

MULTIPLIERS FROM $L^1(G)$ TO SPACES OF LIPSCHITZ TYPE

by

Hans G. Feichtinger

Institut für Mathematik
der Universität Wien
Strudlhofgasse 4
A-1090 Wien
Austria

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1. Introduction. In the present paper spaces of Lipschitz type on metric, locally compact abelian groups are defined. These spaces are derived from arbitrary homogeneous Banach spaces on these groups. The "degree of smoothness" is described in terms of a solid Banach space Y of functions on $(0,1]$. It is shown that slight extra-conditions on Y imply that these spaces of Lipschitz type are again homogeneous Banach spaces. The main result of the present paper is a characterization of the space of all multipliers from $L^1(G)$ to a space of this type. Together with [3] this note may be considered as a completion to [2]. As in the case of Segal algebras on Abelian groups which are defined by means of properties of their Fourier transforms (cf. [3], Corollary 3.3.) this characterization is reduced to the problem of describing the Köthe dual of the solid Banach space of functions involved in the construction. The paper ends with a series of applications of the general result to classical spaces of Lipschitz-type on the torus and on \mathbb{R}^n . We thereby extend results due to T.S. Quek and Y.H. Yap ([6]). Their paper was the starting point and motivation for the present work.

2. Notations. Throughout this note G is a locally compact abelian group, endowed with a translation invariant metric $|\cdot|$ generating the topology of G . For simplicity we assume $|x| = |-x|$ for all $x \in G$. We shall be mainly concerned with elementary groups of the form $\mathbb{R}^m \times \mathbb{Z}^n \times \mathbb{T}^k$, $m, k, n \in \mathbb{N}$, with the Euclidian metric.

Unexplained notation is taken from [7], [8], and mainly from [2]. The reader is assumed to be familiar with the main result of [2]. Throughout, B denotes a homogeneous Banach space on G (cf. [5], Chap. VI.), i.e. a Banach space (of classes) of measurable functions on G satisfying

$$L1) L_y B \subseteq B \text{ for all } y \in G;$$

$$L2) y \rightarrow L_y f \text{ is continuous from } G \text{ into } B \text{ for all } f \in B;$$

$$L3) \|L_y f\|_B = \|f\|_B \text{ for all } f \in B, y \in G.$$

If furthermore B is a densely embedded into $L^1(G)$ B is called a Segal algebra (see [7]). For $\hat{x} \in \hat{G}$ the operator $M_{\hat{x}}$ is defined by

$$M_{\hat{x}} f(x) := \langle \hat{x}, x \rangle f(x).$$

A Segal algebra B is called character invariant iff $M_{\hat{x}} B \subseteq B$ for all $\hat{x} \in \hat{G}$. It is obvious that B has this property if it is solid, i.e. if

$$\|hf\|_B \leq \|h\|_{\infty} \|f\|_B \text{ for all } h \in L^{\infty}(G), f \in B.$$

For the construction given below we need a solid Banach space Y of measurable functions on the intervall $(0,1]$, satisfying a few conditions. For $t \in (0,1]$ let 1_t and 1^t denote the functions given by

$$1_t(s) = \begin{cases} 1 & 0 < s \leq t \\ 0 & t < s \leq 1 \end{cases} \text{ and } 1^t(s) = 1 - 1_t \text{ on } (0,1].$$

We suppose throughout that Y satisfies

Y1) $\|1_t\|_Y \rightarrow \infty$ for $t \rightarrow 0$.

Sometimes we have to impose another condition:

Y2) $\|1_t h\|_Y \rightarrow 0$ as $t \rightarrow 0$ for any $h \in Y$.

The fact that a continuous increasing function satisfying $\lim_{s \rightarrow 0} h(s) = 0$ belongs to the space Y has to say something

about the smallness of h near zero. Thus, for example,

Y1) implies that there exists such h which does not belong to Y .

Condition Y2) is equivalent to the assumptions that the space of functions having support disjoint to some neighborhood of zero is dense in Y .

Natural candidates for spaces Y arise from weighted L^p -spaces or from a weighted C^0 -space X on $[1, \infty)$ via the transformation

$$X \rightarrow Y = \overset{V}{X} := \{h \mid s \rightarrow h(s^{-1}) \in X\}.$$

For example, $Y = \overset{V}{X}$ satisfying Y1) and Y2) for $X = L^p(\mu)$,

$1 \leq p < \infty$, and μ being any unbounded, positive Radon measure

on $[1, \infty)$, or for $X = C_w^0 = \{g \mid gw \in C^0[1, \infty)\}$, w being a continuous

positive function on $[1, \infty)$ such that $\lim_{t \rightarrow \infty} w(t) = +\infty$, e.g.

$w(t) = t^\alpha$, $\alpha > 0$. For example, spaces of the following form are

used for the classical examples:

$$Y = \{h \mid \int_0^1 [t^{-\alpha} h(t)]^q dt/t\}^{1/q} < \infty\}.$$

Before we are able to define spaces of Lipschitz type to be considered in the present paper we have to fix notations.

$$\Lambda_y f := L_y f - f, \quad y \in G.$$

The modulus of continuity (of first order) of f with respect to B is given by

$$f^B(t) = \sup \{ \|\wedge_y f\|_B, |y| < t \}. \quad (1)$$

f^B defines a positive, continuous, increasing function on $(0, 1]$.
L2) implies $\lim_{t \rightarrow 0} f^B(t) = 0$ for all $f \in B$.

3. Spaces of Lipschitz type

3.1. Definition.

$$\text{Lip}(B, Y) := \{ f \mid f \in B, f^B \in Y \}; \quad (2)$$

$$\|f\|_{\text{Lip}} := \|f\|_B + \|f^B\|_Y. \quad (3)$$

Remark 1. i) For noncompact G one obtains an equivalent definition if one uses spaces \bar{Y} on $(0, \infty)$, as long as a bounded function h on $(0, \infty)$ belongs to \bar{Y} iff its restriction to $(0, 1]$ belongs to Y and $\|1^b\|_Y \leq C < \infty$ for all $b > 0$.

As only the behavior of f^B at the origin will be of relevance we prefer for technical reasons to work with spaces Y defined on $(0, 1]$ (cf. [1], p. 116).

ii) One could as well use moduli of continuity with higher order differences. This would not change very much the arguments given below. It is therefore left to the interested reader to carry out the necessary modifications.

3.2. Theorem. Let B be a homogeneous Banach space on G . Then

i) $(\text{Lip}(B, Y), \|\cdot\|_{\text{Lip}})$ is a Banach-module over $L^1(G)$ satisfying (L3);

ii) If further Y2) is satisfied, $\text{Lip}(B, Y)$ is a homogeneous Banach space. In particular, it is an essential Banach

module over $L^1(G)$ with respect to convolution, and any approximate identity $(u_\alpha)_{\alpha \in I}$ for $L^1(G)$ satisfies $\lim_{\alpha \rightarrow \infty} \|u_\alpha * f - f\|_{\text{Lip}} = 0$ for all $f \in \text{Lip}(B, Y)$.

Proof. Step I: $\sum_{i=1}^{\infty} \|f_i\|_{\text{Lip}} < \infty$ implies the convergence of $\sum f_i$ in B and

$$(\sum f_i)^B \leq \sum f_i^B. \quad (4)$$

Because Y is a solid BF-space it is clear that any absolutely convergent series $\sum f_i$ converges to some $f \in \text{Lip}(B, Y)$, with $\|f\|_{\text{Lip}} \leq \sum_{i=1}^{\infty} \|f_i\|_{\text{Lip}}$. Consequently $\text{Lip}(B, Y)$ is a Banach space.

That $\text{Lip}(B, Y)$ satisfies L3) follows from (G Abelian!):

$$\|\Delta_y L_x f\|_B = \|L_x \Delta_y f\|_B = \|\Delta_y f\|_B \quad (5)$$

which implies

$$(L_x f)^B = f^B. \quad (6)$$

Since, by the assumptions, B is an essential $L^1(G)$ -module we have for $k \in L^1(G)$, $f \in B$

$$(k * f)^B \leq \|k\|_1 f^B. \quad (7)$$

Consequently $\|k * f\|_{\text{Lip}} \leq \|k\|_1 \|f\|_{\text{Lip}}$ for all $k \in L^1(G)$, $f \in \text{Lip}$, i.e. B is a $L^1(G)$ Banach module.

Step II. Suppose now that Y2) is satisfied. We have to show that $\|L_x f - f\|_{\text{Lip}} \rightarrow 0$ for $x \rightarrow 0$. As B satisfies L2) it is sufficient to prove $\|(\Delta_x f)^B\|_Y \rightarrow 0$ as $x \rightarrow 0$ (cf. (3)). Observe that

$$\|\Delta_y \Delta_x f\|_B \leq 2 \|\Delta_x f\|_B \quad \text{for all } x, y \in G \quad (8)$$

on the other hand (4) and (6) imply

$$(\Delta_x f)^B \leq (L_x f)^B + f^B \leq 2f^B. \quad (9)$$

It follows from (8) and (9) that

$$\|(\Delta_x f)^B\|_Y \leq \|(\Delta_x f)^B 1_t\|_Y + \|(\Delta_x f)^B 1_t^t\|_Y \leq 2\|f^B 1_t\|_Y + \|\Delta_x f\|_B \|1_t^t\|_Y \quad (11),$$

for all $x \in G$ and all $t > 0$. Let now $\epsilon > 0$ be given. By Y2) we may choose $t > 0$ such that $\|f^B 1_t\|_Y < \epsilon/4$. Having chosen $t > 0$ we are able to destinate a neighborhood U of zero such that

$$\|\Delta_x f\|_B \leq \epsilon/2 \|1_t^t\|_Y \text{ for all } x \in U.$$

This choice implies $\|(\Delta_x f)^B\|_Y \leq \epsilon$ for all $x \in U$.

Step III. The final statements are true for general homogeneous Banach spaces (via vector-valued integration, cf. [5]). That the usual net (u_β) of normalized characteristic functions of neighborhoods of the identity forms an approximate identity follows from property L2) (cf. the proof for Segal algebras, [8]). Thus $L^1 * B$ is dense in B . Using Cohen's factorization theorem ([4], 32.22) we obtain $L^1 * \text{Lip} = \text{Lip}$, i.e. any $f \in \text{Lip}$ is of the form $f = k * f_1$, $k \in L^1(G)$, $f_1 \in \text{Lip}(B, Y)$. Consequently $\lim_\alpha u_\alpha * f = \lim_\alpha (u_\alpha * k) * f_1 = k * f_1 = f$ for all $f \in B$ and an arbitrary approximate identity (u_α) for $L^1(G)$.

Remark 2. So far we did not say anything about the question whether or not $\text{Lip}(B, Y)$ might become trivial. As it is well known from the ordinary Lipschitz spaces that one must not assume that $f^B(t)$ tends to zero too fast as $t \rightarrow 0$.

To see what we have to do in order to avoid this collapse let us consider the case $B = L^1(G)$. Before we are able to formulate

our result concerning this case let us introduce some notations.

3.3. Let E be any open relatively compact subset of \hat{G} . Then we define a function g_E on $(0,1]$ by

$$g_E(t) := \sup_{|y| \leq t} \sup_{\hat{x} \in E} |\langle y, \hat{x} \rangle - 1|. \quad (12)$$

It is not difficult to see that g_E is a bounded, continuous function on $(0,1]$ vanishing at zero. Moreover, the speed of convergence to zero is essentially independent of the choice of E . In fact, let E_1, E_2 be two such sets. Then the corresponding functions are equivalent, $g_1 \sim g_2$, i.e. there exists a constant $C = C(E_1, E_2)$ such that

$$C^{-1} g_1(t) \leq g_2(t) \leq C g_1(t) \text{ for } t \in (0,1]. \quad (13)$$

Proof. Step I. Let us assume for a moment that $E_1 = E$ is a compact neighborhood of the identity and that $E_2 = E + E = \{\hat{x} + \hat{y}; \hat{x}, \hat{y} \in E\}$. The inequality $g_1 \leq g_2$ is evident in this case. By a compactness argument there exists $n_0 \in \mathbb{N}$ and $(\hat{x}_i)_{i=1}^{n_0} \in E$ such that $E + E \subseteq \bigcup_{i=1}^{n_0} \hat{x}_i + E$. Consequently

$$\begin{aligned} \sup_{\hat{x} \in E_2} |\langle y, \hat{x} \rangle - 1| &\leq \max_i \sup_{\hat{x} \in E} |\langle y, \hat{x}_i + \hat{x} \rangle - 1| \\ &\leq \max_i \sup_{\hat{x} \in E} (|\langle y, \hat{x} \rangle - 1| + |\langle y, \hat{x}_i \rangle - 1|) \\ &\leq 2 g_1(t) \text{ for all } |y| \leq t. \end{aligned}$$

It follows that $g_2 \leq 2 g_1$, i.e. $g_1 \sim g_2$.

Step II. As \hat{G} is compactly generated it can be proved by induction that for E_1 above and E_2 an arbitrary compact subset of \hat{G} there is a constant C such that $g_2 \leq C g_1$.

Step III. It remains to show that for any open set $E_2 \subseteq \hat{G}$ there is some neighborhood E_1 of the identity such that for some $C' < \infty$ $g_1 \leq C'g_2$. To this aim note that

$$\sup_{\hat{x} \in E_2} |\langle y, \hat{x} \rangle - 1| = \sup_{\hat{x} \in E_2} |\langle y, -\hat{x} \rangle - 1|,$$

and therefore $g_2 = g_3$, with $E_3 = E_2 \cup (-E_2)$. Now $E_3 + E_3$ contains a neighborhood E_1 , and by the arguments of Step I $g_1 \leq 2g_3$.

3.4. We are now in the position to introduce certain spaces Y^* and Y_0^* that will be of importance in the sequel: Let E_0 be any fixed open, relatively compact subset of \hat{G} and set $g_0 := g|_{E_0}$. Then

$$\begin{aligned} Y^* &:= \{h \mid h/g_0 \in L^\infty(0,1]\} \\ \ddot{Y}_0^* &:= \{h \mid h/g_0 \in C^0(0,1]\} \end{aligned} \tag{14}$$

Y^* , endowed with the norm $\|h\|_{Y^*} := \|h/g_0\|_\infty$ is a solid BF-space on $(0,1]$. As a consequence of the above considerations the space Y^* is independent of the choice of E_0 (up to equivalence of the norms).

3.5. Proposition. i) $\text{Lip}(L^1, Y^*)$ is a dense Banach ideal of $L^1(G)$;
ii) $\text{Lip}(B, Y_0^*) = \{0\}$, if G is nondiscrete.

Proof. i) It is clear from Theorem 3.2. i) that $\text{Lip}(L^1, Y^*)$ is a Banach ideal in $L^1(G)$. As a consequence of [7], Chap. 5, § 1.1. $\text{Lip}(L^1, Y^*)$ contains any $f \in L^1(G)$ that can be written as $f = g^*h$, $g, h \in L^2(G)$, $\text{supp } \hat{g} \subseteq E$, $\text{supp } \hat{h} \subseteq E$ for some compact subset $E \subseteq \hat{G}$. That functions of this type span a dense subspace of $L^1(G)$ is obvious.

ii) Let $f \neq 0$ be given. Then there is some $A > 0$ and an open subset $E \subseteq \hat{G}$ such that $|\hat{f}(\hat{x})| \geq A$ for all $\hat{x} \in E$. This implies $f^{L^1}(t) = \sup_{|y| \leq t} \|L_y f - f\|_1 \geq \sup_{|y| \leq t} \sup_{\hat{x} \in E} |\hat{f}(t)| |\langle y, \hat{x} \rangle - 1| \geq A g_E(t)$.

It follows (cf. (14)) that $f^{L^1} \notin Y_c$.

Remark 3: The proof of ii) is a modification of the proof of [7], Chap. 1, § 2.1.

3.6. Corollary. Let B be a homogeneous Banach space. Suppose Y satisfies Y1) and $Y^* \subseteq Y$. Then we have

- i) $\text{Lip}(B, Y)$ is a dense subspace of B ;
- ii) If further Y2) is satisfied and if B is a Segal algebra, then $\text{Lip}(B, Y)$ is again a Segal algebra.

Proof. Let $f \in B$, $\epsilon > 0$ be given. Then there exists $u \in L^1$ such that $\|u * f - f\|_B < \epsilon$. According to Proposition 3.5. i) there is some $v \in \text{Lip}(L^1, Y^*)$ with $\|u - v\|_1 \leq \epsilon / \|f\|_B$. $\text{Lip}(B, Y^*) * B \subseteq \text{Lip}(B, Y^*) \subseteq \text{Lip}(B, Y)$ (cf. the proof of Theorem 3.2. i)) implies $v * f \in \text{Lip}(B, Y)$.

ii) is an immediate consequence of i).

There is another result involving the assumption $Y^* \subseteq Y$:

3.7. Lemma. Let B be a solid Segal algebra. Suppose $Y^* \subseteq Y$ and that Y1), Y2) are satisfied. Then $\text{Lip}(B, Y)$ is a character invariant Segal algebra.

Proof. Let $\hat{x} \in \hat{G}$ and $f \in \text{Lip}(B, Y)$ be given. Then

$$\begin{aligned} \|I_y M_{\hat{x}} f - M_{\hat{x}} f\|_B &\leq \|M_{\hat{x}} L_y f - M_{\hat{x}} f\|_B + \|M_{\hat{x}} L_y f - I_y (M_{\hat{x}} f)\|_B \\ &\leq \|I_y f - f\|_B + |\langle y, \hat{x} \rangle - 1| \|I_y f\|_B. \end{aligned} \quad (15)$$

Thus (15) implies $(M_{\hat{x}} f)^B(t) \leq f^B(t) + \|f\|_B g_E(t)$ for any compact set E containing \hat{x} . As g_E belongs to $Y^* \subseteq Y$ this implies $(M_{\hat{x}} f)^B \in Y$, i.e. $M_{\hat{x}} f \in \text{Lip}(B, Y)$.

4. Multipliers from $L^1(G)$ into $\text{Lip}(B, Y)$.

For the proof of the next main result we need the following

4.1. Lemma.

$O \text{Lip}(B, Y) := \{f \mid \|f\|_{\text{Lip}} \leq 1\}$ is a bounded, equicontinuous subset of B , i.e. given $\epsilon > 0$ there exists a neighborhood U of B such that

$$\|1_Y f - f\|_B \leq \epsilon \text{ for all } y \in U, f \in O \text{Lip}. \quad (16)$$

Consequently $\|u * f - f\|_B \leq \epsilon$ for all $f \in O \text{Lip}$, $u \in L^1(G)$, $\|u\|_1 = 1$ with $\text{supp } u \subseteq U$. (17)

Proof. Let $f \in \text{Lip}(B, Y)$, $\|f\|_{\text{Lip}} \leq 1$ be given. As f^B is an increasing function we have for any $t > 0$

$$1 \geq \|f^B\|_Y \geq \|f^{B, t}\|_Y \geq f^B(t) \|1^t\|_Y. \quad (18)$$

By the use of Y1) (18) implies $f^B(t) \leq \|1^t\|_Y^{-1} \rightarrow 0$, i.e. assertion i) is proved. ii) is an immediate consequence of i).

4.2. Definition. Let (u_j) be an approximate identity for the Banach algebra $C^0(0, 1]$, i.e. a sequence $(u_j)_{j \geq 1}$ such that $\lim_{j \rightarrow \infty} \|u_j f - f\|_\infty = 0$ for all $f \in C^0(0, 1]$, e.g.

$$u_j(t) = \begin{cases} 0 & \text{for } t \in (0, (2j)^{-1}) \\ \text{linear on } & [(2j)^{-1}, j^{-1}] \\ 1 & \text{for } t \in [j^{-1}, 1] \end{cases} \quad (19)$$

Given a solid space Y we define

$$\tilde{Y} := \{f \mid f \text{ measurable, } \sup_j \|u_j f\|_Y < \infty\}.$$

Remark 4. It is not difficult to prove that \tilde{Y} coincides with the Köthe dual of Y . Thus $Y = \tilde{Y}$ iff Y has the weak Fatou property (cf. [11], section 65) in particular, if Y is reflexive as a Banach space (see [11], section 73).

For $Y = C_g^0 := \{h \mid \lim_{s \rightarrow 0} h(s)/g(s) = 0\}$ one has $\tilde{Y} = L_g^\infty := \{h \mid h/g \in L^\infty(0,1]\}$. In any case, Y is a closed subspace of \tilde{Y} .

4.3. Theorem. Suppose that B is closed in \tilde{B}_4 (e.g. if B is an essential $C^0(G)$ -module), and $Y^* \subseteq Y$. Then the space of bounded linear operators from $L^1(G)$ into B commuting with translations can be identified with $\text{Lip}(B, \tilde{Y})$, i.e.

$$(L^1, \text{Lip}(B, Y)) = \text{Lip}(B, \tilde{Y})$$

Proof. Step I. Let $T \in (L^1, \text{Lip}(B, Y))$ be given. Then $T \in (L^1, B)$, and by the assumptions (cf. [2], Theorem 3.3) there exists a (translation-bounded) Radon measure $\mu \in \tilde{B}_4$ such that $Tk = \mu * k$ for all $k \in L^1(G)$. First of all we want to show that μ belongs to B .

As T is a bounded operator one has $\|Tk\|_{\text{Lip}} \leq \|T\|$ for all $k \in L^1(G)$, $\|k\|_1 \leq 1$. According to 4.1, (17), there exists $u \in L^1(G)$ such that

$$\|u * \mu * k - \mu * k\|_B < \epsilon \text{ for all } k \in L^1(G), \|k\|_1 \leq 1 \quad (20)$$

By the definition of \tilde{B}_4 this implies $\|u * \mu - \mu\|_{\tilde{B}_4} < \epsilon$. Since $u * \mu \in L^1 * \tilde{B}_4 \subseteq B$, and since B is closed in \tilde{B}_4 this implies $\mu = f \in B$.

Step II. We now want to prove that $f \in \text{Lip}(B, \tilde{Y})$. Recall that

$$\|f * u_\alpha\|_{\text{Lip}} \leq \|T\| \text{ for all } \alpha, \quad (21)$$

(u_α) being a bounded approximate identity in $L^1(G)$. Thus the functions

$$f_{\alpha}^B(t) := \sup_{|y| \leq t} \|(L_y f - f) * u_{\alpha}\|_B = \sup_{|y| \leq t} \|\Delta_y(f * u_{\alpha})\|_B$$

satisfy

$$\|f_{\alpha}^B\|_Y \leq \|T\| \text{ for all } \alpha. \quad (22)$$

Now $\{\Delta_y f | y \in K\}$ is an equicontinuous subset of B for any compact subset of $K \subseteq G$ (cf. the proof of Lemma 4.1.) and therefore $\|\Delta_y f * u_{\alpha}\|_B$ converges to $\|\Delta_y f\|_B$ uniformly on compact sets of $(0, 1]$. As Y is a solid BF-space and the approximate units (u_j) (cf. 4.2.) have compact supports we have by (22)

$$\|u_j f^B\|_Y = \lim_{\alpha} \|u_j f_{\alpha}^B\|_Y \leq \|T\| \text{ for all } j \geq 1 \quad (23)$$

This implies $f \in \text{Lip}(B, \tilde{Y})$. I) and II) give

$$(L^1, \text{Lip}(B, Y) \subseteq \text{Lip}(B, \tilde{Y}). \quad (24)$$

Step III. In view of (24) it remains to show $L^1 * \text{Lip}(B, \tilde{Y}) \subseteq \text{Lip}(B, Y)$.

Now $\text{Lip}(L^1, Y^*) * B \subseteq \text{Lip}(B, Y)$ (cf. 3.6.). Let now $f_1 \in L^1(G)$, $f_2 \in \text{Lip}(B, \tilde{Y})$ be given. By Proposition 3.5. there exists a sequence $(f_n)_{n \geq 3}$ in $\text{Lip}(L^1, Y^*)$ such that $f_n \rightarrow f_1$ in $L^1(G)$.

Since

$$\|k * f_2\|_{\text{Lip}(B, Y)} \leq \|k\|_{\text{Lip}(L^1, Y^*)} \|f_2\|_B \text{ for } k \in \text{Lip}(L^1, Y^*), f_2 \in B,$$

$(f_n * f_2)$ is a Cauchy sequence in $\text{Lip}(B, Y)$. The completeness of $\text{Lip}(B, Y)$ implies $f_1 * f_2 \in \text{Lip}(B, Y)$, and the proof is complete.

4.4. Corollary. With the assumptions of Theorem 4.3. the following is true

i) $(L^1, \text{Lip}(B, Y)) = (L^1, \text{Lip}(B, \tilde{Y})) \cong \text{Lip}(B, \tilde{Y})$, in particular $(L^1, \text{Lip}(B, Y)) = \text{Lip}(B, Y)$ if Y is a reflexive Banach space;

If further Y satisfies Y2) one also has

ii) $L^1 * \text{Lip}(B, \tilde{Y}) = \text{Lip}(B, Y) = L^1 * \text{Lip}(B, Y)$;

iii) $f \in \text{Lip}(B, Y) \iff f \in \text{Lip}(B, \tilde{Y})$ and $\lim_{|x| \rightarrow 0} \|\Delta_x f\|_{\text{Lip}(B, \tilde{Y})} = 0$;

If B is a solid Segal algebra the chain of equations given in i) can be extended to

iv) $(L^1, \text{Lip}(B, Y)) = (\text{Lip}(B, Y))^{\sim} = \text{Lip}(B, Y)_4^{\sim}$; and

v) the relative completion of $\text{Lip}(B, Y)$ in $L^1(G)$ coincides with $\text{Lip}(B, \tilde{Y})$.

Proof. i) follows from the equation $\tilde{Y} = \tilde{\tilde{Y}}$;

ii) stems from Theorem 3.2. i) and Theorem 4.3.;

iii) follows from Theorem 3.2. ii) and from ii);

iv) is a consequence of i) and Lemma 3.7. above, together with Theorems 3.9. and 3.3. of [2];

v) follows from iv) by means of Lemma 4.1. of [2], since $\text{Lip}(B, \tilde{Y}) \subseteq L^1$ for any Segal algebra B .

5. Applications. In this section the general result is applied to typical spaces of Lipschitz type on $G = \mathbb{R}^n$ or \mathbb{T} that have been considered in the literature.

First of all we observe that $g_0(t) \sim t$ as $t \rightarrow 0$ for these groups.

This follows from [7], Chap. 1, § 2.1. and Chap. 5, § 1.1.

in the case $G = \mathbb{R}^n$, $n \geq 1$, and for the n -torus the same result is available. We have thus

$$Y^* = L_1^{\infty} := \{h | s \mapsto h(s)/s \in L^{\infty}(0, 1]\}.$$

Let now $L_{\alpha}^q(0,1]$, $1 \leq q < \infty$ denote the space

$$\{h \mid \|h\|_{q,\alpha} = [\int_0^1 (t^{\alpha} |h(t)|)^q dt/t]^{1/q} < \infty \}.$$

Then it is not difficult to verify that $L_{\alpha}^q(0,1]$ contains Y^* and satisfies Y1) and Y2) for $0 < \alpha < 1$. The same is true for

$$Y = C_{\alpha}^0 = \{h \mid h \text{ stetig, } \lim_{s \rightarrow 0} h(s)/s^{\alpha} = 0\}, \quad 0 < \alpha < 1.$$

This allows us to prove the following results concerning multipliers from $L^1(\mathbb{T})$ to the Lipschitz spaces Λ_{α} , λ_{α} , Λ_{α}^p or λ_{α}^p respectively (compare [12], Chap. II, § 3 (for the definitions)).

5.1. Theorem. Let $0 < \alpha < 1$ be given. Then

- i) $(L^1, \Lambda_{\alpha}) = (L^1, \lambda_{\alpha}) = \Lambda_{\alpha} = \tilde{\lambda}_{\alpha}$
- ii) $(L^1, \Lambda_{\alpha}^p) = (L^1, \lambda_{\alpha}^p) = \Lambda_{\alpha}^p = \tilde{\lambda}_{\alpha}^p$ for $1 \leq p < \infty$.

Remark. The above result is not new. It coincides with the main result of [6] (compare also [10], Corollary 2.8.).

Proof. In view of the above remarks the result follows directly from Corollary 4.4., as soon as one has checked the following equations:

$$\begin{aligned} \Lambda_{\alpha} &= (C^0, L_{\alpha}^{\infty}), & \lambda_{\alpha} &= (C^0, C_{\alpha}^0); \\ \Lambda_{\alpha}^p &= (L^p, L_{\alpha}^{\infty}), & \lambda_{\alpha}^p &= (L^p, C_{\alpha}^0), \quad 1 \leq p < \infty; \end{aligned}$$

In [1] more general Lipschitz spaces $\Lambda(\theta, r, q, B)$, with $B = C(\mathbb{T})$ or $B = L^p(\mathbb{T})$, $1 \leq p \leq \infty$ are considered. One easily verifies that $\Lambda(\theta, 1, q, B) = \text{Lip}(B, L_{\theta}^q)$. We thus have in the notation of [1]:

5.2. Theorem.

$$(L^1, \text{Lip}(\theta, t, q, B)) = \text{Lip}(\theta, t, q, B) \text{ for } 1 \leq q < \infty \quad 0 \leq \theta < 1.$$

If one carries out the whole program for the difference operator Δ^r of r -th order, $r \geq 2$, instead of $\Delta = \Delta^1$ one can derive similar results for $\Lambda(\theta, r, q, B)$, $r \in \mathbb{N}$. We conclude the applications with a result concerning the family of Lipschitz spaces $\Lambda(\alpha, p, q)$ on \mathbb{R}^n studied in [9].

5.3. Theorem. Let $0 < \alpha < 1$, $1 \leq p, q < \infty$ be given. Then

$$(L^1, \Lambda(\alpha, p, q; \mathbb{R}^n)) = \Lambda(\alpha, p, q; \mathbb{R}^n).$$

Proof. It follows from [9], Theorem 4, G that $\Lambda(\alpha, p, q; \mathbb{R}^n)$ can be identified with the space $\text{Lip}(L^p(\mathbb{R}^n), L^q_{\alpha})$ in our setting. The result thus follows directly from 4.4. i).

Concluding remark. We do not intend to give here more applications. Instead, we only emphasize that the formulation of the general result admits a variety of further applications. Moreover, it turns out that assumption L3) has not been of importance, and the results of this paper remain true for more general Banach spaces of measurable functions satisfying only L1) and L2), such as certain weighted L^p -spaces. One only has to replace throughout $L^1(G)$ by a suitable Beurling algebra $L^1_w(G)$.

R e f e r e n c e s

- [1] P.L. BUTZER - K.L. SCHERER, Approximationsprozesse und Interpolationsmethoden, Bibl. Institut, Mannheim/Zürich, 1968.
- [2] H.G. FEICHTINGER, Multipliers from $L^1(G)$ to a homogeneous Banach space, Journ. Math. Anal. Appl. 61 (1977), 341 - 356.
- [3] H.G. FEICHTINGER, Results on Banach ideals and spaces of multipliers, Math. Scand. 41 (1977), 315 - 324.

- [4] E. HEWITT and K.A. ROSS, Abstract Harmonic Analysis II (Grundlehren Math. Wiss. 152), Springer-Verlag, Berlin-Heidelberg-New York, 1970.
- [5] Y. KATZNELSON, An Introduction to Harmonic Analysis, J. Wiley & Sons, New York, 1968.
- [6] T.S. QUEK and Y.H. YAP, Multipliers from $L^1(G)$ to a Lipschitz space, preprint.
- [7] H. REITER, Classical Harmonic Analysis and Locally Compact Groups, Oxford University Press, London, 1968.
- [8] H. REITER, L^1 -algebras and Segal algebras, Lecture Notes in Math., Vol. 231, Springer, Berlin-Heidelberg-New York, 1971.
- [9] M.H. TAIBLESON, On the theory of Lipschitz spaces on Euclidian n-space, I; J. Math. Mech. 13 (1964), 407 - 480.
- [10] H.C. WANG, Homogeneous Banach algebras, Lecture Notes in Pure and Appl. Math., 29, M. Dekker, 1977.
- [11] A.C. ZAAANEN, Integration, North Holland Publ. Co., Amsterdam, 1967.
- [12] A. ZYGMUND, Trigonometric series, vol. I, Cambridge University Press, 1968.