MULTIDIMENSIONAL IRREGULAR SAMPLING OF BAND-LIMITED FUNCTIONS IN $L^p$-SPACES

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It is the purpose of this note to present a qualitative approach to irregular variants of the so-called Sampling theorem for band-limited functions on $\mathbb{R}^m$. The basic assertion is the following: Given a compact subset $\Omega \subseteq \mathbb{R}^m$ there is critical sampling rate $\delta_0 = \delta_0(\Omega) > 0$ such that any band-limited $f \in L^p(\mathbb{R}^m)$ with spec $f \subseteq \Omega$ can be completely reconstructed from the sampling values $(f(x_i))_{i \in I}$ at any $\delta_0$-dense discrete family of points $X = (x_i)_{i \in I}$. The reconstruction will be obtained by an iterative procedure yielding a sequence of smooth approximations of $f$, convergent to $f$ in the $L^p$-sense for $1 \leq p < \infty$.

1. Introduction.

One of the most important mathematical results for information theory and digital signal processing is the famous sampling theorem (due to Shannon, Whittacker, Kotel'nikov and others, cf. [1] and the references given there). It states that an important class of smooth functions - the so-called band-limited functions - can be completely recovered from the sampling values on a sufficiently fine regular lattice by means of the cardinal series. On the other hand the Theorem of Plancherel-Polya (cf.[7]) states that the discrete $L^p$-sum of the sampling values $(f(x_i))_{i \in I}$ defines an equivalent norm on the band-limited $L^p$-function under suitable conditions. Thus $f$ is uniquely determined by its sampling values (cf. also [8]); however, no method of reconstruction is provided and this result has no practical consequences.

As a positive answer to this problem, which is also of interest for digital signal processing, we presented in a series of recent papers (cf. [3],[4],[5]) a very general method to reconstruct band-limited functions from their irregular sampling values. The result given so far use spline-type approximation operators, followed by convolutions (filters) with band-limited auxiliary functions. In the present note a new elegant approach is taken, avoiding the use of an auxiliary function. It also allows to use more general filters (including many, which are not band-limited), which yields much better decay properties, hence better localization properties of the iterative procedure, which is of interest if one thinks of parallel processing.
2. Notations

In order to present the basic ideas of our approach as clear as possible (involving some features not present in the more detailed and quantitative presentations obtained earlier) we restrict our discussion to band-limited functions in $L^p(\mathbb{R}^m)$ for $1 \leq p < \infty$. More precisely, we shall consider for a fixed compact set $\Omega \subseteq \mathbb{R}^m$ and $p$ as above the space

$$B^p(\Omega) := \{ f \in L^p(\mathbb{R}^m), \text{ suppf} \subseteq \Omega \},$$

where $\text{suppf}$ has to be understood as the support of the Fourier transform of $f$ in the sense of tempered distributions (at least for $p > 2$). The $p$-norm of $f$ is written as $\|f\|_p := (\int_{\mathbb{R}^m} |f(x)|^p dx)^{1/p}$. We recall that convolution on $\mathcal{K}(\mathbb{R}^m) := \{ f, \text{ complex-valued, continuous, suppf compact} \}$ is given pointwise for $f, g \in \mathcal{K}(\mathbb{R}^m)$ through the formula

$$f \ast g(x) := \int_{\mathbb{R}^m} f(y-x)g(y)dy,$$

and satisfies $\|f \ast g\|_p \leq \|f\|_1 \|g\|_p$ for $p=1$. In particular, $(L^1, \| \cdot \|_1)$ is a Banach algebra with respect to convolution. The translation operators are given by $T_x f(z) := f(z-x)$ for $f \in \mathcal{K}(\mathbb{R}^m)$.

3. Local Properties of Band-limited Functions.

In the course of our discussions the following two auxiliary functions, associated in a canonical way to any locally bounded (hence any continuous) function will be important (we write $B_\delta(x)$ for the ball of radius $\delta$ around $x \in \mathbb{R}^m$, $B(x)$ for $B_1(x)$; $U \subseteq B(0)$ denotes an arbitrary neighborhood of zero).

3.1. Definition. The **local maximal function** associated to $f$ is given by

$$f^\#(x) := \sup_{z \in B(x)} |f(z)|.$$

The **local $U$-oscillation** of $f$ is given by

$$\text{osc}_U f(x) := \sup_{z, y \in x + U} |f(z) - f(y)|.$$

We shall use the symbol $\text{osc}_\delta f$ if $U = B_\delta(0)$. With the help of these notations we may introduce a family of new spaces by

$$C^p(\mathbb{R}^m) := \{ f \text{ continuous, complex-valued on } \mathbb{R}^m, f^\# \in L^p(\mathbb{R}^m) \} \text{ for } 1 \leq p < \infty.$$

It is left to the reader to verify that one has

3.2. Lemma. 1) For $1 \leq p < \infty$ the spaces $C^p(\mathbb{R}^m)$ are Banach spaces with respect to their natural norm $f \mapsto \|f^\#\|_p$;

2) For $1 \leq p \leq q < \infty$ there is a continuous embedding $C^p(\mathbb{R}^m) \hookrightarrow C^q(\mathbb{R}^m)$;

3) The space $\mathcal{K}(\mathbb{R}^m)$ is dense in $C^p(\mathbb{R}^m)$ for $1 \leq p < \infty$. In particular, the
spaces $C^p(\mathbb{R}^m)$ are continuously embedded into $(C^0(\mathbb{R}^m), \| \cdot \|_\infty)$, the space of continuous complex-valued functions vanishing at infinity, with the sup-norm.

iv) $C^1(\mathbb{R}^m) \hookrightarrow L^1 \cap C^0(\mathbb{R}^m) \hookrightarrow L^r(\mathbb{R}^m)$ for any $r \geq 1$.

The following facts will be relevant for our proofs:

3.3. Proposition. 1) The following inequality holds pointwise:

\[(f \ast g)^\# \leq |f| \ast g^\#;\]

ii) Therefore the following convolution relations hold true (together with corresponding norm estimates):

\[(3.4) \quad L^1 \ast C^p \subseteq C^p \quad \text{for } 1 \leq p < \infty.\]

\[(3.5) \quad L^p \ast C^1 \subseteq C^p \]

Proof. The verification of 1) is left to the reader, and (3.4) follows from the continuity of the convolution product resulting from the inclusion $C^p \subseteq C^0$, which gives $L^1 \ast C^p \subseteq C^0$. (3.5) follows in a similar way, using now the fact that $L^p \ast C^1 \subseteq L^p \ast L^p' \subseteq C^0(\mathbb{R}^m)$, where $1/p' + 1/p = 1$.

As a consequence for band-limited functions we obtain the following

3.4. Proposition. 1) For any compact set $\Omega \subseteq \mathbb{R}^m$ one has $B^p(\Omega) \hookrightarrow C^p(\mathbb{R}^m)$, i.e. there exists a constant $C^p_\Omega > 0$ such that

\[(3.6) \quad \|f^\#\|_p \leq C^p_\Omega \|f\|_p \quad \text{for all } f \in B^p(\Omega).\]

Proof. It is sufficient to choose some $h \in C^1$ such that $\hat{h}(t) = 1$ on $\Omega$ (e.g. $h$ may be taken to be the inverse Fourier transform of a convolution product of two characteristic functions of rectangles in $\mathbb{R}^m$, or of some function $g$ in the Schwartz space $\mathcal{S}(\mathbb{R}^m)$ with $g(t) = 1$ on $\Omega$). It follows that $f = h \ast f$ for all $f \in B^p(\Omega)$, and thus by (3.3) $\|f^\#\|_p \leq \|h^\#\|_1 \|f\|_p$, q.e.d.

Similar assertions can be made with respect to the oscillation.

3.5. Proposition. 1) The following pointwise inequality holds true:

\[(3.7) \quad \text{osc}_\Omega(f \ast h) \leq |f| \ast \text{osc}_\Omega h;\]

ii) For any compact set $\Omega \subseteq \mathbb{R}^m$ there exists $C^1_\Omega > 0$ such that

\[(3.8) \quad \|\text{osc}_\delta(f)\|_p \leq \delta \cdot C^1_\Omega \|f\|_p \quad \text{for all } f \in B^p(\Omega).\]

Proof. The verification of 1) is left to the reader. In order to prove ii) it is then sufficient to show for some $h \in C^1(\mathbb{R}^m)$ (as in the proof of Prop.3.4) that one has $\|\text{osc}_\delta(h)\|_1 \leq \delta \cdot C^1_\Omega$. For the one-dimensional case this is veri-
fied by observing that the mean value theorem implies
\[ |h(z) - h(y)| \leq 2\delta |h'(\xi)| \text{ for some } \xi \text{ between } z \text{ and } y, \text{ hence} \]
\[ |\text{osc}_\delta(h)| \leq 2\delta(h')^\# \text{ for } h \in C^1_\delta(\mathbb{R}). \]
In the general m-dimensional setting the same kind of estimate can be obtained by replacing the simple derivative by the absolute value (norm) of the gradient of h (cf. [4] for details). The desired estimate is then obtained for any h satisfying \(|\text{grad}|^\# \in L^1(\mathbb{R}^m)\) (e.g. for h \in \mathcal{F}(\mathbb{R}^m)).

Before we can prove a version of the Plancherel-Polya theorem with these ingredients we have to describe more precisely the discrete sets \(X = \{x_i\}_{i \in I}\) of interest in this context.

3.6. Definition. Given a compact neighborhood \(U\) of zero a family \(X\) is called \(U\)-dense in \(\mathbb{R}^m\) if the family \((x_i + U)_{i \in I}\) covers \(\mathbb{R}^m\) (in the case \(U = B_\delta(x)\) we speak of \(\delta\)-density). It is called relatively separated if there is a uniform bound \(C_d\) for the number of points \(x_i\) in any of the balls \(B(x)\) (independent of \(x\)). It is called well-spread if it is \(\delta\)-dense (for some \(\delta > 0\)) and relatively separated.

We shall only consider neighborhoods \(U \subseteq B(0)\) in the sequel.

3.7. Theorem. i) For any relatively separated family \(X = \{x_i\}_{i \in I}\) in \(\mathbb{R}^m\) there exists a constant \(C = C(C_d)\) such that
\[
(\sum_{i \in I} |f(x_i)|^p)^{1/p} \leq C \|f\|^\#_p \text{ for all } f \in C^p(\mathbb{R}^m).
\]
ii) For any compact set \(\Omega \subseteq \mathbb{R}^m\) there exists a neighborhood \(\delta_\Omega > 0\) such that for any relatively separated and \(\delta_\Omega\)-dense family \(X\) in \(\mathbb{R}^m\) the expression \((\sum_{i \in I} |f(x_i)|^p)^{1/p}\) defines an equivalent norm on \(B^p(\Omega)\).

Proof. i) We start with the general observation that for any \(\delta > 0\) the expression \((\sum_{i \in I} |f(x_i)|^p)^{1/p}\) defines a norm equivalent to \(\|\sum_{i \in I} |f(x_i)| x_{B_\delta(x_i)}\|_p\) in the given situation (where \(x_M\) denotes the indicator-function of the set \(M\)). In fact, if the family \(X\) has the property that \((B(x_i))_{i \in I}\) defines a family of pairwise disjoint sets \((\sum_{i \in I} |f(x_i)|^p)^{1/p}\) can be interpreted, up to some constant, as the \(L^p\)-norm of the function \(f_X := \sum_{i \in I} |f(x_i)| x_{B_\delta(x_i)}\).

For arbitrary relatively separated sets \(X\) the same estimate (up the constant depending only on \(C_d\)) is obtained, splitting \(X\) into a finite union of discrete sets \(X^k, 1 \leq k \leq k_\Omega\) of the above type.
1) Since we have the pointwise inequality $|f_x| \leq C \cdot f^\#$, it follows

$$\|f_x\|_p \leq \|f^\#\|_p.$$  

11) The relevant estimate from below is based on the following inequality:

$$|f(x)| \leq |f(x_1)| + \text{osc}_U f(x_1) \text{ for all } x \in x_1 + U; \text{ upon summation we obtain}$$

$$\sum_{1 \in I} |f(x_1)| x_{1+U} \geq \sum_{1 \in I} (|f| x_{1+U} - \text{osc}_U f(x_1) x_{1+U});$$

Taking $p$-norms on both sides this gives

$$\sum_{1 \in I} \|f(x_1) x_{1+U}\|_p \geq \sum_{1 \in I} \|f\| x_{1+U} - \sum_{1 \in I} \text{osc}_U f(x_1) x_{1+U}\|_p \geq$$

$$\geq \|f\|_p - \sum_{1 \in I} \text{osc}_U f(x_1) x_{1+U}\|_p \quad \text{(because } X \text{ is } \delta\text{-dense}) := *).$$

But

$$\|\sum_{1 \in I} \text{osc}_U f(x_1) x_{1+U}\|_p \leq \|\text{osc}_U (f^\#)\|_p \leq \delta \cdot C^2 \|f\|_p$$

(by a modification of 3.5.). For a suitable choice of $U_0$ we may ensure that for any $U \subseteq U_0$ the above estimate may be continued by $* \geq \|f\|_p - 0.5 \cdot \|f\|_p = 0.5 \cdot \|f\|_p$, as was required.

As an immediate consequence we have the following uniqueness theorem:

3.8. Corollary. In the situation of Theorem 3.7, the complete vanishing of a band-limited function $f$ at all points $(x_1)_{1 \in I}$ of any $U_0$-dense family $X$ implies that $f$ is identically zero.

It is an interesting consequence of the theory of frames (as developed in [2]; cf. also [8] for a treatment on nonharmonic Fourier series) that in the $L^2$-case the norm equivalence allows at least on the analytic level to reconstruct $f$ completely from its sampling values. In fact, denoting for a moment by $g$ the function $f^{-1} \chi_\Omega$ (thus $\hat{g}$ equals the indicator function of $\Omega$), the norm equivalence can be reinterpreted as the fact that the operator $D: f \mapsto \sum_{1 \in I} f(x_1) L_{x_1} g$ is positive and satisfies the following inequality:

$$A \cdot \text{Id} \leq D \leq B \cdot \text{Id} \text{ for suitable values } A, B > 0.$$  

It follows that $\text{Id} - B^{-1} \cdot D \leq B^{-1} (A - B) \cdot \text{Id}$, hence $D$ is invertible (by the usual Neumann series) and $D^{-1} = B^{-1} \sum_{n=1}^\infty (\text{Id} - B^{-1} \cdot D)^n$. Hence

$$f = D^{-1} \circ D(f) = \sum_{1 \in I} f(x_1) D^{-1}(L_{x_1} g),$$

This reconstruction, however, is only of theoretical importance because it has the same drawbacks as the classical sampling cardinal series. The kernel $g$ in the series $\sum_{1 \in I} f(x_1) L_{x_1} g$ has as poor decay properties as the sinc-function. Thus this reconstruction has a rather bad stability behavior and it is useless for numerical analysis.
A Reconstruction Algorithm

The last remark raises the problem of finding efficient reconstruction algorithms. We present such a method in this section, with the following desirable properties: a) good localization; b) good stability with respect to numerical errors; c) great generality, going far beyond the Hilbert space case. Since our space is limited here we present the qualitative theory in the setting of $L^P$-spaces, which does not require too many technical details.

In the first step of the approximation we need the simple fact, that step functions are reasonable good approximations for smooth functions.

**Lemma.** Given a family $X = \{x_j\}_{j \in I}$ in $\mathbb{R}^m$, we denote by $V_X f$ the most natural step function associated to the family $\{f(x_j)\}_{j \in I}$, given by

$$V_X f(x) = \sum_{i \in I} f(x_i) \chi_{V_i},$$

where $V_i$ is the so-called Voronoi region of nearest neighbors,

$$V_i := \{ x \mid |x-x_j| < |x-x_1| \text{ for all } j \neq 1 \}.$$

Then one has for any $U$-dense family $X = \{x_j\}_{j \in I}$:

$$|V_X f - f| \leq \text{osc}_\delta f;$$

and

$$|V_X f| \leq f^\#.$$

In particular $\|V_X f - f\|_p \leq \|\text{osc}_U f\|_p \to 0$ for $U \to \{0\}$, for any $f \in C^P$.

The following theorem is the main result of this paper. It gives (in the proof) an algorithm which allows to reconstruct the band-limited function from its sampling values.

**Theorem.** Let $\Omega$ be a compact subset of $\mathbb{R}^m$ and $h \in C^1(\mathbb{R}^m)$ with $\hat{h}(t) \equiv 1$ on $\Omega$ be given. Then there exists some compact neighborhood $U$ of the origin and $C = C(h, U)$ such that the following is true: Given any family $X = \{x_j\}_{j \in I}$ which is $U$-dense in $\mathbb{R}^m$ there is a bounded linear operator $B$ on $C^P(\mathbb{R}^m)$ with

$$f = B(V_X f \ast h) \text{ for all } f \in B^P(\Omega),$$

and $\|B\| \leq C$ (for all $p$, $1 \leq p < \infty$). In particular, complete reconstruction of $f \in B(\Omega)$ from its sampling values is possible.

**Proof.** Our first observation concerns the fact that we have $f = f \ast h$ for $f \in B^P(\Omega)$, and therefore $V_X f \ast h$ appears as a reasonable and smooth approximation to $f$. In fact, the "remainder" operator $R: f \mapsto (f - V_X f) \ast h$ will be very useful for us. It is bounded on $C^P(\mathbb{R}^m)$, since by (3.3) and (4.3)

$$\|Rf\|^\# = \|(f - V_X f) \ast h\|^\# \leq |f - V_X f| \ast h^\# \leq 2f^\# \ast h^\#.$$
which implies upon taking norms

\[ \| R^2 f \|_{C^p} = \| R^2 f \|_{p} \leq 2 \cdot f^\# \|_{p} \cdot h^\# \|_{1} = (2 \cdot h^\# \|_{1}) \cdot f \|_{C^p} . \]

Another pointwise estimate based on (4.2) is even more important:

\[ |R^2 f| \leq |f - \nabla_X f| \cdot |h| \leq \text{osc}_U (f) \cdot |h| . \]

It allows us to verify that \( R^2 \) is a contraction on \( C^p \) (given sufficient density of the family \( X \) only). Applying (4.6) to \( R^2 f \) and using (3.3) yields

\[ (R^2 f)^\# = (\text{osc}_U (R^2 f) \ast h)^\# \leq \text{osc}_U (R^2 f) \ast h^\# , \]

which gives together with the following estimate (involving (3.7) and (4.3))

\[ \text{osc}_U (R^2 f) \leq |f - \nabla_X f| \ast \text{osc}_U (h) \leq 2 f^\# \ast \text{osc}_U (h) \]

the following combined estimate

\[ (R^2 f)^\# \leq f^\# \ast (2 \cdot h^\# \ast \text{osc}_U (h)) . \]

Taking \( p \)-norms on both sides we obtain the decisive estimate

\[ \| R^2 f \|_{C^p} = \| (R^2 f)^\# \|_{p} \leq \| f \|_{C^p} \cdot 2 \cdot h^\# \ast \text{osc}_U (h) \|_{1} . \]

Since \( \| \text{osc}_U (h) \|_{1} \rightarrow 0 \) for \( U \rightarrow \{0\} \) for any \( h \in C^1 \) it is now clear that \( R^2 \) is a contraction, i.e. \( \| R^2 \| \leq \gamma < 1 \) for \( U \) small enough.

The decisive step is the following identity which holds true for \( n \geq 0 \):

\[ f = R^{n+1} f + (\sum_{k=0}^{n} R^k)(\nabla_X f \ast h) . \]

It is proved by induction. Since it is true for \( n = 0 \) by the definition of \( R \) we assume that it is true for \( 1, \ldots, n-1 \). Using the identity \( f = f \ast h \) we obtain

\[ R^nf = R^n (f \ast h) = R^n (\nabla_X f \ast h + (f - \nabla_X f) \ast h) = R^n (\nabla_X f \ast h) + R^{n+1} f , \]

showing that the inductive step can be verified.

Since we now already that \( \| R^{2n} \| \leq \gamma^n \) and thus by (4.6) \( \| R^{2n+1} \| \leq 2 \cdot h^\# \|_{1} \cdot \gamma^n \) for \( U \) small enough it follows that the series \( B := \sum_{k=0}^{\infty} R^k \) is convergent (on the operator algebra over \( C^p \) for any \( p \geq 1 \)) and

\[ \| B \| \leq (1 - \gamma)^{-1} (1 + 2 \cdot h^\# \|_{1} ) =: C . \]

Finally we obtain (4.4) by taking limits in (4.12):

\[ f = (\sum_{k=0}^{\infty} R^k)(\nabla_X f \ast h) = B (\nabla_X f \ast h) , \quad \text{and our proof is complete.} \]

4.3 Corollary. In the above situation there exists a bounded family \( \{ e_i \}_{i=1}^{\infty} \) in \( C^1 \) such that any \( f \in B^p (\Omega) , 1 \leq p < \infty \), can be written as

\[ f = \sum_{i=1}^{\infty} f(x_i) e_i , \]

with unconditional convergence in \( C^p (\mathbb{R}^m) \), hence locally uniformly and in \( L^p \).
Proof. Recall that \( V_X^f = \sum_{i \in I} f(x_i) \chi_{V_i} \) (with unconditional convergence in \( L^p \), i.e. as limit of partial sums over finite subsets of the index set \( I \)). Using now (3.5) we observe that \( V_X^f \ast h \) is well defined and coincides with the series \( \sum_{i \in I} f(x_i) \chi_{V_i} \ast h \), which converges in \( C^P \) (1 \( \leq p < \infty \)). Applying now the operator \( B \) (which acts boundedly on \( C^P \)) we observe that
\[
B(V_X^f \ast h) = \sum_{i \in I} f(x_i) B(\chi_{V_i} \ast h).
\]
Thus setting \( e_i := B(\chi_{V_i} \ast h) \) we have proved the desired result, since boundedness in \( C \) follows from the estimate
\[
\|e_i\|_{C^1} = \|B(\chi_{V_i} \ast h)\|_{C^1} \leq \|B\| \|\chi_{V_i} \ast h\|_1 \leq C \|\chi_{V_i}\|_1 \|h\|_1 \leq C \|U\| \cdot \|h\|_1
\]
for all \( i \in I \). This completes the proof.

4.4. Remark. The last result is of particular interest in the case of periodic sampling, i.e. if there is a finite set \( F \subseteq \mathbb{R}^m \) and same lattice \( \Lambda \) such that \( X \) is of the form \( X = F + \Lambda \). In that case it is only necessary to calculate explicitly a finite collection of \( e_i \)'s (the other ones being of the form \( T_{\lambda} e_i \) for some \( \lambda \in \Lambda \)).

5. References


More hints on existing literature are given in the above references.