epsilon$ for $\beta_1, \beta_2 \geqslant \beta_0$, and the proof is complete.

It is clear that by the principle of symmetry assumption (ii) of Theorem 2.3 can be replaced by

(ii′) $\{y \rightarrow R_y f, f \in M\}$ is an equicontinuous family of functions from G into B;

(With R_y being the right translation operator: $R_y f(x) := f(xy^{-1}) \Delta^{-1}(y)$.) In a more general situation (ii′) seems to be even more significant.

THEOREM 2.4. *Let B be a pseudosymmetric Segal algebra. Then $M \subseteq B$ is compact iff it is closed and satisfies* (i), (ii′), *and* (iii).

Proof. If B is a pseudosymmetric Segal algebra $y \rightarrow R_y f$ is a continuous function from G into B for all $f \in B$. Thus, by the uniform boundedness principle, the function $y \rightarrow \| R_y \|$ (the norm of the operator R_y on B, which is bounded by the closed graph theorem) is bounded on compact subsets of G. Therefore (ii′) is a necessary condition. On the other hand, by the definition of a pseudosymmetric Segal algebra, B contains continuous, positive functions $h \neq 0$ with arbitrary small support. Using these functions it follows from (ii′) that for $\epsilon > 0$ there is some $h \in B$ with $\| f - f * h \|_B < \epsilon/4$ for all $f \in M$. Using this fact and Theorem 2.2(ii) instead of Theorem 2.2(iii) the proof proceeds as in Theorem 2.3.

THEOREM 2.5. *Let B be a symmetric Segal algebra on G. Then (L^1, B) is isometrically isomorphic to the space $\tilde{B}_2 = \{\mu, \ = v\text{-}\lim f_\alpha, (f_\alpha)$ is tight and bounded in $B\}$ with the infimum of $\sup_\alpha \| f_\alpha \|_B$ over all families (f_α) subject to the above conditions as norm $\| \mu \|_2$.*

Proof. This theorem follows from the second part of the proof of Theorem 1.5 and from Theorem 2.2(iii), showing that $\mu \in \tilde{B}_2$ defines a multiplier from $L^1(G)$ to B and satisfies $\| f * \mu \|_B \leqslant \| f \|_1 \| \mu \|_2^\sim$ for all $f \in L^1(G)$.

If we suppose that B is an essential $C^0(B)$-module with respect to pointwise multiplication, we can obtain results that are similar to the above result. Thus, for the rest of this section we assume that B is a homogeneous Banach space and an essential $C^0(G)$-module. The following result may serve as background information:

LEMMA 2.6. *Let B be a homogenous Banach space which is a $C^0(G)$-module. Then $K(G)$ is continuously embedded in B. Moreover B is an essential $C^0(G)$-module if and only if $K(G)$ is dense in B.*

If B is an essential $C^0(G)$-module we can introduce the concept of tightness in B.

DEFINITION. A subset $F \subseteq B$ is called tight (in B), if for every $\epsilon > 0$, there exists some $k \in K(G)$, $0 \leqslant k(x) \leqslant 1$, such that $\|f - fk\|_B < \epsilon$ for all $f \in F$ holds.

Remarks. Note that for $B = L^1(G)$ the two definitions of tightness are equivalent. It follows from the assumptions that any finite subset F of B is tight in B. By approximation this result can be extended to compact subsets of B. Essentially as in the proof of Lemma 2.1, it can be shown that $F_1 * F_2$ is tight in B if F_1 and F_2 are bounded, tight subsets of $M(G)$ and B, respectively.

DEFINITION. A subset $F \subseteq B$ is called β-tight iff $\{g * f \mid f \in F\}$ is a tight subset of B for every $g \in L^1(G)$.

Of course a bounded subset of B is β-tight if and only if $(k * f_\alpha)$ is tight in B for all $k \in K(G)$. It should be observed that a β-tight subset of $L^1(G)$ need not be tight in $L^1(G)$ (take $G = \mathbb{R}$ and $\{f_n\} = \{L_n r_n\}$, where r_n denotes the nth Rademacher function on $[0, 1]$). It is true that any tight set is β-tight in B (cf. Lemma 2.1).

THEOREM 2.7. *Let B be an essential $C^0(G)$-module and (f_α) be a bounded, β-tight net in B with $\mu = v\text{-}\lim f_\alpha$. Then we have $\|f * f_\alpha - f * \mu\|_B \to 0$ for all $f \in L^1(G)$.*

Proof. Let $k \in K(G)$, $\epsilon > 0$ be given. Then, by the assumptions, there is some $k_0 \in K(G)$, $0 \leqslant k_0(x) \leqslant 1$ with $\|(k * f_\alpha)(1 - k_0)\|_B < \epsilon$ for all α. As in the proof of Theorem 2.2(i) one can see that $(k * f_\alpha)$ converges to $k * \mu$ uniformly on compact sets, i.e., $(k * f_\alpha) k_0$ converges to $(k * \mu) k_0$ in $K(G)$ and hence in B, i.e., $\|(k * f_\alpha - k * \mu) k_0\|_B < \epsilon$ for all $\alpha, \beta \geqslant \alpha_0$. This implies $\|(k * f_\alpha - k * f_\beta\|_B < 4\epsilon$ for all $\alpha, \beta \geqslant \alpha_0$. Thus, $(k * f_\alpha)$ is a Cauchy net in B which has, of course, $k * \mu$ as its limit. The extension to $f \in L^1(G)$ can be obtained as in the proof of Theorem 2.2(iii).

It is routine to check that $\bar{B}$ is a Banach space which inherits from B the property of being a left Banach module over $M(G)$ with respect to convolution.

THEOREM 2.8. *Let B be an essential $C^0(G)$-module. A subset $M \subseteq B$ is relatively compact in B iff the following three conditions are satisfied;*

(i) *M is bounded in B;*

(ii) *$\{y \to L_y f, f \in M\}$ is an equicontinuous family of functions from G into B;*

and

(iii) *M is tight in B.*

The proof of Theorem 2.8 is the same as that of Theorem 2.3, using Theorem 2.7 instead of Theorem 2.2.

THEOREM 2.9. *Let B be an essential $C^0(G)$-module. Then (L^1, B) is isometrically isomorphic to the space $\tilde{B}_3 = \{\mu, \mu = v\text{-}\lim f_\alpha, (f_\alpha)$ β-tight and bounded in $B\}$ with the infimum of $\sup_\alpha \| f_\alpha \|_B$ over all families subject to the above conditions as norm $\| \mu \|_3^\sim$.*

The proof is left to the reader.

It should be observed that the results of the second part of this section apply to many of the pseudosymmetric Segal algebras defined on general locally compact groups. In fact, it can be shown that a Segal algebra which is a $C^0(G)$-module is pseudosymmetric if and only if it is right invariant and contains $K(G)$ as a dense subspace.

Note that the tightness assumptions are, of course, irrelevant if G is a compact group. In Section 4 we shall point out that Theorem 2.5 is a generalization of the main result of [10]. The theorems on relative compactness in B (Theorems 2.3, 2.4, and 2.8) extend Weil's criterion for relatively compact subsets of $L^p(G)$, $1 \leqslant p < \infty$ [5, p. 269], to a large class of translation invariant Banach spaces of functions on G. (Observe that we did not use the special case $B = L^1(G)$ in order to obtain the general result.) In the next section, we shall show that in Theorem 2.9, β-tightness cannot be replaced by tightness, nor can it be dropped in the general case.

3. FURTHER IDENTIFICATION THEOREMS

In this section we show that for many symmetric Segal algebras the (annoying) tightness condition of Theorem 2.5 can be dropped, although we shall show at the end of this paper that it is not superfluous in general. Besides, we mention that in particular Theorem 3.9 is of great interest because it appears as an essential step in the proof on nonfactorization for a class of Segal algebras on Abelian groups (for details see [6]).

DEFINITION. The space of all Radon measures that are a vague limit of a bounded net (f_α) in B shall be denoted by $\tilde{B}$. With the usual norm $\| \mu \|^\sim = \inf\{\sup_\alpha \| f_\alpha \|_B, \mu = v\text{-}\lim f_\alpha\}$, $\tilde{B}$ is a normed space.

Note that for any homogeneous Banach space B, $\tilde{B}$ is a subspace of $T(G)$ by Lemma 1.2. If B is contained in $L^1(G)$ (e.g., if B is a Segal algebra) we have $\tilde{B} \subseteq M(G)$.

It is the aim of this section to show that, under certain assumptions, $\tilde{B}$ can be identified with (L^1, B). Before we can do this, we give further characterizations of (L^1, B).

Throughout this section (u_α) denotes a (fixed) two-sided approximate unit of $L^1(G)$ with norm 1. To avoid complications we assume that the functions u_α have common compact support.

Definition. Let B be a homogeneous Banach space on G. Then $\{\mu \mid \mu \in R(G), (u_\alpha * \mu) \text{ is bounded in } B\}$ is denoted by $\tilde{B}_4$. $\tilde{B}_4$ is endowed with the norm $\|\mu\|_4^\sim = \sup_\alpha \|u_\alpha * \mu\|_B$.

Since we have $\mu = v\text{-}\lim u_\alpha * \mu$ for $\mu \in \tilde{B}_4$ it is evident that $\tilde{B}_4$ is a subspace of $\tilde{B}$. Thus for a Segal algebra B we have $\tilde{B}_4 = \{\mu \mid \mu \in M(G), (u_\alpha * \mu) \text{ is bounded in } B\}$, and in this second definition we could replace (u_α) by an arbitrary two-sided approximate unit of norm 1 in $L^1(G)$, in particular we might assume that $(u_\alpha) \subseteq B$ (which implies already $(u_\alpha * \mu) \subseteq B$ for all $\mu \in M(G)$). In view of this consideration the following result extends Theorem 2.3 of [4].

Theorem 3.1. *Let B be a symmetric Segal algebra. Then the spaces (L^1, B) and $\tilde{B}_4$ are isometrically isomorphic as Banach spaces.*

Proof. In view of Theorem 1.5 it will be sufficient to prove that $\mu \in \tilde{B}_4$ defines a multiplier T from $L^1(G)$ to B with $\|T\| \leqslant \|\mu\|_4^\sim$. We only have to show that for $f \in L^1(G)$, $(f * u_\alpha * \mu)$ is a Cauchy net in B. Let $\epsilon > 0$ be given. Then there is some $g \in B$ with $\|f - g\|_1 \leqslant \epsilon / \sup_\alpha \|u_\alpha * \mu\|_B$. This implies $\|f * u_\alpha * \mu - f * u_\beta * \mu\|_B \leqslant \|f * u_\alpha * \mu - g * u_\alpha * \mu\|_B + \|g * u_\alpha * \mu - g * u_\beta * \mu\|_B + \|g * u_\beta * \mu - f * u_\beta * \mu\|_B \leqslant 2\|f - g\|_1 \sup_\alpha \|u_\alpha * \mu\|_B + \|g * u_\alpha - g * u_\beta\|_B \|\mu\| \leqslant 4\epsilon$ for $\alpha, \beta \geqslant \alpha_0$, B being a symmetric Segal algebra. Hence $T(f) = f * \mu$ is in B for all $f \in L^1(G)$ and $\mu \in \tilde{B}_4$. The inequality $\|g * u_\alpha * \mu\|_B \leqslant \|g\|_1 \|\mu\|_4^\sim$ now implies $\|T\| \leqslant \|\mu\|_4^\sim$.

For more general homogeneous Banach spaces, which need not be Segal algebras, we have a similar result, if we restrict our attention to SIN-groups.

A locally compact group G is called a SIN-group if G has arbitrary small invariant neighborhoods U of the identity ($xUx^{-1} = U$ for all $x \in G$). For example Abelian and compact groups are SIN-groups. It is well known that $L^1(G)$ has central approximate units (u_α) if (and only if) G is a SIN-group. In the following theorem we shall assume (u_α) to be such an approximate unit.

Theorem 3.2. *Let B be a homogeneous Banach space on a* SIN-*group G. Then the spaces (L^1, B) and $\tilde{B}_4$ are isometrically isomorphic.*

Proof. The essential fact is to show that $f * \mu$ is in B for all $f \in L^1(G)$ and all $\mu \in \tilde{B}_4$. It is easy to see that B is an isometric subspace of $\tilde{B}_4$. The existence of central approximate units in $L^1(G)$ can be used to show that $\tilde{B}_4$ is a left $L^1(G)$-Banach module, since we have for $f \in L^1(G)$ and $\mu \in \tilde{B}_4$:

$$\| f * \mu \|_4^\sim = \sup_\alpha \| u_\alpha * f * \mu \|_B = \sup_\alpha \| f * u_\alpha * \mu \|_B \leqslant \| f \|_1 \| \mu \|_4^\sim .$$

But this together with the inclusion $(u_\alpha * \mu) \subseteq B$ implies that $(f * u_\alpha * \mu)$ is a Cauchy net in B, the limit of which is of course $f * \mu$ with $\| f * \mu \|_B = \| f * \mu \|_4^\sim \leqslant \| f \|_1 \| \mu \|_4^\sim$. The proof is now complete.

For homogeneous Banach spaces on arbitrary locally compact groups another useful condition can be used to prove $(L^1, B) = \tilde{B}_4$.

THEOREM 3.3. *Let B be a homogeneous Banach space such that B is closed in $\tilde{B}$ (is isometrically embedded in $\tilde{B}$). Then the Banach spaces (L^1, B) and $\tilde{B}_4$ are (isometrically) isomorphic.*

Proof. Observe that the condition $(u_\alpha * \mu) \in B$ for all α is part of the definition of $\tilde{B}_4$ (if B is a symmetric Segal algebra and (u_α) is taken from B this is automatically satisfied). First of all let us show that it follows from the assumptions that $\tilde{B}_4$ is a closed subspace of $\tilde{B}$ (with $\| \mu \|_4^\sim = \| \mu \|^\sim$ for all $\mu \in \tilde{B}_4$). Let (f_β) be a bounded net in B with $\mu = v\text{-}\lim_\beta f_\beta$, $\sup_\beta \| f_\beta \|_B \leqslant \| \mu \|^\sim (1 + \epsilon)$. Then we have $u_\alpha * \mu = v\text{-}\lim_\beta u_\alpha * f_\beta$ for all α. Since $(u_\alpha * \mu)$ lies in B and B is closed in $\tilde{B}$ we have

$$\| u_\alpha * \mu \|_B \leqslant C \| u_\alpha * \mu \|^\sim \leqslant C \sup_\beta \| u_\alpha * f_\beta \|_B \leqslant C \sup_\beta \| f_\beta \|_B \leqslant$$
$$\leqslant C \| \mu \|^\sim (1 + \epsilon)$$

(with $C = 1$). The vague convergence of $(u_\alpha * \mu)$ to μ now implies $\| \mu \|_4^\sim = \sup_\alpha \| u_\alpha * \mu \|_B \leqslant C \| \mu \|^\sim (1 + \epsilon)$ (with $C = 1$). Therefore it remains to identify $(\tilde{B}_4, \| \ \|^\sim)$ with (L^1, B). We show again that $(f * u_\alpha * \mu)$ is a Cauchy net in B for $f \in L^1(G)$ and $\mu \in \tilde{B}_4$. We know already that $(f * u_\alpha * \mu)$ is contained in B and bounded. Since B is closed in $\tilde{B}$ and $\tilde{B}$ is a left $L^1(G)$-module we have

$$\| f * u_\alpha * \mu - f * u_\beta * \mu \|_B \leqslant C \| f * u_\alpha * \mu - f * u_\beta * \mu \|^\sim$$
$$\leqslant C \| f * u_\alpha - f * u_\beta \|_1 \| \mu \|^\sim \leqslant C\epsilon \| \mu \|^\sim$$

for all $\alpha, \beta \geqslant \alpha_0$, and the proof is complete.

The following lemma is important, because it shows that Theorem 3.3 is applicable to all homogeneous Banach spaces which are essential $C^0(G)$-modules.

LEMMA 3.4. *Let B be a homogeneous Banach space which is an essential $C^0(G)$-module. Then B is isometrically embedded in $\tilde{B}$.*

Proof. Since the inequality $\| f \|^\sim \leqslant \| f \|_B$ is obvious we only have to show $\| f \|_B \leqslant \| f \|^\sim$ for all $f \in B$. Let us therefore assume that we have $f = v\text{-}\lim f_\alpha$

with $\sup \|f_\alpha\|_B \leqslant \|f\|^\sim + \epsilon$. It follows from the assumption that there is some $k \in K(G)$, $0 \leqslant k(x) \leqslant 1$ with $\|f - kf\|_B < \epsilon$. Then we have $kf = v\text{-}\lim kf_\alpha$ and $\sup \|kf_\alpha\|_B \leqslant \sup \|f_\alpha\|_B \leqslant \|f\|^\sim + \epsilon$. Since (kf_α) is, of course, a tight subset of B, we have by Theorem 2.7 $\|g * kf_\alpha - g * kf\|_B \to 0$ for all $g \in L^1(G)$. Let us now choose $g \in L^1(G)$ with $\|g\|_1 \leqslant 1$ and $\|g * kf - kf\|_B < \epsilon$. Then we have $\|f - g * kf_\alpha\|_B \leqslant \|f - kf\|_B + \|kf - g * kf\|_B + \|g * kf - g * kf_\alpha\|_B < 3\epsilon$ for all $\alpha \geqslant \alpha_0$. This implies $\|f\|_B \leqslant \|g * kf_\alpha\|_B + 3\epsilon \leqslant \|f\|^\sim + 4\epsilon$. Since $\epsilon > 0$ was arbitrary we have $\|f\|_B \leqslant \|f\|^\sim$.

As a first application of Theorem 3.3 we are now able to identify (L^1, C^0).

THEOREM 3.5. *The space* (L^1, C^0) *can be identified with the subspace of* $L^\infty(G)$ *consisting of all functions satisfying* $(*)$ $\lim_{x\to\infty} \int_{xK} g(y)\, dy = 0$ *for all (arbitrary small) compact subsets* $K \subseteq G$.

Proof. Since any multiplier from $L^1(G)$ to $L^\infty(G)$ is given by a function $g \in L^\infty(G)$ it is clear that any multiplier from $L^1(G)$ to $C^0(G)$ must satisfy $(*)$ (convolve g with the characteristic function of the compact set K^{-1}). On the other hand suitable multiples of characteristic functions of compact neighborhoods of the identity from a two-sided approximate unit (u_α) for $L^1(G)$, and $(u_\alpha * g)$ is automatically a bounded subset of $L^\infty(G)$ consisting of continuous function. $(*)$ just implies that g is in $(C^0(G))^\sim_4$, and therefore defines a multiplier from $L^1(G)$ to $C^0(G)$.

Let us now recall Theorem 2.9 with $B = C^0(G)$. Since $\tilde{B}$ coincides with $L^\infty(G)$, but for example constant functions do not define multipliers from $L^1(G)$ to $C^0(G)$, we see that β-tightness cannot be dropped in 2.9. On the other hand, it is not difficult to see that the function $g_0 = \sum_{n=1}^{\infty} L_n r_n$, r_n being once more the nth Rademacher function on $[0, 1]$ is in $L^\infty(\mathbb{R})$ and satisfies $(*)$. Nevertheless g_0 cannot be the vague limit of a tight net in $C^0(R)$ since it takes the value 1 on open sets outside any given compact set. Therefore β-tightness cannot be replaced by tightness in 2.9 in the general case.

THEOREM 3.6. *Let* B *be a symmetric Segal algebra which is an essential* $C^0(G)$*-module. Then* (L^1, B) *is isometrically isomorphic to* $\tilde{B}$.

Proof. By Lemma 3.4, B is an isometric subspace of $\tilde{B}$. Since B is a subspace of $L^1(G)$, $\tilde{B}$ is contained in $M(G)$. Thus we have $(u_\alpha * \mu) \subseteq B$, if we assume that (u_α) is contained in B (B is a dense subspace of $L^1(G)$), for every $\mu \in \tilde{B}$. It follows therefrom that $(u_\alpha * \mu)$ is a bounded subset of B for all $\mu \in \tilde{B}$, in other words $\tilde{B}$ coincides with $\tilde{B}_4$ in this case. Theorem 3.3 now completes the proof.

In order to extend Theorem 3.6 to Segal algebras which are not $C^0(G)$-modules, let us consider Segal algebras on Abelian groups.

DEFINITION. We say that B is (*strongly*) *character invariant* if B is closed under multiplication with characters, i.e., if $\chi f \in B$ for all $\chi \in \hat{G}$ and $f \in B$ (and

$\| \chi f \|_B = \| f \|_B$). H. C. Wang uses the term "character Segal algebra" for a strongly character invariant Segal algebra ([18]).

Since any essential $C^0(G)$-module can (as a consequence of the factorization theorem) be considered as a Banach module over the space $C^b(G)$ of all bounded continuous functions on G any such space is in particular strongly character invariant. Therefore Lemma 3.8 will be a generalization of Lemma 3.4 if we consider Segal algebras on Abelian groups.

LEMMA 3.7. *Let B be a character invariant Segal algebra on an Abelian group G. Then we have*

(i) *$M_\chi : f \to \chi f$ is a bounded operator on B for all $\chi \in \hat{G}$;*

(ii) *the function $\chi \to \| M_\chi \|$ (the norm of this operator on B) is submultiplicative and bounded on compact sets;*

(iii) *$\chi \to \chi f$ is a continuous function from $\hat{G}$ into B for all $f \in B$.*

Proof. (i) follows from the closed graph theorem. In order to obtain (ii) we first observe that (iii) holds for $f \in B$ with $\hat{f} \in K(\hat{G})$ because $\chi \to \chi f$ is continuous from $\hat{G}$ into $L^1(G)$ for all $f \in L^1(G)$ and for any compact subset $K \subseteq G$ the norms of a Segal algebra B and of $L^1(G)$ are equivalent on the set $\{f \mid f \in L^1(G), \operatorname{supp} \hat{f} \subseteq K\} \subseteq B$ [14, Chap. 6, Sect. 2.2]. Since these elements constitute a dense subspace of B [14, Chap. 6, Sect. 2.3] the submultiplicative function $\chi \to \| M_\chi \|$ is measurable. Thus by [3, Theorem A7], (ii) is satisfied. Finally the continuity of $\chi \to \chi f$ can be derived therefrom for all $f \in B$.

LEMMA 3.8. *Let B be a (strongly) character invariant Segal algebra. Then B is closed (isometrically embedded) in $\tilde{B}$.*

Proof. It is only a reformulation of Lemma 3.7(iii) to say that $\hat{B} = \{\hat{f} \mid f \in B\}$ with the norm $\| \hat{f} \|_{\hat{B}} = \| f \|_B$ is a translation invariant Banach space of continuous functions $\hat{G}$ such that $\hat{f} \to L_y \hat{f}$ is a continuous function from $\hat{G}$ into $\hat{B}$ for all $\hat{f} \in \hat{B}$. It follows therefrom that $\hat{B}$ is an essential Banach convolution module over some Beurling algebra $L_w^1(\hat{G})$ (e.g., one can take $w(\chi) = \max(\| M_\chi \|, 1)$). Now let $f \in B$ be given with $f = v\text{-}\lim f_\alpha$, $\sup \| f_\alpha \|_B = \| f \|^\sim$. Then there are some $k \in L_w^1(\hat{G})$ with $\| k \|_{1,w} \leqslant C$ and $\| \hat{f} - k * \hat{f} \|_{\hat{B}} < \epsilon$ and some $u \in B$, $\| u \|_1 = 1$ with $\| \hat{k} f - u * \hat{k} f \|_B < \epsilon$. It is now clear that $\hat{k} f = v\text{-}\lim \hat{k} f_\alpha$ and that $(\hat{k} f_\alpha)$ is tight in $L^1(G)$, $\hat{k}$ being in $C^0(G)$ and (f_α) being a bounded family in $L^1(G)$. Theorem 2.2 now implies $\| u * \hat{k} f_\alpha - u * \hat{k} f \|_B < \epsilon$ for all $\alpha \geqslant \alpha_0$. Altogether we have $\| f - u * \hat{k} f_\alpha \|_B < 3\epsilon$ for all $\alpha \geqslant \alpha_0$. Since $\| u * \hat{k} f_\alpha \|_B \leqslant \| u \|_1 \| k * \hat{f}_\alpha \|_{\hat{B}} \leqslant C \| f_\alpha \|_B \leqslant C \| f \|^\sim$ holds we have $\| f \|^\sim \leqslant \| f \|_B \leqslant C \| f \|^\sim$ for all $f \in B$. It follows from the proof that we can assume $C = 1$ if B is strongly character invariant.

THEOREM 3.9. *Let B be a (strongly) character invariant Segal algebra on a locally compact Abelian group. Then* (L^1, B) *and* $\tilde{B}$ *are (isometrically) isomorphic as Banach algebras.*

Proof. The proof follows the lines of Theorem 3.6, using Theorem 3.3 and Lemma 3.8.

At the end of this section let us indicate a further improvement of Lemma 3.4 that will considerably enlarge the family of spaces to which Theorem 3.9 is applicable. This extension is based on the observation that it would be sufficient to show $\|f\|_B \leqslant C\|f\|^{\sim}$ for all f in a dense subset of B. This in turn can be proved as in Lemma 3.4 if B_K (the functions in B with compact support) is dense in B and if there exists a family (k_γ) of (continuous) functions with compact support satisfying

(i) $\|k_\gamma f\|_B \leqslant C_1\|f\|_B$ for all γ and all $f \in B$;

(ii) for any compact set $K \subseteq G$ there is some γ with $k_\gamma(x) = 1$ for all $x \in K$.

Such a family, even for $C = 1 + \epsilon$, can be found for most of the spaces defined on $G = \mathbb{R}^m$, which are defined by continuity or differentiability properties, such as Lipschitz spaces or Sobolev and Besov spaces. In this case (k_α) has to consist of sufficiently smooth functions.

4. FINAL REMARKS

In this section we indicate some relations of this paper to results in the literature and discuss possible extensions of our results.

(A) First of all we have a few words concerning the use of the vague topology. As already mentioned this topology can be replaced by the w^*-topology on $M(G)$ throughout the text if we work with Segal algebras, because in this case it follows from Wendel's theorem that (L^1, B) is a subspace of $M(G)$. For Segal algebras on Abelian groups at least two further topologies (on $M(G)$) are of interest: the topologies of convergence of the Fourier–Stieltjes transform, either pointwise or uniformly on compact subsets of G. That we can use these topologies as well, follows essentially from Theorem 2.2, showing that these topologies are weaker than (and hence equivalent to) the vague topology on tight, bounded subsets of $L^1(G)$ (cf. [11, Chap. VI, 2, Exercise 3]). Therefore Theorem 2.5, (which can also be directly derived from Theorem 2.2 without Theorem 1.5) can be considered as an extension of Theorem 6 of [10] to non-Abelian groups (since any Segal algebra on an Abelian group is symmetric).

(B) In [4] it has been shown that for a Segal algebra on an Abelian group (L^1, B) is isometrically isomorphic with the relative completion $\tilde{B}^1$ of B in

$L^1(G)$ (cf. [4], where $\tilde{B}^1$ is denoted by $\tilde{B}$), if (L^1, B) is contained in $L^1(G)$. It is not difficult to derive this result from ours:

LEMMA 4.1. *Let B be a symmetric Segal algebra. Then the Banach spaces $\tilde{B}^1$ and $\tilde{B}_2 \cap L^1(G)$ coincide.*

Proof. This lemma is essentially a consequence of Theorem 2.2. Let $f \in \tilde{B}^1$ be given, $f = \lim f_n$ in $L^1(G)$, $\sup \| f_n \|_B < \infty$. Then (f_n) is tight in $L^1(G)$ and hence by 2.2(iii) it is in $(L^1, B) = \tilde{B}_2$. Let now $f \in \tilde{B}_2 \cap L^1(G)$ be given, $f = v\text{-}\lim f_\alpha$, $\sup \| f_\alpha \|_B < \infty$, (f_α) tight in $L^1(G)$. Since B is dense in $L^1(G)$ there is some $g \in B$, $\| g \|_1 = 1$, with $\| f * g - f \|_1 < \epsilon$. On the other hand we have by Theorem 2.2(i) $\| f_\alpha * g - f * g \|_1 < \epsilon$ for all $\alpha \geqslant \alpha_0$. Thus $\| f - f_\alpha * g \|_1 < 2\epsilon$ and $\| f_\alpha * g \|_B \leqslant \sup \| f_\alpha \|_B$ holds, i.e., $f \in \tilde{B}^1$. As a consequence we have the following result:

THEOREM 4.2. *Let B be a symmetric Segal algebra contained in $\bigcup_{p>1} L^p(G)$, then (L^1, B) is isometric isomorphic with $\tilde{B}^1$.*

Proof. It is not difficult to show that $B \subseteq \bigcup_{p>1} L^p(G)$ implies $B \subseteq L^{p}\,(G)$ for some $p_0 > 1$. But this implies $(L^1, B) \subseteq (L^1, L^{p_0}(G)) = L^{p_0}(G)$, and, further, $\tilde{B}_2 = (L^1, B) \subseteq M(G) \cap L^{p_0}(G) \subseteq L^1(G)$, i.e. $\tilde{B}^1 = \tilde{B}_2$.

The result now follows from Theorem 2.5.

(C) It should be emphasized that property (H3) is not essential. It can be shown that a space satisfying (H1) and (H2) is an essential left Banach module over a Beurling algebra $L_w^1(G)$ with respect to ordinary convolution.

In particular the proofs of Theorems 2.6–2.9, and 3.2–3.4 produce in the more general situation characterizations of $H_{L_w^1}(L_w^1, B)$. The proofs of Theorems 2.1–2.4 and 3.1, 3.5–3.8 apply to spaces that one could call "symmetric Segal algebras of Beurling algebras," i.e., dense subspaces of a Beurling algebra $L_w^1(G)$ which are at the same time essential, two-sided $L_w^1(G)$-Banach modules with respect to convolution. We only have to replace $L^1(G)$ by $L_w^1(G)$ throughout. In particular $H_{L^1}(L^1, L^1) = M(G)$ has to be replaced by $H_{L_w^1}(L_w^1, L_w^1)$. Gaudry has proved an analogous result to Wendel's theorem characterizing this space of multipliers as a weighted space of measures [7, Theorem 4]. Tightness in $M(G)$ now has to be replaced by tightness in this space of measures. Nevertheless we can use the vague topology again, since any weight function is locally bounded and locally bounded away from zero. Theorem 3.6 can be extended to this more general context if we assume either that $L_w^1(\hat{G})$ (in the proof of Theorem 3.5) satisfies the condition of Beurling–Domar [14, Chap. 6, Sect. 3.1] (in this case we may assume that $k \in L_w^1(\hat{G})$ has Fourier transform $\hat{k}$ with compact support), or for example that we have $G = \mathbb{R}^n$ and $w(x) \leqslant K(1 + |x|)^a$ for some $a < \infty$ (in this case we can choose k sufficiently smooth with $\hat{k}(x) \leqslant K_1(1 + |x|)^{-a-2}$).

(D) In the case of an Abelian group, tightness of a bounded subset of $L^1(G)$ has an alternative description:

LEMMA 4.3. *Let M be a bounded subset of $L^1(G)$. Then M is tight iff the family of functions from $\hat{G}$ into $L^1(G)$ $\{\chi \to \chi f, f \in M\}$ is equicontinuous.*

Proof. One direction is obvious, $\hat{G}$ being endowed with the compact-open topology. The other conclusion follows from an argument similar to that used in the proof of Lemma 3.4, by showing that there is some $k \in L^1(\hat{G})$ such that $\|f - \hat{k}f\|_1 < \epsilon$ for all $f \in M$ if $\{\chi \to \chi f, f \in M\}$ is an equicontinuous family of functions from G into $L^1(G)$. The tightness of M now follows from the tightness of $\{\hat{k}f, f \in M\}$ for all $k \in L^1(\hat{G})$.

(E) It is not true in general that (L^1, B) coincides with $\tilde{B}$. In particular, there are Segal algebras B such that B is not closed in $\tilde{B}$ (cf. Lemmas 3.4, 3.8). We shall prove this assertion by exhibiting a class of Segal algebras B with $\tilde{B} = M(G)$. A first example of this type has been given by C. Graham (private communication).

Let G be a second countable, Abelian group, and let H be a noncompact, closed subgroup of $\hat{G}$ of measure zero (in $\hat{G}$). Then it is not difficult to check that for p, $1 \leqslant p < \infty$

$$B = B(H, p) = \{f \mid f \in L^1(G), \hat{f}/H \in L^p(H)\}$$

is a proper Segal algebra on G with the norm $\|f\|_B = \|f\|_1 + \|\hat{f}/H\|_p$. In order to prove $\tilde{B} = M(G)$ it will be sufficient to show that on B the norms of $\tilde{B}$ and of $L^1(G)$ are equivalent. According to a result in [8] (cf. also [14, p. 152]) there exists in our situation a sequence (μ_n) of measures on G satisfying (i) $\|\mu_n\| \leqslant 2$; (ii) $\hat{\mu}_n$ equals 1 in a neighborhood of H; (iii) $\|\hat{\mu}_n * f\|_1 \to 0$ for any $f \in L^1(G)$ with $\hat{f}/H = 0$. It follows from (iii) that we have $\hat{\mu}_n(y) \to 0$ for $y \in G \backslash H$, in particular $\hat{\mu}_n(y) \to 0$ a.e. on $\hat{G}$. Lebesgue's theorem on dominated convergence now implies $\langle \mu_n * f, g \rangle = \langle \hat{\mu}_n, \hat{f}\hat{g} \rangle \to 0$ for $g \in A^1(G) = \{f \mid f \in L^1(G), \hat{f} \in L^1(\hat{G})\}$. Since $A^1(G)$ is dense in $C^\circ(G)$ we have $f = v\text{-}\lim(f - \mu_n * f)$ and therefore $\|f\|_1 \leqslant \|f\|^\sim \leqslant \sup_n \|f - \mu_n * f\|_B \leqslant 3\,\|f\|_1$, and the proof is complete.

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