Generalized dual Gabor atoms and best approximations by Gabor family

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Received 13 January 1995; revised 14 December 1995
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Abstract

Let $g$ be a Gabor window of length $N$ and $(a,b)$ be a pair of lattice constants. $a$ and $b$ are considered as time and frequency gaps for a TF-lattice with $N^2/ab$ elements. The corresponding time–frequency shifts of $g$ form a so-called Gabor family. In this paper, we investigate the structural properties of the discrete Gabor transforms. We address the problem of finding the best approximation of a signal $x \in \mathbb{C}^N$ by linear combinations of a Gabor family. We consider critical sampling, oversampling and undersampling, and do not assume that the Gabor family is a frame. For the task we determine the (generalized) dual Gabor atom (GDGA). This amounts to determining the pseudoinverse of the Gabor matrix and can be solved by the conjugate-gradient (CG) algorithm with $O(N)$ complexity for fixed lattice constants $(a,b)$. We provide an easy practical criterion for checking whether a Gabor triple $(g,a,b)$ generates a Gabor frame or not. We propose an efficient algorithm for estimating the Gabor frame bounds and an algorithm for determining tight Gabor atoms.

Zusammenfassung

Es seien $g$ ein Gaborfenster der Länge $N$ und $(a, b)$ ein Paar von Gitterkonstanten; $a$ und $b$ werden als Zeit- und Frequenzlücken für ein Zeit–Frequenz-Gitter mit $N^2/ab$ Elementen betrachtet. Die entsprechenden Zeit–Frequenz-Verschiebungen von $g$ bilden eine sogenannte Gabor-Familie. In diesem Beitrag diskutieren wir die strukturellen Eigenschaften der diskreten Gabortransformationen. Wir sprechen das Problem an, wie die beste Approximation eines Signals $x \in \mathbb{C}^N$ durch lineare Überlagerungen einer Gaborfamilie zu finden ist. Dabei betrachten wir kritische Abtastung wie Unter- und Überlastung und nehmen nicht an, daß die Gabor-Familie einen Rahmen bildet. Zu diesem Zweck bestimmen wir das (verallgemeinerte) duale Gaboratom (GDGA). Das führt dazu, daß die Pseudoinverse der Gabormatrix bestimmt werden muß, was für feste Gitterkonstanten $(a, b)$ mit dem "Conjugate-Gradient" (CG) Algorithmus mit einer Komplexität $O(N)$ zu lösen ist. Wir stellen ein einfaches praktisches Kriterium bereit, mit dem zu testen ist, ob ein Gabortripel $(g, a, b)$ einen Gaborrahmen erzeugt oder nicht. Wir schlagen einen effizienten Algorithmus zur Schätzung der Gaborrahmen-Grenzen vor sowie ein Verfahren zur Bestimmung dichter Gaboratome.

Résumé

Soit $g$ une fenêtre de Gabor de longueur $N$ et $(a, b)$ une paire de constantes de treillis. $a$ et $b$ sont considérés comme étant les intervalles de temps et de fréquence pour un treillis TF de $N^2/ab$ éléments. Les déplacements temps–fréquence correspondants de $g$ forment ce qu’on appelle une famille de Gabor. Dans cet article, nous discutons des propriétés structurelles des transformées de Gabor discrètes. Nous considérons le problème de trouver la meilleure approximation d’un signal $x \in \mathbb{C}^N$ par

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PII S0165-1684(96)00015-1
des combinaisons linéaires d'une famille de Gabor. Nous étudions l'échantillonnage critique, le sur- et le sous-échantillonnage, et ne supposons pas que la famille de Gabor est une trame. Pour cette tâche, nous déterminons l'atome de Gabor dual (généralisé) (GDGA). Ceci revient à déterminer le pseudo-inverse de la matrice de Gabor et peut être résolu en utilisant l'algorithme du gradient conjugué (GC) de complexité $O(N)$ pour des constantes de treillis fixes $(a, b)$. Nous donnons un critère pratique simple pour vérifier si un triplet de Gabor $(g, a, b)$ génère une trame de Gabor ou pas. Nous proposons un algorithme efficace pour estimer les limites de la trame de Gabor et un algorithme pour déterminer les atomes de Gabor.

**Keywords:** Gabor expansion; (Generalized) dual Gabor atom (GDGA); $\mathcal{B}$-symmetry

1. Introduction

The theory of Gabor expansions has been thoroughly studied by many scientists, mostly via the Zak transform, see e.g. [3] and the references cited there. Since Gabor elementary functions are nonorthogonal, the main difficulty of doing Gabor analysis is how to obtain suitable Gabor coefficients efficiently. As we know, these coefficients can be obtained by short-time Fourier transforms (STFT) with a so-called biorthogonal Gabor analysis window. It is the desire of finding efficient determinations of this dual Gabor window (atom) that has attracted considerable attention from both mathematicians and engineers. Actually, the oversampled and the critically sampled Gabor families which are frames have been studied essentially by the following two methods:

- Via the Zak transform and the frame method: We refer to [1-3, 7, 14, 18, 27, 30].
- Via matrix decompositions: We refer to [9, 17, 19-25, 28, 29].

The main purpose of this paper is to consider discrete Gabor transforms generally. Specifically for an arbitrary Gabor triple $(g, a, b)$, we present a general method to obtain the associated (generalized) dual Gabor atom (GDGA) $\tilde{g}$ based on the CG (conjugate-gradient)-method, which can be implemented with $O(N)$ complexity for fixed lattice constants $(a, b)$. For a signal $x$, using the calculated $\tilde{g}$, we are able to determine the best approximation of $x$ by linear combinations of the Gabor family. We propose an algorithm to determine the tight Gabor atom $g_t$ based on the binomial series.

In Section 2, we list some notation which have been widely used in our previous papers [21, 23] and some preliminary results about discrete Gabor expansions. Section 3, the main part of this paper, will discuss the Rotated($\mathcal{B}$)-symmetric structure of the Gabor matrix in detail and will give an easy way of computing the Gabor frame bounds. Section 3 also contains the corresponding numerical algorithms. In Section 4 we provide some numerical results. We draw our conclusions in Section 5.

2. Notation and preliminaries

We use essentially the notation introduced in [21]. For the paper to be self-contained, we recall the main definitions. We use $i = \sqrt{-1}$ to denote the imaginary unit and $j, k, l, a, b$ are always nonnegative integers.

(i) A signal $x = \{x(k)\}_{k=0}^{N-1}$ is considered as an $N$-periodic row vector in the complex space $\mathbb{C}^N$. We use either $x_k$ or $x(k)$ to denote the $(k + 1)$th coordinate of a signal $x$ $(k = 0, 1, \ldots, N-1)$. The inner product of two signals $x, y \in \mathbb{C}^N$ is $\langle x, y \rangle = \sum_{j=0}^{N-1} x(j) \overline{y(j)}$. Signals are treated as row vectors. For a matrix $A \in \mathbb{C}^{N \times N}$, it induces a linear mapping (operator) $A : x \mapsto x * A$. The adjoint operator corresponds to $x \mapsto x * A'$, where $A'$ is the conjugate transpose (or Hermitian transpose) of $A$. We use $A^T$ to denote the transpose of $A$.

(ii) The time translation operator $T_k$ corresponds to the cyclic rotation:

$$T_k(x_0, x_1, \ldots, x_{N-1}) := (x_{-k}, x_{1-k}, \ldots, x_{N-1-k}, x_0, \ldots, x_{N-k-1}).$$

The modulation operator (or the frequency translation operator) is given by

$$M_l(x_0, x_1, \ldots, x_{N-1}) := (x_0, x_1w, x_2w^2, \ldots, x_{N-1}w^{N-1}),$$
where
\[ w = e^{-\frac{2\pi i l}{N}} \text{ for } l \in \mathbb{Z}. \]

(iii) The following rotation operator (introduced in [21, 23]) with the rotation number \( a \) acting on vectors is also used for the cyclic rotation:

\[ \text{rot}(x, a) := (x_{N-a}, x_{N-a+1}, \ldots, x_{N-1}, x_0, \ldots, x_{N-a-1}). \]

If \( y \) is a column vector, we define \( \text{rot}(y, a) := [\text{rot}(y'), a]'. \)

The matrix rotation on matrix \( A \) with a rotation number \( a \) is obtained by rotating all the column vectors of \( A \). Precisely if \( A \in \mathbb{C}^{p \times q} \) and \( A = (A_1, A_2, \ldots, A_q) \), then

\[ \text{rotm}(A, a) := (\text{rot}(A_1, a), \text{rot}(A_2, a), \ldots, \text{rot}(A_q, a)). \]

For any number \( k = 1, 2, \ldots, \) we define inductively \( \text{rotm}^k(A, a) = \text{rotm}^k(\text{rotm}^{k-1}(A, a), a). \)

It is easy to check that \( \text{rotm}^k(A, a) = \text{rotm}(A, ka). \)

(iv) For a signal \( g \in \mathbb{C}^N \), we call \((g, a, b)\) a Gabor triple if both \( a \) and \( b \) are positive integers and divisors of \( N \). We call \( g \) a Gabor atom. The constants \( a \) and \( b \) are called time and frequency gaps. They form a pair of lattice constants. We call \( a = N/a \) and \( b = N/b \) the dual lattice constants.

(v) A Gabor expansion of a signal \( x \) with respect to a Gabor triple \((g, a, b)\) is a series expansion of the form

\[ x = \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} c_{nm} g_{nm}, \]

where the Gabor elementary functions \( g_{nm} \) are given as \( g_{nm} = M_{nb} T_{na} g \) for \( n = 0, 1, \ldots, \tilde{a} - 1 \) and \( m = 0, 1, \ldots, \tilde{b} - 1 \). The collection \( \{g_{nm}\}_{nm} \) is called a Gabor family. \( \{c_{nm}\}_{nm} \) are usually referred to as the Gabor coefficients. The Gabor operator \( S \) associated to \((g, a, b)\) is defined as

\[ Sx = \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} \langle x, g_{nm} \rangle g_{nm}, \quad \text{for } x \in \mathbb{C}^N. \]

Unless the contrary is explicitly stated, we assume from now on that lattice constants \((a, b)\)

satisfy that \( ab \leq N \), which are equivalent to critical sampling and oversampling.

(vi) For a Gabor triple \((g, a, b)\), we define the associated Gabor basic matrix \( GAB(g, a, b) \) as the following \( \tilde{N} \times \tilde{N} \) matrix \( (\tilde{N} = \tilde{a} \tilde{b} = N^2/ab) \):

\[
GAB(g, a, b) = \begin{bmatrix}
g & T_{ag} & \cdots \\
T_{(a-1)a}g & M_hg & M_bT_{ag} \\
& \ldots & \ldots \\
M_{(\tilde{a}-1)a}g & & \\
M_{\tilde{a}a}T_{ag} & & \\
& \ldots & \ldots \\
M_{\tilde{a}\tilde{a}} & & \\
\end{bmatrix}
\]  

Then it is easy to see the following matrix representation for the Gabor operator:

\[ Sx = x \ast S \quad \text{for } x \in \mathbb{C}^N, \]

where \( S = [GAB(g, a, b)]' \ast [GAB(g, a, b)]' \) is called the Gabor matrix. In general, for any two Gabor triples \((g_1, a, b)\) and \((g_2, a, b)\), we call the matrix product \( [GAB(g_1, a, b)]' \ast [GAB(g_2, a, b)]' \) the Gabor matrix associated to \((g_1, a, b)\) and \((g_2, a, b)\).

For a Gabor triple \((g, a, b)\), the associated Gabor operator \( S \) and the Gabor matrix \( S \) may not be invertible. However, \( \text{pinv}(S) \) is uniquely defined, where \( \text{pinv}(S) \) denotes the pseudoinverse [26] (or the Moore–Penrose inverse [16, pp. 432–440]) matrix of \( S \). Then we define the associated generalized dual Gabor atom (GDGA) as

\[ \tilde{g} := g \ast \text{pinv}(S). \]

Since the Gabor matrix \( S \) is positive semidefinite [11, 16, 26], the square root \( S^{1/2} \) is well defined [16, pp. 180–184]. We define the generalized tight Gabor atom (GTGA) as \( g_1 = g \ast \text{pinv}(S^{1/2}) \).

If a Gabor triple \((g, a, b)\) generates a Gabor frame, the associated Gabor operator \( S \) is the usual Gabor frame operator (cf. [12, pp. 635–636] or [6, pp. 58–59]). In this case, \( \text{pinv}(S) = \text{inv}(S) \) is the usual inverse of \( S \). The associated GDGA \( \tilde{g} \) and GTGA \( g_1 \) are the dual and tight Gabor atoms in the usual sense (cf.}
[27], [6] or [12]). In this case, we call $\tilde{g}$ and $g$, the usual dual and tight Gabor atoms.

The following proposition is used to obtain the best approximation of a signal $x \in \mathbb{C}^N$ by linear combinations of a Gabor family $\{g_{nm}\}_{n,m}$. This is equivalent to determine the orthogonal projection of $x$ onto the Gabor space: $\text{Span}\{g_{nm}\}$, where $\text{Span}\{g_{nm}\}$ is the linear space spanned by the Gabor family $\{g_{nm}\}_{n,m}$ [16, pp. 83-87].

**Proposition 1.** Let $\tilde{g}$ be the GDGA associated to $(g,a,b)$. For a signal $x \in \mathbb{C}^N$, the orthogonal projection of $x$ onto $\text{Span}\{g_{nm}\}$ can be represented as

$$P(x) = \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} \langle x, g_{nm} \rangle \tilde{g}_{nm} = \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} \langle x, \tilde{g}_{nm} \rangle g_{nm}. \tag{3}$$

In particular, if $x \in \text{Span}\{g_{nm}\}$, then

$$x = \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} \langle x, g_{nm} \rangle \tilde{g}_{nm} = \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} \langle x, \tilde{g}_{nm} \rangle g_{nm}. \tag{4}$$

Let us sketch the proof of Proposition 1. For further details, we refer to [24]. Let $G = [GAB(g,a,b)]^* [GAB(g,a,b)]$ be the Gabor matrix associated to $(g,a,b)$. It is clear that $G$ and thus $\text{pinv}(G)$ are Hermitian symmetric. For a signal $x \in \text{Span}\{g_{nm}\}$, we see that $x$ is in the row space of the Gabor matrix $G$ (cf. [16, pp. 78-80], [26, pp. 137-140]). Hence, we have (see [16, pp. 428-438], [26, pp. 140-141])

$$\{x * G\} * \text{pinv}(G) = \{x * \text{pinv}(G)\} * G \quad \text{for } x \in \text{Span}\{g_{nm}\}. \tag{5}$$

In order to prove Eqs. (3) and (4), we need to show first that

$$M_{mb} T_{na} (g * \text{pinv}(G)) = (M_{mb} T_{na} g) * \text{pinv}(G),$$

i.e. $\tilde{g}_{nm} = \hat{g}_{nm}$. \tag{6}

In fact, by the commutative property of the Gabor operator with both $T_{a}$ and $M_{b}$, it is easy to see that

$$\tilde{g}_{nm} = M_{mb} T_{na} (g * \text{pinv}(G))$$

$$= M_{mb} T_{na} (g * \text{pinv}(G^2) * G)$$

$$= (M_{mb} T_{na} g) * \text{pinv}(G).$$

This yields that $\tilde{g}_{nm} \in \text{Span}\{g_{nm}\}$. Then using (5), we derive that

$$\tilde{g}_{nm} = M_{mb} T_{na} (g * \text{pinv}(G)) * \{G * \text{pinv}(G)\}$$

$$= M_{mb} T_{na} [g * \text{pinv}(G) * G] * \text{pinv}(G)$$

$$= (M_{mb} T_{na} g) * \text{pinv}(G) = g_{nm} * \text{pinv}(G).$$

This proves Eq. (6). Hence,

$$\sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} \langle x, \tilde{g}_{nm} \rangle g_{nm} = \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} \langle x, g_{nm} * \text{pinv}(G) \rangle g_{nm}$$

$$= \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} \langle x * \text{pinv}(G), g_{nm} \rangle g_{nm}$$

$$= \{x * \text{pinv}(G)\} * G.$$ We conclude from Eqs. (5) and (6) that if $x \in \text{Span}\{g_{nm}\}$

$$x = \{x * G\} * \text{pinv}(G) = \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} \langle x, \tilde{g}_{nm} \rangle g_{nm},$$

and

$$x = \{x * G\} * \text{pinv}(G)$$

$$= \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} \langle x, g_{nm} \rangle (g_{nm} * \text{pinv}(G))$$

$$= \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} \langle x, g_{nm} \rangle \tilde{g}_{nm}.$$ Eq. (4) is proved.

For the general case (3), since $P(x) \in \text{Span}\{g_{nm}\}$, using (4), we see that

$$P(x) = \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} \langle P(x), g_{nm} \rangle \tilde{g}_{nm} = \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} \langle x, g_{nm} \rangle \hat{g}_{nm}$$

$$= \{x * G\} * \text{pinv}(G) = \{x * \text{pinv}(G)\} * G$$

$$= \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} \langle x, \tilde{g}_{nm} \rangle g_{nm}.$$ Similarly,

$$P(x) = \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} \langle x, g_{nm} \rangle \hat{g}_{nm}.$$
Eq. (3) is obtained. The sketch of the proof of Proposition 1 is complete.

Remark. Since Eq. (6) holds without the assumption that \( \{g_{nm}\}_{n,m} \) is a frame, Eq. (6) is an extended version of the result presented in [4, Theorem 3.11], [6, pp. 83], [7, pp. 971].

3. Mathematical setup

As we have seen in [21, 23], the Gabor matrix \( S \) associated to Gabor triples \( (g_1, a, b) \) and \( (g_2, a, b) \) is completely determined by its first \( N \times a \) block matrix \( B \). All the other \( \tilde{a} - 1 \) block matrices are obtained from \( B \) by proper rotations. We state these descriptions in the following theorem.

Theorem A (Qiu and Feichtinger [21, 23]). Let \( (g_1, a, b) \) and \( (g_2, a, b) \) be two Gabor triples, \( S = [GAB(g_1, a, b)]' * [GAB(g_2, a, b)] \) be the associated Gabor matrix. Assume that \( B \) is the first \( N \times a \) block matrix of \( S \). Then \( S \) is of the following block matrix form:

\[
S = [B, \text{rotm}(B, a), ..., \text{rotm}^{\tilde{a} - 1}(B, a)]
\]

for \( r = \tilde{a} - 1 \). (7)

For \( k = 1, 2, ..., \tilde{a} - 1 \), \( \text{rotm}^{\tilde{k}}(B, a) \) is exactly the \( (k + 1) \)-th block matrix of size \( N \times a \). Moreover, the possibly nonzero entries of \( B \) are distributed only in the \( l \)-th diagonals for \( l = 0, \pm \tilde{b}, \pm 2\tilde{b}, ..., \pm (b - 1)\tilde{b} \).

We call \( B \) the first block corresponding to Gabor triples \( (g_1, a, b) \) and \( (g_2, a, b) \). If \( g_1 = g_2 = g \), we call \( B \) the first block matrix associated to \( (g, a, b) \).

Extracting all the zero-entries of \( B \), we obtain a compressed "nonzero"-block matrix \( \tilde{B} \) of size \( b \times a \). The \( l \)-th row-vector is exactly the \( l \)-th diagonal of \( B \) for \( l = 0, \tilde{b}, ..., (b - 1)\tilde{b} \). We call \( \tilde{B} \) the (first) "nonzero"-block matrix corresponding to Gabor triples \( (g_1, a, b) \) and \( (g_2, a, b) \). When \( g_1 = g_2 = g \), we call \( \tilde{B} \) the "nonzero"-block matrix associated to \( (g, a, b) \). By Theorem A and the definition of \( \text{rotm} \), every (sub-)diagonal is \( a \)-periodic. We consider \( \tilde{B} \) as \( a \)-periodic in column-subscripts and \( b \)-periodic in row-subscripts.

More precisely if \( B \) is given (by Theorem A) as

\[
B = \\
\begin{bmatrix}
\begin{array}{cccc}
  c_{1,1} & 0 & \cdots & 0 \\
  0 & c_{2,2} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & c_{a,a} \\
  c_{\tilde{b}+1,1} & 0 & \cdots & 0 \\
  0 & c_{\tilde{b}+2,2} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & c_{b,0,a} \\
\end{array}
\end{bmatrix}
\]

then the \( b \times a \) "nonzero"-block matrix \( \tilde{B} \) is

\[
\tilde{B} = \\
\begin{bmatrix}
\begin{array}{cccc}
  c_{1,1} & c_{2,2} & \cdots & c_{a,a} \\
  c_{\tilde{b}+1,1} & c_{\tilde{b}+2,2} & \cdots & c_{b,0,a} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{(b-1)\tilde{b}+1,1} & c_{(b-1)\tilde{b}+2,2} & \cdots & c_{(b-1)\tilde{b}+a,a} \\
\end{array}
\end{bmatrix}
\]

Since \( \tilde{B} \) is of small size \( b \times a \), \( \tilde{B} \) can be easily built up. We give a sufficient condition with \( \tilde{B} \) for a Gabor triple \( (g, a, b) \) to generate a Gabor frame.

Corollary 1 (Sufficient condition). Let \( \tilde{B} = (u_{i,j})_{b \times a} \) be the "nonzero"-block matrix associated to a Gabor triple \( (g, a, b) \), and assume that for any \( 1 \leq j \leq a \),

\[
u_{1,j} > \sum_{i=2}^{b} |u_{i,j}|,
\]

then \( (g, a, b) \) generates a Gabor frame.

Proof. Condition (8) implies that the Gabor matrix \( S \) associated to \( (g, a, b) \) is strictly diagonal dominant. Therefore, \( S \) is nonsingular and \( (g, a, b) \) does generate a frame. \( \square \)

In the following, we show that if \( g_1 = g_2 = g \) then the Gabor matrix associated to \( (g, a, b) \) has the rotated-symmetry (\( \sigma \)-symmetry) structure.
Theorem 1 (\(\mathcal{R}\)-symmetry structure). Assume that 
\(g_1 = g_2 = g\), and let 
\[ S = [B, \text{rotm}(\tilde{B}), \ldots, \text{rotm}^r(\tilde{B})], \]
where \(r = \tilde{a} - 1\),
be the Gabor matrix associated to \((g, a, b)\). Writing
\[ b_0 = \begin{cases} \frac{1}{2}(b + 1) & \text{if } b \text{ is odd}, \\ \frac{1}{2}(b + 2) & \text{if } b \text{ is even}, \end{cases} \]
then the Gabor matrix \(S\) has at most \(ab_0\) independent nonzero elements. They are from the upper part of the \(N \times a\) block matrix \(B\).

Proof. Using Theorem A, we show first that for \(r = q\tilde{b} + 1, s = (b - q)\tilde{b} + 1\) with \(q = 1, 2, \ldots, b_0 - 1\), the \(s\)th subdiagonal elements of \(B\) are completely determined by the \(r\)th subdiagonal elements of \(B\) via the rotation and the conjugation. Conversely, we can determine the \(r\)th subdiagonal from the \(s\)th subdiagonal. Without loss of generality, we take \(r = \tilde{b} + 1\) and \(s = (b - 1)\tilde{b} + 1\).

Noticing that the diagonal elements of the block matrix are of periodicity \(N = b\tilde{b}\), for \(k = 1, 2, \ldots, a\), the general elements of the \(s\)th subdiagonal can be written as
\[ d_k^s = S_{\{(b - 1)\tilde{b} + k\}, k} \]
\[ = \tilde{b} \sum_{n=0}^{\tilde{a} - 1} T_{na} g((b - 1)\tilde{b} + k - 1) T_{na} g(k - 1) \]
\[ = \tilde{b} \sum_{n=0}^{\tilde{a} - 1} T_{na} g(k - \tilde{b} - 1) T_{na} g(k - 1) \]
\[ = \tilde{b} \sum_{n=0}^{\tilde{a} - 1} T_{na} g(k - \tilde{b} - 1) T_{na} g(k - 1) \]
\[ = \tilde{d}_{k - \tilde{b}}^r. \]

Therefore,
\[ d_k^s = d_{k - \tilde{b}}^r \quad \text{for } k = 1, 2, \ldots, a. \]

This yields that the \((s = (b - 1)\tilde{b} + 1)\)th subdiagonal elements of \(B\) can be obtained via a suitable rotation and the conjugation from the \((r = \tilde{b} + 1)\)th subdiagonal elements of \(B\).

In general, for \(r = q\tilde{b} + 1, s = (b - q)\tilde{b} + 1\) with \(q = 1, 2, \ldots, b_0 - 1\), we can show similarly that
\[ d_k^s = d_{k - q\tilde{b}}^r \quad \text{for } k = 1, 2, \ldots, a. \]

This means that the \(s\)th subdiagonal of \(B\) is completely obtained via a rotation with the rotation number \(q\tilde{b}\) (or the remainder of \(q\tilde{b}\) divided by \(a\)) and the conjugation of the \(r\)th subdiagonal of \(B\).

Hence, all the independent entries of \(S\) are located in the \(r\)th subdiagonal of \(B\), for \(r = q\tilde{b} + 1\) with \(q = 0, 1, \ldots, b_0 - 1\). The total number of independent entries are at most \(ab_0\).

The proof is complete. □

Remark. Since \(ab_0 \approx \frac{1}{2}ab\), the \(\mathcal{R}\)-symmetry of the Gabor matrix has reduced the computational cost by half.

In general, if \(g_1 \neq g_2\), the Gabor matrix corresponding to \((g_k, a, b)\) for \(k = 1, 2\) does not have the \(\mathcal{R}\)-symmetry.

Using the introduced "nonzero"-block matrix, we rewrite Theorem 1 as the following equivalent form.

Theorem 2. Let \(\tilde{B}\) be the "nonzero"-block matrix associated to \((g, a, b)\). Assume that \(\tilde{B}_j\) is the \(j\)th row vector of \(\tilde{B}\) for \(j = 2, \ldots, b_0\). Then the \((b - j + 2)\)th row vector \(\tilde{B}_{b - j + 2}\) of \(\tilde{B}\) can be written as
\[ \tilde{B}_{b - j + 2} = \text{rot}(\tilde{B}_j, \text{mod}(j - 1)\tilde{b}, a)), \]
where \(\text{mod}(j - 1)\tilde{b}, a)\) is the remainder of \((j - 1)\tilde{b}\) divided by \(a\).

In particular if the signal length \(N\) is divisible by \(ab\) (especially for the critical cases), then
\[ \tilde{B}_{b - j + 2} = \tilde{B}_j \quad \text{for } j = 2, \ldots, b_0. \]

If the Gabor atom is a real-valued signal, then
\[ \tilde{B}_{b - j + 2} = \text{rot}(\tilde{B}_j, \text{mod}(j - 1)\tilde{b}, a)) \]
for \(j = 2, \ldots, b_0\).

Theorem 3. Let \(S\) be the Gabor matrix corresponding to a Gabor triple \((g, a, b)\). Then for any \(k = 0, 1, \ldots, S^k\) is \(\mathcal{R}\)-symmetric. More generally, if \(p(z)\) is any real polynomial, then \(p(S)\) also has the \(\mathcal{R}\)-symmetry.
For the proof of Theorem 3, we need the following interesting proposition.

**Proposition 2.** Let $S^{k/2}$ be the linear operator realized by the matrix $S^{k/2}$ for $k = 0, 1, 2, \ldots$. Then $S^{k/2}$ for $k = 0, 1, 2, \ldots$ commute with both the time translation operator $T_a$ and the frequency modulation operator $M_b$.

**Proof.** If $k = 0$, the fact is trivial. We only need to consider the cases for $k \geq 1$.

For $k = 1$, $S$ is the associated Gabor operator. It is known that $S$ commutes with both $T_a$ and $M_b$.

Since $S$ is positive semidefinite, $S^{k/2}$ and $S^{k}$ are well defined. Assume that $U$ is the unitary matrix diagonalizing $S$, then there is a diagonal matrix $D = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_N)$ such that

$$S = U' \ast D \ast U,$$

where $\alpha_1 \geq \alpha_2 \geq \cdots \alpha_N \geq 0$, $\text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_N)$ denotes the diagonal matrix with $\alpha_j$ ($j = 1, 2, \ldots, N$) being its diagonal entries. From Eq. (10), it is easy to see that

$$S^{k/2} = U' \ast D^{k/2} \ast U$$

$$= U' \ast \text{diag}(\alpha_1^{k/2}, \alpha_2^{k/2}, \ldots, \alpha_N^{k/2}) \ast U. \quad (11)$$

By the polynomial interpolation theory [13], there is an interpolation polynomial $L(x)$ such that

$L(\alpha_j) = \alpha_j^{k/2}$, for $j = 1, 2, \ldots, N$.

From Eq. (11), we can easily check that

$$S^{k/2} = U' \ast \text{diag}(L(\alpha_1), L(\alpha_2), \ldots, L(\alpha_N)) \ast U$$

$$= U' \ast L(D) \ast U = L(S). \quad \Box$$

Then by the fact that $S$ commutes with both $T_a$ and $M_b$, we deduce that $S^{k/2}$ commutes with both the time-translation operator $T_a$ and the modulation operator $M_b$. \quad \Box

With the same argument given above, we obtain the following corollary.

**Corollary 2.** If we define $S^{-k/2} = (\text{pinv}(S))^{k/2}$ for $k = 1, 2, \ldots$, then $S^{k/2}$ for $k = 0, \pm 1, \pm 2, \ldots$ commute with both $T_a$ and $M_b$.

**Proof of Theorem 3.** Set $h = S^{(k-1)/2}g$, then the Gabor operator $S_h$ associated to $(h, a, b)$ is defined as

$$S_h x = \sum_{m=0}^{\tilde{b}-1} \sum_{n=0}^{\tilde{a}-1} \langle x, h_{nm} \rangle h_{nm} \quad \text{for } x \in \mathbb{C}^N.$$

On the other hand, by Proposition 2,

$$S^k x = S^{(k-1)/2} (S (S^{(k-1)/2} x))$$

$$= S^{(k-1)/2} \left\{ \sum_{m=0}^{\tilde{b}-1} \sum_{n=0}^{\tilde{a}-1} \langle S^{(k-1)/2} x, g_{nm} \rangle g_{nm} \right\}$$

$$= \sum_{m=0}^{\tilde{b}-1} \sum_{n=0}^{\tilde{a}-1} \langle x, S^{(k-1)/2} g_{nm} \rangle S^{(k-1)/2} g_{nm}$$

$$= \sum_{m=0}^{\tilde{b}-1} \sum_{n=0}^{\tilde{a}-1} \langle x, h_{nm} \rangle h_{nm}.$$

Thus,

$$S_h x = x \ast S^k \quad \text{for } x \in \mathbb{C}^N.$$

This implies that

$$S^k = [\text{GAB}(h, a, b)]' \ast [\text{GAB}(h, a, b)].$$

Therefore, $S^k$ is $R$-symmetric by Theorem 1.

For any real polynomial $p(z)$, $p(S)$ also has the $R$-symmetry. \quad $\Box$

The 1-norm [11, 13, 16] of a matrix $A = (a_{i,j})_{n \times m}$ is defined as

$$\|A\|_1 = \max_{1 \leq j \leq m} \sum_{i=1}^{n} |a_{i,j}|.$$

It is easy to compute, especially for a small-size matrix. We show that the 1-norm of the small "nonzero"-block matrix can be used to estimate the Gabor frame upper bounds.

**Theorem 4.** (Gabor frame bounds). Suppose $(g, a, b)$ generates a Gabor frame, $S$ is the associated Gabor frame operator. Let $\tilde{g}$ be the dual Gabor atom, $\tilde{B}$ and $B_d$ be the $b \times a$ "nonzero"-block matrices associated to $(g, a, b)$ and $(\tilde{g}, a, b)$, respectively. Then

$$\frac{1}{||\tilde{B}||_1} \leq ||S|| \leq ||\tilde{B}||_1.$$  

(12)
Proof. Assume that $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are the Gabor matrices associated to $(g, a, b)$ and $(\tilde{g}, a, b)$, respectively, then
\[||\mathcal{S}|| = ||\mathcal{S}|| \quad \text{and} \quad ||\mathcal{S}^{-1}|| = ||\mathcal{S}^{-1}|| = ||\tilde{\mathcal{S}}||.\]
Since
\[\mathcal{S} = [B, \text{rotm}(B, a), \ldots, \text{rotm}^r(B, a)]\]
and $B$ is obtained from $\tilde{B}$ by filling in some proper zero-entries, it is clear that
\[||\text{rotm}^k(B, a)||_1 = ||B||_1 = ||\tilde{B}||_1.\]
Hence,
\[||\mathcal{S}||_1 = ||B||_1 = ||\tilde{B}||_1.\]
Similarly,
\[||\tilde{\mathcal{S}}||_1 = ||\tilde{B}||_1.\]
Because the norm (spectral norm) of $\mathcal{S}$ is majorized (see [13; 16, pp. 358–362]) by the 1-norm of $\mathcal{S}$, we see that
\[\frac{1}{||\tilde{B}||_1} \leq ||S||_1 \leq ||\tilde{B}||_1.\]
Therefore,
\[\frac{1}{||\tilde{B}||_1} \leq ||S||_1 \leq ||\tilde{B}||_1.\]
\[\square\]
Remark. (1) Let $\lambda_{\max}$ and $\lambda_{\min}$ be the maximal and minimal eigenvalues of the Gabor matrix $\mathcal{S}$, then $\lambda_{\max}$ and $\lambda_{\min}$ are the best upper and lower Gabor frame bounds. Tables 1 and 2 show that the approximate Gabor frame bounds $U_B := ||\tilde{B}||_1$ and $L_B := 1/||\tilde{B}||_1$ calculated from Theorem 4 approximate well to $\lambda_{\max}$ and $\lambda_{\min}$.

(2) With the simple and efficient formula presented for the Gabor frame bounds, the condition number of the Gabor matrix can be estimated as
\[\text{cond}(\mathcal{S}) \leq ||\tilde{B}||_1||\tilde{B}||_1,\]
which can be used to check the quality (that is, the stability of the associated Gabor expansion and synthesis) of a Gabor signal.

Corollary 3. (1) If $a = 1$, then the "nonzero"-block corresponding to $(g, a, b)$ is a vector $\tilde{B} = (t_j)_{j=1}^{d} \in \mathbb{C}^d$.
\[||\tilde{B}||_{\infty} = \max_{j=1}^{d} t_j \quad \text{is the best Gabor upper bound.}\]

(2) If $a = 1$, let $\tilde{B}_{\mathcal{F}}$ be the "nonzero"-block corresponding to $(\mathcal{F}(g), b, 1)$, then the best Gabor upper bound is obtained as $||\tilde{B}_{\mathcal{F}}||_{\infty}/N$.

Proof. Since the Gabor matrix associated to $(g, a, 1)$ is a nonnegative diagonal matrix [23, Corollary 8], the first result is obvious, while the second result is deduced from [23, Theorem 3]. \[\square\]

Algorithm 1 (Nonzero-block and Gabor matrix). Let $(g, a, b)$ be a Gabor triple and $\mathcal{S}$ be the associated Gabor matrix. Then we can construct $\mathcal{S}$ with only $\overline{a}(ab_0) \approx \frac{1}{\sqrt{2}}Nb$ multiplications and via proper rotations and the conjugation by the following steps:

Step 1. We calculate the first $b_0$ row-vectors of the $b \times a$ "nonzero"-block matrix $\tilde{B}$. The total computation needs only $(ab_0)\overline{a} = Nb_0 \approx \frac{1}{\sqrt{2}}Nb$ multiplications. The $(k, j)$-th general entry of the "nonzero"-block $\tilde{B}$ can be computed as
\[\tilde{B}_{k,j} = \tilde{B} \sum_{n=0}^{a-1} T_{n, \text{ag}}((k-1)\tilde{b} + j - 1)T_{n, \text{ag}}(j - 1)\]
for $j = 1, 2, \ldots, a$ and $k = 1, 2, \ldots, b_0$.

Step 2. We obtain the remaining $b - b_0$ row-vectors in the lower part of $\tilde{B}$. If $\tilde{B}_j$ denotes the $j$th $a$-vectors row vector of $\tilde{B}$ for $j = 1, 2, \ldots, b$, then for $j = 2, \ldots, b_0$,
\[\tilde{B}_{b - j + 2} = \text{rotm}(\tilde{B}_j, \text{mod}(j - 1)\tilde{b}, a)).\]

Step 3. Filling in some extra zero-entries into $\tilde{B}$, we get the first $N \times a$ block matrix $B$ of the Gabor matrix.

Step 4. By the rotation operator $\text{rotm}$ with the rotation number $a$, we can successively obtain the other $a - 1$ block matrices of the same size $N \times a$. The Gabor matrix $\mathcal{S}$ is built up.

Algorithm 2 (Gabor matrix–vector multiplication). Let $(g, a, b)$ be a Gabor triple, $\mathcal{S}$ and $\tilde{B}$ be the associated Gabor operator and Gabor matrix, $\tilde{B}$ be the associated "nonzero"-block matrix of size $b \times a$ and $x$ be an arbitrary signal. Then the general elements of $y = SX = x * S := (y_0, y_1, \ldots, y_{N-1})$ can be computed from $\tilde{B} = (c_k)_b \times a$ by
\[y_j = \sum_{p=1}^{b} x_{j+(p-1)b} c_{p,j}.\]
The derivation of this algorithm is similar to the proof of Algorithm 2 in [23]. With the precalculated "nonzero"-block \( \bar{B} \), we only need \( Nb \) multiplications to perform Algorithm 2. We should also notice that if \( S \) is the Gabor matrix corresponding to \( (g_k, a, b) \) for \( k = 1, 2, \ldots, \tilde{a} \), i.e., \( S = [\text{GAB}(g_1, a, b)]^* \cdot [\text{GAB}(g_2, a, b)] \), then Algorithm 2 does also hold. In practical applications, we do not need to build the \( N \times N \) Gabor matrix \( S \). We only need to work with the small \( b \times a \) "nonzero"-block matrix \( \bar{B} \) by Algorithm 2.

We now present the CG-method for obtaining the GDGA \( \bar{y} \) associated to a Gabor triple \((g, a, b)\). The standard CG-method is stated in the following [11, p. 523]

**Algorithm A** (Conjugate gradient [11]). If \( A \) is positive definite, then we have the following iterative algorithm for computing \( x \) so that \( x^*A = \gamma \), for a vector \( y \): Starting with \( x_0 = 0 \), \( r_0 = \gamma \), \( p_1 = r_0 \) the iterative step is as follows, for \( k \geq 1 \),

- Set \( \beta_0 = 0 \) and \( \beta_k = \frac{r_{k-1}^2 - r_{k-2}^2 - 1}{r_{k-2}^2} \),
- Set \( p_1 = r_0 \) and \( p_k = r_{k-1} + \beta_k p_{k-1} \).
- Set \( \alpha_k = \frac{r_{k-1}r_k}{p_k * \mathbf{A} * p_k^T} \).
- Set \( x_k = x_{k-1} + \alpha_k p_k \).
- Set \( r_k = r_{k-1} - \alpha_k p_k * \mathbf{A} \).

It is known that \( N \) is the upper bound of the number of iteration steps and \( x_N \) is the exact solution to the linear system \( x * \mathbf{A} = y \) if the algorithm is performed in finite precision. In this case, we say that the CG-algorithm is convergent after \( N \) iterations.

For a Gabor triple \((g, a, b)\), let \( S \) be the associated Gabor matrix corresponding to \((g, a, b)\). We show in the following theorem that the CG-method can always be applied and be convergent after at most \( ab \) iterations to obtain the associated GDGA \( \tilde{g} \) by solving the linear equation \( \tilde{g} * S = g \). Using the Gabor matrix-vector multiplication algorithm 2, the product \( p_k * S \) for each CG-cycle can be implemented in only \( O(N) \) operations.

**Algorithm 3** (Generalized dual Gabor atom (GDGA)). Let \( S \) be the Gabor matrix corresponding to a Gabor triple \((g, a, b)\). If we replace both \( \mathbf{A} \) and \( y \) by \( S \) and \( g \) in Algorithm A, respectively, then the CG-algorithm is convergent after \( s \) \((s \leq ab)\) iterations and the resulting \( x_s = \tilde{g} \) is the solution for \( \tilde{g} * S = g \). Moreover, \( \tilde{g} = x_s \) is exactly the GDGA corresponding to \((g, a, b)\), that is, \( \tilde{g} = g * \text{pinv}(S) \).

In particular if \( g \) is real-valued, the algorithm is convergent after at most \( ab_0 \) iterations.

In order to derive Algorithm 3, we need the following lemma.

**Lemma 1.** Write
\[
P = \text{Span}\{S' : l = 0, 1, \ldots\}
\]
and
\[
P_* = \text{Span}\{x * S' : l = 0, 1, \ldots\}, \text{ for } x \in \mathbb{C}^N.
\]

Then both \( P \) and \( P_* \) (especially, \( P_g \)) are at most \( ab \)-dimensional subspaces of \( \mathbb{C}^{N \times N} \) and \( \mathbb{C}^N \), respectively. If \( g \) is a real-valued Gabor atom, then the dimensions of both \( P \) and \( P_g \) over \( \mathbb{R} \) are at most \( ab_0 \).

**Proof.** For \( l = 0, 1, \ldots \), let \( B_l \) be the first \( N \times a \) block matrix of \( S' \). By Theorem 3, \( S' \) are all completely determined by \( B_l \) which have at most \( ab \) nonzero entries. Because the nonzero entries of \( B_l \) for \( l = 0, 1, \ldots \), are in the same positions, both \( P \) and \( P_* \) are at most \( ab \)-dimensional subspaces of \( \mathbb{C}^{N \times N} \) and \( \mathbb{C}^N \), respectively.

If \( g \) is real-valued, by Theorem 1 and Theorem 3, for any \( l = 0, 1, \ldots, r = (b - q)b + 1 \)-th subdiagonal of \( B_l \) can be obtained by \( s = qb + 1 \)-th subdiagonal of \( B_l \) via only the rotation with the rotation number \( qb \). This yields that the dimension of \( P \) and consequently the dimension of \( P_g \) over \( \mathbb{R} \) are at most \( ab_0 \).

**Proof of Algorithm 3.**

**Step 1.** We show that the CG-procedure works and converges after at most \( ab \) iterative steps.

Following the arguments in [11, pp. 516–523], we only need to show that \( \alpha_k \) is always defined. Equivalently, we need to show the following:

\[
\text{if } r_{k-1} \neq 0, \text{ then } p_k * S * p_k^T \neq 0.
\] (13)

In fact, suppose that there exist \( p_k \) and \( r_{k-1} \) such that \( p_k * S * p_k^T = 0 \) but \( r_{k-1} \neq 0 \), then \( p_k * [\text{GAB}(g, a, b)]' * [\text{GAB}(g, a, b)] * p_k = 0 \), which yields that \( p_k * [\text{GAB}(g, a, b)]' = 0 \). Since \( S = [\text{GAB}(g, a, b)]' * [\text{GAB}(g, a, b)] \), \( g \) is in the row space of \( \text{GAB}(g, a, b) \). Inductively, it is easy to check that \( p_k \) is in the row space of \( \text{GAB}(g, a, b) \). We deduce that (see [11, pp. 50]), \( p_k = 0 \). This implies that

\[
r_{k-1} + \beta_k p_{k-1} = 0.
\]

But \( r_{k-1} p_{k-1}' = 0 \) holds (see [11, Theorem 10.2.2]), \( r_{k-1} = 0 \) follows. So we get a contradiction and Eq. (13) is true.

Therefore, if \( r_{k-1} \neq 0 \), then we always have that \( p_k * S * p_k^T \neq 0 \). By [11, Theorem 10.2.2 and Theorem 10.2.3], after \( k - 1 \) steps of the CG-algorithm, the residuals

\[
r_0, r_1, \ldots, r_{k-1} \in \text{Span}\{g, g * S, \ldots, g * S^{k-1}\}
\]

are mutually orthogonal. By Lemma 1, the dimension of the subspace \( \text{Span}\{g * S' \} \) for \( l = 0, 1, \ldots, \) \( \in \mathbb{C}^N \) is at most \( ab \). Hence, there exists \( s \) \((s \leq ab)\) such that the \((s + 1)\)-th residual \( r_s = 0 \), which yields that \( x_s * S = g \). Write \( \tilde{g} = x_s \), then \( \tilde{g} * S = g \).

If \( g \) is a real-valued signal, the algorithm is convergent after \( ab_0 \) steps by Lemma 1.

**Step 2.** We show that the solution \( \tilde{g} \) to the linear system \( \tilde{g} * S = g \) based on the CG-method is exactly the GDGA \( \gamma = g * \text{pinv}(S) \).
By the CG-iteration steps, we see that there exists a (real) polynomial \( p(t) \) with at most \( ab \) degree such that

\[
\bar{g} = g \star p(S) \quad \text{and} \quad \bar{g} \star S = g.
\]

We show that \( p(S) = \text{pinv}(S) \) and therefore \( \bar{g} = g \star \text{pinv}(S) \). Since \( S \) and \( p(S) \) are Hermitian and commute with each other, with the definition of the pseudoinverse [11, 16, 26], we only need to show that

\[
S \star p(S) \star S = S. \quad (14)
\]

Let \( S \) be the Gabor operator associated with \((g, a, b)\). Since \( S \) commutes with both the time translation \( T_u \) and the frequency modulation \( M_v \), for any \( k, l = 0, 1, \ldots, N - 1 \), we have

\[
g_{k,l} \star p(S) \star S = (p(S)g_{k,l}) = (p(S)g)_{k,l} = (S(p(S))g)_{k,l} = (g \star p(S) \star S)_{k,l} = (ar{g} \star S)_{k,l} = g_{k,l}.
\]

Thus,

\[
g_{k,l} \star p(S) \star S = g_{k,l} \quad \text{for} \quad k, l = 0, 1, \ldots, N - 1. \quad (15)
\]

By the definition of the Gabor basic matrix (see Eq. (2)), Eq. (15) yields that

\[
\text{GAB}(g, a, b) \star p(S) \star S = \text{GAB}(g, a, b). \quad (16)
\]

Since \( S = [\text{GAB}(g, a, b)]^* [\text{GAB}(g, a, b)] \), Eq. (14) follows immediately if we multiply both sides of Eq. (16) by \([\text{GAB}(g, a, b)]^*\). The proof is complete. \( \Box \)

Remark.

- **Complexity analysis:** Algorithm 3 is always applied with Algorithm 2. By Algorithm 2, we do not need to build up the full \( N \times N \) Gabor matrix \( S \), we only need to compute and save the associated \( b \times a \) “nonzero”-block matrix \( \tilde{B} \). This largely saves the storage space. Using Algorithm 2, the calculation of \( p_k \star S \) for every CG-iteration requires only \( Nb \) multiplications and \( Nb \) additions. Since \( \tilde{B} \) can be precalculated with only \( b_0(a + N) \) multiplications and \( b_0N \) additions in total, Algorithm 3 can be performed fast. By Algorithm 3, the upper bound for the number of CG-iteration steps is \( ab \). Therefore, the total operations for the computation of the GDGA \( \bar{g} \) is about \( 2ab(bN + 4N) \). If the lattice constants \((a, b)\) are fixed, then we obtain an \( O(N) \) CG-algorithm.

- **Jacobi-method:** When the Gabor atom is given as a typical Gaussian function (for example, the Gabor atom shown in Fig. 1, the associated Gabor matrix is (strictly) diagonal dominant. We use the Jacobi-method [11, 13] instead to improve the implementation. In this case, the computations and decompositions of Gabor matrices can also be realized by the associated “nonzero”-block matrices. Fig. 1 shows the comparison of convergent rates between the Jacobi-method and the CG-method.

- **PCG-method:** In practice, for a given Gabor triple \((g, a, b)\), the Gabor atom \( g \) is well localized. Even if the associated Gabor matrix \( S = (s_{i,j})_{N \times N} \) does not have good diagonal dominance, we can use the preconditioned conjugate gradient (PCG) algorithm [11] with the preconditioner

\[
P = \text{diag}(s_{1,1}, s_{2,2}, \ldots, s_{N,N})
\]

to improve Algorithm 3. Equivalently, if we set \( S_0 = S \star P^{-1} \) and \( g_0 = g \star P^{-1} \), then we need to solve the equation \( \bar{g} \star S_0 = g_0 \) by the CG-algorithm. Since \( S_0 = S \star P^{-1} = (s_{i,j}/s_{kk})_{N \times N} \) is a Gabor-type matrix [22], all the matrix–vector multiplications are easily performed. In fact, if \( \tilde{B} = (t_{k,l})_{b \times a} \) is the “nonzero”-block matrix of \( S \), then \( \tilde{B}_0 = (t_{k,l}/s_{kk})_{b \times a} \) is the one corresponding to \( S_0 \). Therefore, all the matrix–vector multiplication with \( S_0 \) can be simply performed with \( \tilde{B}_0 \) by Algorithm 2.

- **Undersampled cases:** With the relationship between the undersampled Gabor transforms and the oversampled Gabor transforms (cf. [23, Theorem 5] and details in [24]), we can also compute the associated GDGA \( \bar{g} \), when \((g, a, b)\) is given as a Gabor triple with \( ab > N \). Without the assumption that \((g, \tilde{b}, \tilde{a})\) generates a Gabor frame, we have extended the algorithms proposed in [23, 24].

**Corollary 4 (Gabor frame condition).** Let \( \bar{g} \) be the associated GDGA to a Gabor triple \((g, a, b)\), \( \tilde{B}_0 = (u_{k,i})_{b \times a} \) be the “nonzero”-block matrix associated
with the Gabor triples \((g, a, b)\) and \((\tilde{g}, a, b)\). Then \((g, a, b)\) generates a Gabor frame if and only if

\[
u_{k,j} = \delta_{1,k} = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k \neq 1, \end{cases}
\]

for \(k = 1, \ldots, a\) and \(j = 1, \ldots, b_0\).

Proof. Let \(S_0 = [GAB(g, a, b)]^* [GAB(\tilde{g}, a, b)]\). Then \((g, a, b)\) generates a Gabor frame if and only if \(S_0\) is an identity matrix. The condition (17) follows.

Now suppose that condition (17) is satisfied, we show that \((g, a, b)\) generates a Gabor frame. Since there exists a real polynomial \(p(t)\) such that \(\tilde{g} = g \ast \text{pinv}(S) = g \ast p(S)\), following the similar argument to the proof of Theorem 3, we can verify that the Gabor matrix \(S_0\) can be identified with \(p(S) \ast S\). By Theorem 3, \(S_0\) satisfies the \(R\)-symmetry and the condition (17) implies that \(S_0\) is an identity matrix. Therefore, \((g, a, b)\) generates a Gabor frame. \(\Box\)

Remark. For an arbitrary Gabor triple \((g, a, b)\), we can always check whether \((g, a, b)\) generates a Gabor frame or not via two steps:

1. Determine \(\hat{g}\) by Algorithm 3.
2. Test only \(ab_0 \approx \frac{1}{2} ab\) number of conditions (17).

In the following, we show that the GDGA \(\hat{g}\) obtained by Algorithm 3 is the closest one to the Gabor atom \(g\) in the sense of Qian and Chen [19]. Theorem 5 can be seen as an extended discrete version of Proposition 3.2 in the recent work of Janssen [15].

**Theorem 5.** Assume that \(\hat{g}\) and \(S\) are the GDGA and the Gabor matrix associated to a Gabor triple \((g, a, b)\), then \(\hat{g} = g \ast \text{pinv}(S)\) is the unique solution of the problem

\[
\text{minimize } \frac{\gamma - g}{\|g\|} \text{ over } \gamma \in \mathbb{C}^N
\]

with \(\gamma = [GAB(\hat{g}, \hat{b}, \hat{a})]^t = \frac{ab}{N^a}. \tag{18}\)

Proof. Write \(\mathcal{M} = \text{Span}\{g_{nm}\}\), then \(\{g_{nm}\}\) is a frame of \(\mathcal{M}\). Let \(S\) and \(S_{\mathcal{M}}\) be the Gabor operators associated with \(\{g_{nm}\}\) in \(\mathbb{C}^N\) and in \(\mathcal{M}\), respectively. Then \(S_{\mathcal{M}}\) is invertible. It is easy to check that

\[
S_{\mathcal{M}} = S_{|\mathcal{M}}
\]
and

\[ S_{\mu} x = x \ast S \quad \text{for} \quad x \in \mathcal{M}. \]

\[ S_{\mu}^{-1} \text{ has the following matrix representation:} \]

\[ S_{\mu}^{-1} x = x \ast \text{pinv}(S) \quad \text{for} \quad x \in \mathcal{M}. \]

Hence, \( \tilde{g} = g \ast \text{pinv}(S) = S_{\mu}^{-1} g \) is the usual dual Gabor atom associated to \((g, a, b)\) in \( \mathcal{M} \). Similar to Janssen’s argument [15, p. 17], we can show that \( \tilde{g} \) is the unique solution to the problem (18). In fact, setting \( g_t = S_{\mu}^{-1/2} g \), then \( g_t \in \mathcal{M} \) and

\[ g_t := g \ast (\text{pinv}(S))^{1/2} = g \ast \text{pinv}(S^{1/2}) \]

is exactly the GTGA corresponding to \((g, a, b)\). For any \( \gamma \) satisfying the biorthogonality \( \gamma \ast [\text{GAB}(g, \tilde{b}, \tilde{a})]^t = \frac{ab}{N} e \), we can express \( g_t \) as

\[ g_t = \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} \langle g_t, \gamma_{nm} \rangle g_{nm}. \]

On the other hand, \( g_t \in \mathcal{M} \) can also be expressed as

\[ g_t = \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} \langle g_t, \tilde{g}_{nm} \rangle g_{nm}. \]

Since \((g, a, b)\) generates a frame of \( \mathcal{M} \) and \( \tilde{g} \) is the dual Gabor atom associated to \((g, a, b)\) in \( \mathcal{M} \), the frame property (cf. [6, Proposition 3.2.4; 12, Proposition 2.1.5] or originally [8, Lemma VIII]) yields that

\[ ||\tilde{g}|| = \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} |\langle g_t, \gamma_{nm} \rangle|^2 = \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} |\langle g_t, \tilde{g}_{nm} \rangle|^2 \]

\[ \leq \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} |\langle g_t, \gamma_{nm} \rangle|^2 = \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} |\langle g_t, \tilde{g}_{nm} \rangle|^2 \]

\[ = ||\gamma||^2 \leq ||g||^2, \]

where \( \gamma, \tilde{g} \) is the orthogonal projection of \( g \) onto \( \mathcal{M} \). Since \( \gamma \ast [\text{GAB}(g, \tilde{b}, \tilde{a})]^t = \frac{ab}{N} e \) implies that \( \langle g, g \rangle = ab/N \),

\[ \left| \left| \frac{\gamma}{||\gamma||} - \frac{g}{||g||} \right| \right| = 2 - 2\text{Re} \frac{\langle \gamma, g \rangle}{||\gamma|| ||g||} \]

\[ = 2\left( 1 - \frac{ab}{N||\gamma|| ||g||} \right). \]

It follows that \( \tilde{g} = g \ast \text{pinv}(S) \) is the unique solution to the problem (18). \( \square \)

**Remark.** By choosing a different initial vector \( x_0 \), we can modify Algorithm 3 to obtain desirable biorthogonal Gabor analysis atoms. In fact, let \( h \) be a signal, if we intend to obtain a biorthogonal Gabor atom \( \tilde{g}_0 \) which is close to \( h \), then we set \( x_0 = g - h \ast S \) and apply Algorithm 3. Assume \( \tilde{g}_0 \) is the output, then \( \tilde{g} = \tilde{g}_0 + h \) is what we want.

**Algorithm 4** (Tight Gabor atom). Given a Gabor triple \((g, a, b)\) which generates a frame, the following procedure is used to compute the associated tight Gabor atom \( g_t \):

**Step 1.** By Algorithm 1, we obtain the “nonzero”-block matrix \( \tilde{B} \) of size \( b \times a \) and the Gabor frame upper bound \( U_b \) by Theorem 4.

**Step 2.** We calculate the dual Gabor atom and henceforth the lower Gabor frame bound \( L_b \).

**Step 3.** Set \( r = \frac{2L_b}{L_b + U_b} \), then \( \tilde{B}_r := r\tilde{B} \) is associated to the Gabor frame operator \( S_r := rS \) which satisfies that

\[ \frac{2L_b}{L_b + U_b} \leq ||S_r|| = r||S|| \leq \frac{2U_b}{L_b + U_b}. \]

This yields that

\[ ||I - S_r|| \leq \frac{U_b - L_b}{L_b + U_b} := \gamma < 1. \]

**Step 4.** Since \( ||I - S_r|| \leq \gamma < 1 \),

\[ S_r^{-1/2} = \sum_{k=0}^{\infty} \alpha_k (I - S_r)^k, \]

where

\[ \alpha_0 = 1, \quad \alpha_k = \frac{h(h-1) \cdots (h-k+1)}{1 \cdot 2 \cdots k} \]

for \( h = -\frac{1}{2} \) and \( k = 1, 2, \ldots \).

Hence,

\[ S_r^{-1/2} = \sqrt{r} \sum_{k=0}^{\infty} \alpha_k (I - S_r)^k. \]

Therefore, the tight Gabor atom can be obtained as \( g_t = S_r^{-1/2} g \).
Iteratively, set $g_t^{(0)} = \sqrt{r}g$, for $n \geq 0$,

$$g_t^{(n+1)} = \sqrt{r}a_{n+1} (g_t^{(n)} - S_r g_t^{(n)}) .$$

Then we have

$$\lim_{n \to \infty} g_t^{(n)} = g_t.$$

The error estimation after $n$ iterations is

$$||g_t^{(n)} - g_t|| \leq \frac{\sqrt{r} \gamma^{n+1}}{1 - \gamma} ||g|| .$$

In fact,

$$||g_t^{(n)} - g_t|| = \left|\sqrt{r} \sum_{k=n+1}^{\infty} \alpha_k (I - S_r)^k g \right|$$

$$= \left| \sqrt{r} (I - S_r)^{n+1} \sum_{k=0}^{\infty} \alpha_k a_{k+n+1} (I - S_r)^k g \right|$$

$$= \frac{\sqrt{r} \gamma^{n+1}}{1 - \gamma} ||g|| .$$

**Remark.** Even if $(g,a,b)$ does not generate a Gabor frame, Algorithm 4 also works.

**Algorithm 5 (Best approximation).** Assume that $(g,a,b)$ is a Gabor triple and $\bar{g}$ is the associated GDGA, we obtain the best approximation of a signal $x$ by linear combinations of the Gabor family via the following steps:

1. **Step 1.** We obtain the “nonzero”-block of size $b \times \alpha$ corresponding to the Gabor triples $(g,a,b)$ and $(\bar{g},a,b)$.
2. **Step 2.** Applying Algorithm 2 and Proposition 1, we obtain $x_{\text{approx}}$.

In particular, if $(g,a,b)$ generates a frame or a signal $x$ is in the Gabor space $\text{span}\{g_{nm}\}$, then we can reconstruct a signal $x$ via the above two steps.

**Remark.** In the two-dimensional case, if the Gabor atom is given as a 2-D separable one, then we can calculate the orthogonal projection of an image onto the 2-D Gabor space by using Proposition 1 and the higher dimensional results presented in [23].

4. Numerical results

This section is devoted to numerical simulations. The performances of the previously proposed algorithms are illustrated. Given an arbitrary Gabor triple, the numerical constructions of the associated GDGA, GTGA and the best approximation of a given signal are shown. All the numerical experiments were carried out using MATLAB 4.0 on a Sun4 Station.

As we have demonstrated, the algorithms proposed in this paper are in a very general sense. We do not assume that a given Gabor family is a frame. We do not restrict that the signal length is the power of 2. Signals can be either real valued or complex valued. Gabor atoms (Gabor window functions) can be chosen arbitrarily. Since Gabor atoms with Gaussian types are typical (they are minimizing the uncertainty principle [5]), most of our examples are experimented with Gabor atoms of Gaussian shapes.

When a Gabor atom is well concentrated, the associated Gabor matrix is strongly diagonal dominant. We use the Jacobi-method instead of the CG-algorithm to speed up the convergence. In Fig. 1, the Gabor atom is of a Gaussian shape. The signal length is 128 and the lattice constants $(a,b)$ are $(8,8)$.

Since Gabor window functions are usually with good localizations, the associated Gabor matrices have some diagonal dominance (even if they are not strictly diagonal dominant). In this case, the simple PCG-method is employed to improve the ordinary CG-algorithm. Fig. 2 compares the convergent rates of the PCG, CG and the ordinary frame-method. Both PCG and CG algorithms work much faster, while PCG is better than CG. Meanwhile, if $(g,a,b)$ does not generate a frame, the frame method does not work. However, the CG-method can always be applied to obtain the GDGA.

Tables 1 and 2 illustrate the comparison between the best Gabor frame bounds and the estimated ones calculated according to Theorem 4. For Table 1, we take a Gabor atom with the Gaussian shape shown in Fig. 3. For Table 2, we choose a Gabor atom with the exponential shape shown in Fig. 4. Lattice constant pairs are $(12,12)$, $(10,10)$, $(8,8)$ and $(8,6)$ for both Tables 1 and 2. It is clear that the calculated Gabor bounds $U_b$ and $L_b$ approximate well to the best Gabor
bounds $\lambda_{\text{max}}$ and $\lambda_{\text{min}}$. From the Gabor upper and lower bounds, we can predetermine the stability of the Gabor syntheses with the associated dual Gabor atoms shown in Figs. 3 and 4. In fact, the data in the column with $U_h/L_h = \kappa_\infty$ [11] estimate the condition number of the associated Gabor matrices. If $U_h/L_h$ is small, then the corresponding Gabor synthesis is stable.

The last two columns of Tables 1 and 2 show clearly why the CG-method converges much better than the frame-method [6, 7]. In fact, Theorem 10.2.5 in Golub [11] shows that the CG convergence rate is majorized by

$$
\sigma_1 := \sqrt{U_h - \sqrt{L_h}} = \frac{\sqrt{\kappa_\infty} - 1}{\sqrt{\kappa_\infty} + 1},
$$

while the frame-method convergence rate depends on

$$
\sigma_2 := \frac{U_h - L_h}{U_h + L_h} = \frac{\kappa_\infty - 1}{\kappa_\infty + 1}.
$$

The data in the last two columns of Tables 1 and 2 show that $\sigma_1$ is much smaller than $\sigma_2$. Therefore, the CG-method converges much faster.

Fig. 3 illustrates a Gabor atom and the associated dual and tight Gabor atoms. The signal length $N$ is 480 and the lattice constants are (i) $(12, 12)$, (ii) $(10, 10)$, (iii) $(8, 8)$ and (iv) $(8, 6)$. For a random signal, the reconstruction relative errors with the dual and tight Gabor atoms are of order $10^{-15}$.

Fig. 4 shows the exponential Gabor atom and the dual (tight) Gabor atoms corresponding to lattice constants: (i) $(12, 12)$, (ii) $(10, 10)$, (iii) $(8, 8)$ and (iv) $(8, 6)$. The signal length is 480. For a random signal, the reconstruction relative errors with the dual (tight) Gabor atoms are of order $10^{-14}$.

Next we show some experiments with Gabor triples that do not generate Gabor frames.

Given a Gabor triple $(g, a, b)$ and a signal $x$ with the same signal length as $g$. If $x$ is in the Gabor space $\text{Span}\{g_{nm}\}$, then we get the reconstructed signal of $x$ by Algorithm 5. If $x$ is not in the Gabor space $\text{Span}\{g_{nm}\}$, then we can only obtain the best approximation $x_{\text{approx}} \in \text{Span}\{g_{nm}\}$ of the signal $x$.

Fig. 5 illustrates the Gabor atom and the associated GDGA $\tilde{g}$ and GTGA $g_t$. The signal length $N$ is 1200.
The Gabor atom with respect to the lattice constants \((a, b) = (25, 16)\) does not generate a frame. In Fig. 6 we show

(I) the original signal from the Gabor space \(\text{Span}\{g_{mn}\}\) and the reconstructed signal, the (relative) reconstruction error is of order \(10^{-14}\),

(II) the original Chirp signal and the best approximate Chirp by the linear combinations of the Gabor family.

Fig. 7 shows the original and the best approximate image (Lena) of size \(128 \times 128\) with linear combinations of the 2-D Gabor family. A 2-D Gabor atom and the associated GDGA are shown in Fig. 8. The 2-D lattice constants \((a, b)\) are taken to be \(a = (16, 8)\) and \(b = (8, 4)\).

Fig. 9 gives another numerical simulation for the 2-D case. The original image given as a rectangular matrix of size \(463 \times 331\) and the best approximate image by the linear combination of the 2-D Gabor family are illustrated. The 2-D Gabor atom of size \(480 \times 480\) with the 2-D lattice constants \((a, b)\) for \(a = (30, 24)\) and \(b = (15, 10)\) does not generate a 2-D Gabor frame.

5. Conclusion

We have developed the \(\mathcal{R}\)-symmetry feature of Gabor matrices. The \(\mathcal{R}\)-symmetric property reduces the computational complexity by approximately half, for building up the Gabor matrix, its first \(N \times a\) block matrix and the associated \(b \times a\) "nonzero"-block matrix. The algorithms presented in [21, 23] are improved. In practice, we do not need to construct the full \(N \times N\) Gabor matrix \(\mathcal{S}\), the Gabor matrix–vector multiplications can be realized completely by the small \(b \times a\) \((ab \leq N)\) "nonzero"-block matrix \(\tilde{S}\). We save the storage space. On the other hand, for any Gabor triple \((g, a, b)\), we have presented a general approach to compute the GDGA \(\tilde{g}\) based on the \(O(N^2)\) CG-method and an algorithm to calculate the tight Gabor atom based on the binomial series. We have shown that \(\tilde{g}\) is the closest biorthogonal window function in the sense of Qian.
Fig. 4. A Gabor atom of the exponential shape and the associated dual and tight Gabor atoms associated with four different lattice constants pairs: (i) (12, 12), (ii) (10, 10), (iii) (8, 8) and (iv) (8, 6). The signal length is 480.

Fig. 5. A Gabor atom and the associated GDGA and GTGA. The signal length $N = 1200$ and the lattice constants $(a, b) = (25, 16)$. 
and Chen [19]. The algorithms presented in this paper have the following important applications:

1) to check whether a Gabor triple \((g, a, b)\) generates a Gabor frame or not;

2) to predetermined the quality of a Gabor triple \((g, a, b)\), i.e. the stability of Gabor synthesis with \((g, a, b)\);

3) to obtain the best approximation of a signal \(x \in \mathbb{C}^N\).

Acknowledgements

The author acknowledges fruitful discussions with Prof. Hans G. Feichtinger. The author would like to thank the hospitality of the Department of Mathematics of the University of Vienna where the author began to prepare the paper. Special thanks to Prof. K. Gröchenig for carefully reading the paper and making good suggestions. The author would like to thank...
Fig. 8. A 2-D Gabor atom and the associated GDGA.

Fig. 9. A 2-D Gabor atom, an original image and the best approximate image.
Mr. K. Marinelli for setting up the computer facility enabling to do numerical experiments.

References


