THE MINIMAL STRONGLY CHARACTER INVARIANT SEGAL ALGEBRA, II

by

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(preliminary version)

Introduction.

In the first part of this paper a new Segal algebra $S_0(G)$ has been defined for arbitrary locally compact abelian groups. This Segal algebra has several remarkable properties. Among others it is the minimal strongly character invariant Segal algebra and the only known Segal algebra having feeble factorization. In the present note we continue investigations concerning $S_0(G)$. We shall give further equivalent norms for $S_0(G)$ in section 1. Using these new characterizations several results concerning multipliers can be derived in section 2. Among others it is shown that $(L^1, S_0) \cong S_0$, and that $(S_0, S_0) = (S_0, L_1)$. Further, it is shown that at least for $G = \mathbb{R}^n$ and for any $p < \alpha^*$ there exists some continuous function $f$ not belonging to $L^p(G)$, which nevertheless satisfies $f \ast S_0(G) \ast S_0(G)$. This result substantially improves recent results due to Krogstad and Tewari ([11], [17]) who have shown that there exist Segal algebras on non-compact abelian groups admitting multipliers which are not given by convolution with some bounded measure. Finally, section 3 contains several results concerning the dual space of $S_0(G)$. Among others it is shown that it is properly contained in the space $Q(G)$ of all quasimeasures on $G$, but nevertheless contains most of the spaces of multipliers which are of interest in harmonic analysis. At the end of this paper applications to the Fourier transform of unbounded measures are given.

Throughout this note we assume that the reader is familiar with the content and the notations used in [7].
§ 1. Equivalent norms on $S_o(G)$

In [7] already several different, equivalent norms for $S_o(G)$ have been given. In this paragraph we define further equivalent norms which are particularly useful when working with multipliers.

**Theorem 1.1.** Let $G$ be a locally compact abelian group, and let any $g \in S_o(G)$, $g \not\in \mathcal{Q}$ be given (e.g. $g \in A_Q(G)$, or $g \in L^1(G), \hat{g} \in K(\hat{G})$). Then $f$ belongs to $S_o(G)$ if and only if the function $f^g$, given by 

$$f^g(y) := \|L_y g \cdot f\|_{A(G)}$$

belongs to $L^1(G)$; further, the norm $f \mapsto \|f\|_1 := \|f^g\|_1$ is equivalent to the original norm on $S_o(G)$.

**Proof.** The proof is given in several steps.

**Step I.** By the structure theory for locally compact abelian groups $G$ is of the form $G = \mathbb{R}^m \times H$, $H$ containing some open, compact subgroup $J$ (cf. [10], Vol. I, 24.29 and 24.30). Let now $t$ denote the trapezoid function on $G = \mathbb{R}$, given by $c[0,1] * 2c[-1/4, 1/4]$. We write $c_j$ for the characteristic function of $J = H$ and set $g_o := t \circ \ldots \circ t \circ c_j$, i.e.

$$g_o(x_1, \ldots, x_m, h) := t(x_1) \cdot t(x_2) \cdot \ldots \cdot t(x_m) c_j(h).$$

Then $g_o$ belongs to $A(G)$ and has compact support, i.e. $g_o \in S_o(G)$. Moreover, there exists a discrete subset $(y_j)_{j \in I} \subseteq \mathbb{Z}^m \times H$ such that $\sum_{j \in I} L_{y_j} g_o = 1$. Our first assertion is now the following one

$$\|f\|' := \sum_{j \in I} \|L_{y_j} g_o f\|_{A(G)}$$

defines an equivalent norm on $S_o(G)$.

**Proof.** Since we might assume that the compact set $Q$ used in the definition of $S_o(G)$ contains $\text{supp } g_o$, it is clear that $\|f\|_{S_o} \leq C_1 \|f\|'$ for all $f \in S_o(G)$. Let us now prove the converse.
In view of the definition of $S_o(G)$ it will be sufficient to show that there exists $C_2 > 0$ such that

$$
\sum_{j \in I} \| L_y g_o \|_{A(G)} \leq C_2 \| f \|_{A(G)}
$$
for all $f \in A(G), y \in G$.

For simplicity we may assume here that $Q = [0,1]^m \times J$. By reduction to the case $G = \mathbb{R}^m$ the estimate follows immediately from the observation that there exists $n_o < \infty$ such that for any $y \in G \cap \Delta_j$ supp $g_o \not= \emptyset$ for at most $n_0$ different values $(y_j)_{j \in I}$.

Thus, one may take, for example, $C_2 = n_o \| g_o \|_{A(G)}$.

**Step II.** $f g_o \in L^1(G)$ and $\| f \|_{S_o(G)} \leq C_4 \| f g_o \|_{1}$ for all $f \in S_o(G)$, with $f g_o(y) = \| L_y g_o f \|_{A(G)}, y \in G$.

**Proof.** There exists some finite set $I_o$ such that $\sum_{j \in I_o} \| L_y g_o \|_{A(G)} = 1$ for all $x \in Q \cdot \text{supp } g_o$.

For $y \in y_k \Delta_j$ this implies

$$
\sum_{j \in I_o} \| L_y g_o \|_{A(G)} \leq \sum_{j \in I_o} \| L_y g_o \|_{A(G)} = 1
$$

This gives further

$$
\| f g_o \|_{1} = \int f g_o \ dy \leq \sum_{k \in I} \sum_{j \in I_o} \| L_y g_o f \|_{A(G)}
$$

$$
\leq \sum_{k \in I} \sum_{j \in I_o} \| L_y g_o \|_{A(G)} \leq C_4 \| f \|_{S_o(G)}
$$

**Step III.** Let $g \in S_o(G), g \not= 0$ be given. Then the norm $f \to \| f g_o \|_{1}$ is equivalent to the norm given by $f \to \| f g_o \|_{1}$.

**Proof.** Any $g \in S_o(G)$ has a representation of the form $g = \sum_{n \geq 1} f_n, \text{supp } f_n \subseteq [1/4,3/4]^m \times J, (f_n)_{n \geq 1} \subseteq A(G)$, with $\| f_n \|_{A(G)} \leq C_5 \| g \|_{S_o(G)}$. 

This gives
\[ f_n(y) = \| L_y f_n \|_{A(G)} = \| L_{y_n} f_n g_0 \|_{A(G)} \leq \| f_n \|_{A(G)} \| g_0 \|_{1}, \]
and further
\[ \| f \|_1 \leq \sum_{i \in I} \| L_y f_i \|_1 = \| f_i \|_1 \]
\[ \leq (\| L_y f_i \|_{A(G)} \| g_0 \|_{1}) \leq c \| g \|_{S_0} \| f \|_1. \]

Conversely, suppose that \( f \in L^1(G) \) for some \( g \in S_0(G), \ g \neq 0. \) In view of the inequality \( \| f \|_1^2 \leq \| g \|_{A(G)} \| f \| \) we may suppose without loss of generality that \( g \) is positive. It is then possible to find a finite sequence \( (f_n)_{n=1}^\infty \) such that \( g_n(x) \geq \delta > 0 \) for all \( x \in \text{supp } g \). It follows that there exists \( g \in A(G) \) such that \( g \in \text{supp } g \). This implies
\[ \| f \|_1 = \| f g \|_1 \leq \| g \|_{A(G)} \| g \|_{A(G)} \| f \|_1 \leq c \| f \|_1, \]
and the proof is complete.

**Step IV.** For some \( g \in S_0(G), f \in L^1(G) \) implies
\[ \sum_{j \in I} \| L_{y_j} f \|_{A(G)} \leq c \| f \|_1. \]

**Proof.** Let \( V \) be some open set in \( G \) such that \( y \cap y_j = \emptyset \) for \( i 
eq j \) (e.g., \( V = (0, 1/4)^{i \times j} \). Choose now \( g \in S_0(G) \) such that \( g(xy) = 1 \) for all \( x \in V^{-1} \) \( \text{supp } g \). This implies
\[ \| L_{y_j} g \|_{A(G)} = \| L_{y_j} g \|_{A(G)} \leq \| L_{y_j} g \|_{A(G)} \] for all \( y \in V \).

This gives in turn
\[ \| f \|_1 \geq \sum_{j \in I} \| L_{y_j} g \|_{A(G)} \] for all \( y \in V \).

**Remark.** It should be mentioned that the equivalence of the norm
given in the above theorem and the original norm on $S_0(G)$ has an analogon in the equivalence of the two norms on $W(G)$ defined in [5] and in [13] respectively (cf. [4], Theorem 4). Using once more results from [7] we derive another further equivalent norm on $S_0(G)$.

**Theorem 1.2.** Let $S$ be any strongly character invariant Segal algebra on $G$, and let $g \in S_0(\mathbb{R}^m)$, $g \neq 0$ be given. Then the norm $\|g\|_S$ given by

$$\|g\|_S = \int_{\hat{G}} \|M_y g \ast f\|_S \, dy$$

defines an equivalent norm on $S_0$.

**Proof.** In view of Theorem 1 of [7] it will be sufficient to prove:

1) $\|g\| = \|f\|_S$ and $\|g\|_S$ are equivalent norms on $S_0(G)$; and

2) $\|g\|_S \leq C \|f\|_S$ for all $f \in S_0(G)$.

In order to verify 1) we observe that

$$\|M_y g \ast f\|_S = \|L_y \hat{g} \tilde{f}\|_A(G)$$

for all $f, g \in S_0(G)$.

Since, by Theorem 5 of [7], the Fourier Transform defines an isomorphism from $S_0(G)$ onto $S_0(\hat{G})$, the result follows from Theorem 1 above. We now prove 2). By Theorem 3 of [7] there exists $C_0 > 0$ such that for $f \in S_0(G)$ there exist two sequences $(f_n)_{n \geq 1}$, $(g_n)_{n \geq 1}$ in $S_0(G)$ such that

$$f = \sum_{n=1}^{\infty} f_n \ast g_n \quad \text{and} \quad \sup_{n=1} \|g_n\|_S \|f_n\|_S \leq C_0 \|f\|_S.$$
By means of 1) this implies
\[
\int_G |M_y g*f| \|S_0 \, d\nu \leq \sum_{n=1}^\infty \|M_y g*f_n \| \|g_n\|_{S_0} \\
\leq \sum_{n=1}^\infty \int_G |M_y g*f_n| \|g_n\|_{S_0} \\
\leq C_y C_\Omega \sum_{n=1}^\infty \|f_n\| \leq C\|f\|_{S_0},
\]
and the proof is complete.
2. Results on multipliers.

In the present section we shall use the different characterizations of $S_o(G)$ given in [7] as well as in §1 in order to derive results concerning various spaces of multipliers. Recall that a bounded linear operator between two translation invariant spaces $B_1$ and $B_2$ is called a multiplier (we write $T \epsilon (B_1,B_2)$) if $T \mathcal{L}_y = \mathcal{L}_y T$ for all $y \epsilon G$. Our main interest concerns the spaces $(L^1,S_o),(S_o,S_o)$, and $(S_o,L^1)$.

**Theorem 2.1.** There is a natural isomorphism from $S_o(G)$ onto $(L^1,S_o)$, given by: $f \rightarrow T_f: T_f(g) = f*g$ for all $g \epsilon L^1(G)$.

**Proof.** Since $S_o(G)$ is a Segal algebra it is obvious that $j : f \rightarrow T_f$ defines a contractive embedding from $S_o(G)$ into $(L^1,S_o)$. It therefore remains to prove that $j$ is surjective and that $\|f\|_{S_o} \leq \|T\|$ (operator norm of $T$). Since $S_o \subseteq L^1 \cap A(G) \subseteq L^2(G)$ any $T \epsilon (L^1,S_o)$ is necessarily of the form $T = T_f$ for some $f \epsilon L^1 \cap A(G)$ (cf. Theorem 3.1. of [6]). We have to prove that $f$ belongs to $S_o(G)$. Suppose the contrary. Then for any $n \epsilon N$ there exists a finite sum (cf. Step I of the proof of Theorem 1.1) such that

$$\sum_{j=1}^{k} \|L_y g \epsilon f\|_{A(G)} \geq (n+1)\|T\|.$$  

On the other hand we have

$$\|u*f\|_{S_o} \leq \|T\| \quad \text{for all} \quad u \epsilon L^1(G), \quad \|u\|_1 = 1.$$  

If we choose $u$ such that $\|f-u*f\|_{A(G)} \leq \|T\|/\|g\|_{A(G)} \cdot k$ for $1 \leq j \leq k$, we have

$$C \cdot \|u*f\|_{S_o} \geq \sum_{j=1}^{k} \|L_y g \epsilon (u*f)\|_{A(G)} \geq n\|T\|.$$  

This yields a contradiction and the proof is complete.

**Corollary 2.2.** Suppose a bounded sequence $(f_n)_{n \geq 1} \subseteq S_o(G)$ is given, such that for some measure $\mu \epsilon M(G)$

$$\lim_{n \rightarrow \infty} \int_G f_n(x)k(x)dx = \mu(k) \quad \text{for all} \quad k \epsilon K(G).$$  

Then $\mu = f_0 dx$ for some $f_0 \epsilon S_o(G)$, with $\|f_0\|_{S_o} \leq \sup_n \|f_n\|_{S_o}$. 
Proof. By the definition of \( \tilde{\mathcal{B}} \) (cf. [5]) \( \mu \) belongs to \( S_0(\mathbb{G}) \). On the other hand the character invariance of \( S_0(\mathbb{G}) \) implies \( (L^1, S_0) = \tilde{\mathcal{B}} \).

([5], Theorem 3.9). The result then follows from Theorem 2.1.

We now turn our attention to the space \((S_0, S_0)\).

**Theorem 2.3.** Let \( S, S_1, S_2 \) be Segal algebras on \( \mathbb{G} \), and suppose that \( S_i \) is strongly character invariant. Then

\[
(S_1, S_2) \subseteq (S_0, S_0), \quad \text{and} \quad (S_0, S_0) = (S_0, S_1) = (S_0, L^1)
\]

Proof. Since \( S_0 \) is the minimal strongly character invariant Segal algebra ([7], Theorem 1), we have \((S_1, S_2) \subseteq (S_0, L^1)\) and \((S_0, S_0) \subseteq (S_0, S_1)\). It thus remains to prove \((S_0, L^1) \subseteq (S_0, S_0)\). We shall use the relation \( S_0 \otimes S_0 = S_0([7]) \), Theorem 3). Let \( T \in (S_0, L^1) \) and \( f \in S_0(\mathbb{G}) \) be given. Then

\[
f = \sum_{n=1}^{\infty} f_n * g_n, \quad \text{where} \quad (f_n)_{n \geq 1}, (g_n)_{n \geq 1} \subseteq S_0(\mathbb{G}) \quad \text{and} \quad \sum_{n=1}^{\infty} \| f_n \|_{S_0} \| g_n \|_{S_0} \leq C \| f \|_{S_0}.
\]

This implies \( Tf = \sum_{n=1}^{\infty} f_n * Tg_n \in S_0(\mathbb{G}) \), since

\[
\| Tf \|_{S_0} \leq \sum_{n=1}^{\infty} \| f_n \|_{S_0} \| Tg_n \|_{S_0} \leq \| T \|_{(L^1, S_0)} \sum_{n=1}^{\infty} \| f_n \|_{S_0} \| g_n \|_{S_0} < \infty,
\]

and the proof is complete.

Remark. Only recently it has been shown that there exist Segal algebras \( S(\mathbb{G}) \) on certain noncompact abelian groups which admit multipliers that are not given by convolution with some bounded measure (cf. [11], Theorem 3.9, Corollary 3.8, and the main result of [17]). For example, Krogstad has proved that exist 'true' pseudomeasures with compact support which define multipliers on \( W^p(\mathbb{G}) \). Furthermore, he gave some \( g \in L^1_{\text{loc}}(\mathbb{R}^m) \) such that \( g * w(\mathbb{R}) \in W(\mathbb{R}) \), although \( g \notin L^1(\mathbb{G}) \) (nevertheless \( g \in L^p(\mathbb{R}) \) for any \( p > 1 \)). Since these Segal algebras are all strongly character invariant it follows from the above theorem that \((S_0, S_0) \nsubseteq M(\mathbb{G})\) in general. It is our next target to construct a family of multipliers on \( S_0(\mathbb{G}) \) which are not necessarily given by bounded measures.
To this end we need a characterization of the pointwise multipliers of \( S_0(G) \).

**Theorem 2.4.** Let \( h \) be some (continuous) function on \( G \). Then \( hf \in S_0(G) \) for all \( f \in S_0(G) \) if and only if

\[
(*) \quad \sup_{y \in G} \| L_y g \cdot h \|_{A(G)} = C_h < \infty \quad \text{for some } g \in S_0(G), \ g \neq 0.
\]

**Remark.** Making use of Step III of the proof of Theorem 1.1 one shows that \((*)\) is satisfied for all \( g \in S_0(G) \) if and only if it is satisfied for some \( g \in S_0(G), \ g \neq 0, \) e.g., \( g \in A(G) \cap K(G) \).

**Proof of Theorem 2.4.** Let \( h \) be given satisfying \((*)\). Then it follows from Theorem 1.1 that for any \( f \in S_0(G) \) the following is true:

\[
\|fh\|_{S_0} \leq C \int_G \| L_y g \|_{A(G)} \| f \|_{A(G)} \, dy \\
\leq C \sup_{y \in G} \| L_y g \|_{A(G)} \int_G \| f \|_{A(G)} \, dy \\
\leq C' \cdot C_h \| f \|_{S_0},
\]

i.e., \( M_h : f \mapsto fh \) defines a bounded linear operator on \( S_0(G) \).

Suppose now that \( hf \in S_0(G) \) for all \( f \in S_0(G) \). By the closed graph principle, \( M_h \) defines a bounded, linear operator on \( S_0(G) \). Consequently

\[
\sup_{y \in G} \| L_y g \|_{A(G)} \leq \| M_h \|_{\text{sup} \| L_y g \|_{S_0}} \leq \| M_h \| \| g \|_{S_0} < \infty \quad \text{for all } y \in G,
\]

and the proof is complete.

**Proposition 2.5.** Let \( g \in S_0(\mathbb{R}^m) \), and sequences \( (a_k)_{k \in \mathbb{Z}^m} \in l^\infty(\mathbb{Z}^m) \), \( (x_k)_{k \in \mathbb{Z}^m} \subseteq \mathbb{R}^m \), and \( (y_k)_{k \in \mathbb{Z}^m} \subseteq \mathbb{R}^m \), with \( y_k \in k+\mathbb{Z}^m, k \in \mathbb{Z}^m \), be given. Set

\[
\sigma = \sum_{k \in \mathbb{Z}^m} a_k x_k y_k g.
\]

Then \( \sigma \) is well defined as a pseudomeasure and \( T_\sigma : f \mapsto \sigma \) defines a multiplier on \( S_0(\mathbb{R}^m) \).

**Proof.** We have
\[
| \sum_{k \in \mathbb{Z}^m} a_k (I_{x_k} M_y g)^\wedge(t) | \leq \sup_{k \in \mathbb{Z}^m} |a_k| \sum_{k \in \mathbb{Z}^m} |I_{y_k} \hat{g}(t)|.
\]

Since the space \( P(G) = \mathcal{A}(G) \) is isomorphic to \( L^\infty(\hat{G}) \) (see [12], Theorem 4.2.2) via Fourier transform it will be sufficient to show that the finite partial sums of the right hand sum are uniformly bounded on \( \mathbb{R}^m \) and converge absolutely and uniformly on compact subsets of \( \mathbb{R}^m \).

This can be verified by observing that \( \hat{g} \in S_o(\mathbb{R}^m) \subset L^1(\mathbb{R}^m) \) implies
\[
\sum_{k \in \mathbb{Z}^m} |I_{y_k} \hat{g}(t)| \leq \sup_{t \in \mathbb{R}^m} \max_{z \in \mathbb{R}^m} |\hat{g}(t+z)| = \|\hat{g}\|_{L^1} < \infty.
\]

The function \( h \) given by
\[
h(t) = \sum_{k \in \mathbb{Z}^m} a_k (I_{x_k} M_y g)^\wedge(t)
\]
is thus well defined and bounded on \( \mathbb{R}^m \), with \( \|h\|_{L^\infty} \leq \|g\|_{S_o} \).

Consequently, \( \sigma \) is well defined as a pseudomeasure on \( \mathbb{R}^m \) and by definition the equality \( (\sigma * f)^\wedge = h^\wedge \) holds true for all \( f \in S_o(\mathbb{R}^m) \).

Now we have to show that \( f \mapsto \sigma * f \) defines a multiplier on \( S_o(\mathbb{R}^m) \). By Theorem 5.1 of [7] and by the above identity it has to be proved that
\[
\|L_y g_0 \sum_{k \in \mathbb{Z}^m} L_{y_k} \hat{g}\|_{\mathcal{A}(G)} \leq \sum_{k \in \mathbb{Z}^m} \|L_y g_0 L_{y_k} \hat{g}\|_{\mathcal{A}(G)} \leq C h
\]
for all \( y \in G \) (of Theorem 2.4). Since \( \hat{g} \) belongs to \( S_o(\mathbb{R}^m) \), and in view of the definition of \( S_o(G) \) it will be sufficient to prove that
\[
\sum_{k \in \mathbb{Z}^m} \|L_y g_0 L_{y_k} g_1\|_{\mathcal{A}(G)} \leq C \|g_1\|_{\mathcal{A}(G)} \quad \text{for all } y \in G,
\]
and all \( g_1 \in A_{Q_0}(\mathbb{R}^m), Q_0 = [0,1]^m \). Since this inequality follows in turn from the fact that \( |y_{Q_0} \cap y_{k+Q_0}| \neq 0 \) at most \( 3^m \) different values of \( (y_k)_{k \in \mathbb{Z}^m} \) our proof is complete, if we choose \( C = 3^m \).
Corollary 2.6. Let \( 1 \leq p < \infty \) be given. Then there exists \( f, g \in C^0(\mathbb{R}^m) \), \( f \in L^p(\mathbb{R}^m) \) such that \( \mathsf{T}_f : g \mapsto g \ast f \) defines a multiplier on \( S_0(\mathbb{R}^m) \).

**Proof.** Choose \((a_k)_{k \in \mathbb{Z}^m} \) such that \( \lim_{k \to \infty} |a_k| = 0 \), but \((a_k)_{k \in \mathbb{Z}^m} \) is in \( L^1(\mathbb{Z}^m) \). Then, by Theorem 2.5,

\[
f := \sum_{k \in \mathbb{Z}^m} \mathsf{T}_{a_k} \mathsf{L}_{x_k} M_k g
\]

has the required properties, if \( g \in \Lambda_2(G) \), and \((x_k)_{k \in \mathbb{Z}^m} \) are chosen such that \( x_k \cap x_{k+1} = \emptyset \) for \( k \perp 1 \in \mathbb{Z}^m \).

With some modifications the result of Corollary 2.6 can be extended to arbitrary nondiscrete, noncompact, locally compact abelian groups.

**Proposition 2.7.** i) There exists \( f_1 \in C^0(\mathbb{R}^m) \) such that \( f_1 \ast S_0(\mathbb{R}^m) \subseteq S_0(\mathbb{R}^m) \), but \( f_1 \ast f_2 \notin L^1(\mathbb{R}^m) \) for some \( f_2 \in W(\mathbb{R}^m) \).

ii) There exists \( f_3 \in C^0(\mathbb{R}^m) \), \( f_3 \notin L^1(\mathbb{R}^m) \), such that \( f_3 \ast W(\mathbb{R}^m) \subseteq S_0(\mathbb{R}^m) \).

**Proof.** For \( m = 1 \) our arguments are the following ones:

i) By Theorem 2.5 any pseudomeasure \( \sigma \) of the form

\[
\sigma := \sum_{k \in \mathbb{Z}} a_k \mathsf{L}_{x_k} M_k g
\]

defines a multiplier on \( S_0(\mathbb{R}) \) via convolution. We choose \( a_k = (\log |k|)^{-1} \), \( x_k = 5k \), and \( g \) the trapezoid function given by \( g := 0_{[0,1]} * 0_{[0,2]} \). Then the series defining \( \sigma \) is in fact norm convergent in \( C^0(\mathbb{R}) \).

Let us denote its sum by \( f_1 \). In order to show that \( f_1 \ast f \) need not belong to \( L^1(\mathbb{R}) \) for all \( f \in W(\mathbb{R}) \) let us observe that \( f_1 \ast f \notin L^1(\mathbb{R}) \) for all \( f \in W(\mathbb{R}) \) implies that \( f_1 \ast f \) defines a bounded linear operator from \( W(\mathbb{R}) \) into \( L^1(\mathbb{R}) \) by the closed graph theorem.

A direct calculation with \( h_0 = 0_{[0,1]} \in L^1 \cap L^\infty(\mathbb{R}) \) gives

\[
| h_0 * M_{2k+1} s(y) | = \left| \int_{y}^{y+1} M_{2k+1} (x) dx \right| \geq | \sin(2k+1) \pi x | \left| y^{0+1} \right|
\]

for \( y \in [0,1] \). Consequently

\[
|| h_0 * M_{2k+1} s ||_1 \geq \int_0^{1/2k+1} \sin(2k+1) \pi t dt \geq \pi / 2k+1
\]
for some \( \delta > 0 \) and all \( k \in \mathbb{N} \). This implies
\[
\| \sum_{-n}^{n} \left[ \log (2k+1) \right]^{-1} h_0 \ast L_{5k} g \|_1 \leq c_n \quad \text{for } n \to \infty.
\]

Let us now fix \( n \). Then it is possible to choose some \( u_n \in L^1(\mathcal{D}) \), \( \| u_n \|_1 = 1 \), \( \text{supp } u_n \subseteq [0, 1] \), such that
\[
\| u_n \ast h_0 \ast M_{2k+1} g - h_0 \ast M_{2k+1} g \|_{L^1} \leq \frac{c_n}{4n}
\]
for \( -n \leq k \leq n \). If we set \( h_n := u_n \ast h_0 \) we have \( \| h_n \|_{L^1} \leq 3 \) for all \( n \). On the other hand we have
\[
\| h_n \ast f \|_{L^1} \geq \sum_{k=-n}^{n} \left[ \log (2k+1) \right]^{-1} \| h_n \ast M_{2k+1} g \|_1 \geq c_n \frac{n}{2}
\]
This yields a contradiction. Thus our assertion is proved.

ii) By i) there exist \( f_1 \in C^0(\mathbb{R}) \) and \( f_0 \in W(\mathbb{R}) \) such that \( f_1 \ast f_0 \in L^1(\mathbb{R}) \), but \( f_1 \ast S_o(\mathbb{R}) \in \mathcal{D}_o(\mathbb{R}) \).

If we set \( f_2 := f_1 \ast f_0 \) it is obvious that \( f_2 \in C^0(\mathbb{R}) \setminus L^1(\mathbb{R}) \). On the other hand we have \( f_2 \ast f = f_1 \ast (f_0 \ast f) \in S_o(\mathbb{R}) \) for all \( f \in W(\mathbb{R}) \), since \( f_0 \ast f \) belongs to \( S_o(\mathbb{R}) \) (see [7], Corollary 4).

The above results are easily extended to \( \mathbb{R}^m \). In fact, let \( m \) functions \( f_i, 1 \leq i \leq m \) on \( \mathbb{R} = \mathbb{R}^1 \) be given. Then \( \bigotimes_{i=1}^m f_i \), given by \( \bigotimes_{i=1}^m f_i(x_1, \ldots, x_m) = \bigotimes_{i=1}^m f_i(x_i) \) satisfies
\[
\| \bigotimes_{i=1}^m f_i \|_1 = \prod_{i=1}^m \| f_i \|_1, \quad \| \bigotimes_{i=1}^m f_i \|_W = \prod_{i=1}^m \| f_i \|_W, \quad \text{and } \| \bigotimes_{i=1}^m f_i \|_S_o \leq \prod_{i=1}^m \| f_i \|_S_o.
\]
One has further \( \bigotimes_{i=1}^m f_i \ast h = \bigotimes_{i=1}^m f_i \ast h_i \).

The result for \( G = \mathbb{R}^m \) is therefore obtained by replacing \( f_1 \) by \( \bigotimes_{i=1}^m f_i \), and so on.

**Corollary 2.8.** \((W(\mathbb{R}^m), S_o(\mathbb{R}^m))\) is a proper subspace of \((S_o(\mathbb{R}^m)), L^1(\mathbb{R}^m))\), and \((W(\mathbb{R}^m), L^1(\mathbb{R}^m))\) is a proper subspace of \((S_o(\mathbb{R}^m)), L^1(\mathbb{R}^m))\).
3. The dual space of $S_o(G)$

This section consists of two parts. In the first part some properties of $S_o(G)$ are stated and the relationship to the spaces $P(G)$ and $Q(G)$ of all pseudomeasures and quasimeasures respectively are explained. Besides, several applications in the theory of multipliers are given. In the second part the notion of Fourier transform is extended to $S_o(G)'$, and the connections to other results on the Fourier transform of pseudomeasures, or certain unbounded measures are established.

Before we go into the details let us recall a few definitions which are of common use in the theory of multipliers (cf. [12], 4.2 and 5.1).

The Banach space $P(G)$ of all pseudomeasures on $G$ is defined as the dual space of $A(G)$. The space $Q(G)$ of all quasimeasures on $G$ is defined as the topological dual of the locally convex topological vector space $D(G)$, which in turn is given as inductive limit over the family of spaces $D_K(G)$, $K \subset G$ compact:

$$D_K(G) := \{ h \mid h = \sum_{i=1}^{\infty} f_i \ast g_i, \quad f_i, g_i \in K(G), \quad \text{supp } f_i \cup \text{supp } g_i \subset K, \quad \sum_{i=1}^{\infty} \| f_i \|_{\infty} \| g_i \|_{\infty} < \infty \}.$$ 

For any given $K \subset G$ $D_K(G)$ is a Banach space with respect to the norm given by

$$\| h \|_K := \inf \left\{ \sum_{i=1}^{\infty} \| f_i \|_{\infty} \| g_i \|_{\infty} \right\},$$

the infimum being taken over all admissible representations of $h$.

Besides, we are concerned with $M(G) = C^0(G)'$, the space of bounded measures, which is a subspace of the space $R(G) = K(G)'$ of all Radon measures on $G$. $T(G) \subset R(G)$ denotes the space of all translation bounded measures on $G$. By definition a measure $\mu$ belongs to $T(G)$ iff $\sup_{y \in G} |\mu(yK)| < \infty$ for any compact set $K \subset G$. 


It can be shown without difficulty that $T(G)$ can be identified with the dual space of Wiener's algebra $W(G)$ as defined in [4] (cf. [3] for the case $G = \mathbb{R}$). Further, recall that a Banach space $B$ is called a homogeneous Banach space on $G$ if $B$ is continuously embedded in $L^1_{\text{loc}}(G)$, and satisfies $\|L_y f\|_B = \|f\|_B$ for all $y \in G$, $f \in B$, and $\lim_{y \to 0} \|L_y f - f\|_B = 0$ for all $f \in B$ (see [5], § 1).

For the sake of completeness let us state a recent result concerning $D(G)$ due to Cowling:

**Theorem 3.1** ([3], Theorem 3.1)

$$D(G) = A(G) \cap K(G)$$

If $A(G) \cap K(G)$ is considered as the inductive limit of the family $(A_K(G))_{K \in G}$, $K$ compact, then $D(G)$ and $A(G) \cap K(G)$ are isomorphic as topological vector spaces.

**Theorem 3.2**

The following scheme of inclusions between the spaces introduced above holds true:

A) $D(G) \subset K(G) \subset S^0(G) \subset W(G) \subset C^0(G)$

B) $Q(G) \supset R(G) \supset T(G) \supset M(G)$

If $G$ is nondiscrete and noncompact, then all inclusions stated above are proper.

**Proof.** The inclusions constituting system A) are more or less obvious. Their verification is left to the reader. Since all inclusions in A) are in fact continuous, dense imbeddings the inclusions of system B) follow by duality.

It is obvious that the inclusions $K(G) \subset W(G)$ and $D(G) \subset S^0(G)$ are proper if $G$ is noncompact. Since $S^0(G) \subset W(G) \subset L^1(G)$, the properness
of the inclusions $W(G) \subseteq C^0(G)$ and $S_0(G) \subseteq A(G)$ follows from the fact that $A(G) \cap L^1(G) = A(G)$ and $C^0(G) \cap L^1(G) = C^0(G)$ for noncompact groups. Since there exists $f \not\in K(G)$, $f \not\in A(G)$ for nondiscrete groups it is clear that the remaining inclusions of part A) are proper.

That the inclusions in part B) are all proper follow therefrom by duality.

We continue this section with a characterization of the elements of $S_0(G)'$ among all quasimeasures.

**Proposition 3.3**

Let $2 \leq p < \infty$, $\sigma \in Q(G)$ be given. Then $\sigma$ extends to a bounded, linear functional on $S_0(G)$ iff for some open, relatively compact set $Q \subseteq G$ there exists $c > 0$ such that

$$\left| \sigma(I_y f * g) \right| \leq c \| f \|_p \| g \|_p$$

for all $y \in G$, $f, g \in K(G)$, with $\text{supp } f \cup \text{supp } g \subseteq Q$.

**Proof.** Using the characterization of $S_0(G)$ given in Corollary 4 of [7], together with assumption (*) one sees that $|\sigma(h)| \leq C_1 \| h \|_{S_0}$ for all $h \in S_0(G)$. Since $D(G)$ is a dense subspace of $S_0(G)$, $\sigma$ extends to a bounded linear functional on $S_0(G)$, and the proof is complete.

**Remark.** The above characterization is equivalent to the following condition:

$\sigma \in Q(G)$ belongs to $S_0(G)'$ iff $\sigma * h \in C^b(G)$ for all $h \in D(G)$.

It is thus closely related to the characterization of the elements of $T(G)$ among all Radon measures given in [1], Theorem 1.1.

**Proposition 3.4**

Let $\sigma \in Q(G)$ be given, such that $\sigma(f) \geq 0$ for all $f \in D(G)$ with $f \geq 0$. Then $\sigma$ belongs to $R(G)$. In a similar way 'positivity' of $\sigma \in S_0(G)'$ [P(G)] implies $\sigma \in T(G)$ [P(G)].
Proof. We give the proof for $\sigma \in S_0(G)'$. The proof for $\sigma \in \mathcal{Q}(G)$ or $\sigma \in \mathcal{P}(G)$ is very similar. In view of the duality $T(G) = W(G)'$ and by the definition of $W(G)$ (cf. [47]) it will be sufficient to show that there exists $C > 0$ such that

$$|\sigma(L_y f)| \leq C\|f\|_{\infty} \text{ for all } f \in \mathcal{A}_Q(G), \ y \in G.$$  

Let therefore $f \in \mathcal{A}_Q(G)$ be given. Since $\mathcal{A}(G)$ has bounded approximate units it has the factorization property (cf. [10], 32.22). Therefore there exist $f_1, f_2 \in \mathcal{A}(G)$, such that $f = f_1 \overline{f_2}$, with

$$\|f_1\|_{\mathcal{A}(G)} \cdot \|f_2\|_{\mathcal{A}(G)} \leq 2\|f\|_{\mathcal{A}(G)}.$$  

We may suppose that

$$\|f_1\|_{\mathcal{A}(G)} = \|f_2\|_{\mathcal{A}(G)} = (2\|f\|_{\mathcal{A}(G)})^{1/2}.$$  

Let $f \in \mathcal{A}_Q(G)$ be given. For some neighborhood $V$ of the identity we may choose $h \in \mathcal{A}(G)$ such that $h(x) \equiv 1$ for all $x \in Q$ and supp $h \subseteq VQ = : Q'$.

We set

$$f = f \overline{h} = \frac{f}{\|f\|^{1/2}_{\mathcal{A}(G)}} \frac{h^{1/2}}{\|h\|^{1/2}_{\mathcal{A}(G)}} = : \|h\|^{1/2}_{\mathcal{A}(G)}g_1 \overline{g_2},$$

The identity

$$g_1 \overline{g_2} = \frac{1}{4} \sum_{k=0}^{3} \frac{2}{k!} |g_1 + ig_2|^2$$

implies $f = \frac{\|h\|_{\mathcal{A}(G)}}{4} \sum_{k=0}^{3} \frac{2}{k!} f_{\overline{k}}$, with $f_{\overline{k}} = |g_1 + ig_2|^2 \geq 0$ and

$$\|f_{\overline{k}}\|_{\mathcal{A}(G)} \leq (\|g_1\|_{\mathcal{A}(G)} + \|g_2\|_{\mathcal{A}(G)})^2 \leq 4\|f\|_{\mathcal{A}(G)}.$$  

In view of this decomposition it will be sufficient to prove

$$\text{(**) for } f \geq 0, \ f \in \mathcal{A}_Q(G).$$

This assertion is easily established, since $0 \leq f(x) \leq \|f\|_{\infty} h(x)$ for all $y \in G$ implies
\[ 0 \leq \sigma(I_y f) \leq \|f\|_\infty \quad \sigma(I_y h) \leq (\|\sigma\|_{S_0(G)}, \|L_y h\|_{S_0}) \|f\|_\infty \text{ for all } y \in G. \]

Remark. By the above Proposition the span of the 'positive' elements of \( S_0(G) \) coincides with \( T(G) \).

We now come to applications of \( S_0(G) \)' in the theory of multipliers.

It is shown that \( S_0(G)' \) can be identified with various spaces of multipliers.

**Theorem 3.5**

\[ (W(G), T(G)) = (W^2(G), T(G)) = (S_0(G), S_0(G)') = (S_0(G), L^\infty(G)) = (L^1(G), S_0(G)') = (S_0(G), c^b(G)) \cong S_0(G)'. \]

**Proof.** By Theorem 3 \( \alpha \) [7] we have

\[ S_0 = S_0 \otimes S_0 = W \otimes W = W^2 \otimes W. \]

Since any Segal algebra has approximate units that are bounded in \( L^1(G) \) it is obvious that \( S_0 = L^1 \otimes S_0 = S_0 \otimes L^1 \). An appropriate modification of the proof of Theorem 3.3 of [16] reveals that

\( B_1 \otimes B_2 \) (as defined in the above context) can be identified with the \( L^1(G) \)-module tensor product \( B_1 \otimes_{L^1} B_2 \) of \( B_1 \) and \( B_2 \) if \( B_1 \) is a Segal algebra on \( G \) and \( B_2 \) is a homogeneous Banach space on \( G \). Since the multipliers from \( B_1 \) to \( B_2 \) are exactly the \( L^1(G) \)-module homomorphisms in this situation the result (except the last but one equation) follows from the identity

\[ \text{Hom}_{L^1} (B_1, B_2') = (B_1 \otimes_{L^1} B_2)' \]  

(cf. [15], 2.12 and 2.13).

That \( (S_0(G), L^\infty(G)) = (S_0(G), c^b(G)) \) is proved as follows. By the factorization theorem ([10], 32.22) any \( f \in S_0(G) \) can be written as \( f = g \ast f_1, \quad g \in L^1(G), \quad f_1 \in S_0(G) \). This implies for \( f \in S_0(G) \) and \( T \in (S_0, L^\infty) \):

\[ Tf = T(g \ast f_1) = g \ast Tf_1 \in L^1(G) \ast L^\infty(G) = c^b(G), \]

and the proof is complete.
Theorem 3.6

Let $B_1, B_2$ be two strongly character invariant, homogeneous Banach spaces on $G$. If $B_1 \cap K(G)$ is dense in $B_1$, then

$$(B_1, B_2) \in (S_o(G), C^b(G)) = S_o(G)' .$$

Proof. It is obvious that $(L^1(G) \cap B_1, \| \cdot \|_1 + \| \cdot \|_{B_1})$ is a strongly character invariant, homogeneous Banach space contained in $L^1(G)$. By means of Wiener's theorem ([14], Chap. 6, § 1) the character invariance implies that $L^1(G) \cap B_1$ is dense in $L^1(G)$, i.e. $L^1(G) \cap B_1$ is a Segal algebra. This $S_o \subseteq L^1(G) \cap B_1 \subseteq B_1$ by Theorem 1 of [7]. On the other hand, any homogeneous Banach space is contained in $T(G)$ (compare [1], Theorem 1.2). Therefore any $T \epsilon (B_1, B_2)$ belongs to $(S_o, T(G))$, which coincides with $(S_o, C^b(G))$ and $S_o(G)'$ by Theorem 3.5. In order to prove that we have in fact an injection it will be sufficient to prove that $S_o(G)$ is dense in $B_1$. Let $f \epsilon B_1, \epsilon > 0$ be given. Then there exists $k \epsilon K(G) \cap B_1$ with $\| k - f \|_{B_1} < \epsilon/2$. Now there exists $u \epsilon S_o(G)$ such that

$\| u * k - f \|_{B_1} < \epsilon/2$. Thus $\|u * k - f\|_{B_1} < \epsilon$, and $u * k \epsilon S_o(G) \ast L^1(G) = S_o(G)$. This completes the proof.

Remark. In the case of abelian groups Theorem 3.6 shows that one may replace $Q(G)$ by $S_o(G)$ for most results concerning multipliers (cf. [12], § 5.1, or [3], Theorem 5.1).
As we shall see immediately the Fourier transform defined above coincides with all classical extensions of the ordinary Fourier transform on their proper domain, such as $L^p(G)$, $1 \leq p \leq 2$ (Hausdorff-Young), $B(G)$ (Fourier Stieltjes algebra), or $P(G)$. Besides, any $\mu \in T(G) \subseteq S_0(G)$' has a Fourier transform in this sense, and consequently any $f \in L^p(G)$, $1 \leq p \leq \infty$ (cf. [12], § 5.4), as well as various spaces of multipliers (cf. Theorem 3.8 above). We do not check these assertion separately. Instead, we show that our notion of Fourier transform is in fact an extension of the notion given in [1].

**Definition (cf. [1], p. 8):** A measure $\mu \in R(G)$ is called "transformable", if there exists some $\hat{\mu} \in R(\hat{G})$ such that

$$\int_G k^\ast k^\circ(x) d\mu(x) = \int |\hat{k}(y^{-1})|^2 d\hat{\mu}(y) \text{ for all } k \in K(G).$$

The uniquely determined measure $\hat{\mu}$ is called "Fourier transform" of $\mu$ (Recall that this definition involves the condition $\hat{k} \in L^2(\hat{G})$ for all $k \in K(G)$).

We now come to another application of $S_0(G)$' within harmonic analysis. The fact that the Fourier transform maps $S_0(G)$ onto $S_0(\hat{G})$ ([7], Theorem 5) allows the definition of the Fourier transform of elements of $S_0(G)$'. The extended notion of Fourier transform obtained in this way includes most of the earlier extensions of the ordinary Fourier transform.

**Definition:** The Fourier transform $\hat{\sigma} = F\sigma$ of an element $\sigma \in S_0(G)$' is given by means of

$$\langle F\sigma, f \rangle = \langle \sigma, Ff \rangle \text{ for all } f \in S_0(\hat{G}).$$

**Theorem 3.7** The application $F: \sigma \rightarrow \hat{\sigma}$ defined above establishes an (isometric) isomorphism from $S_0(G)$' onto $S_0(\hat{G})'$. The inverse application $F^{-1}$ is given by $\langle F^{-1}\sigma, f \rangle = \langle \sigma, F^{-1}(f) \rangle$ for all $f \in S_0(G)$.
Proof. That $F$ is well defined and has the required properties follows immediately from Theorem 5.1). Since we may suppose that $F$ is an isometry from $S_o(G)$ onto $S_o(\hat{G})$ (use the norm $\|f\|_{S_o} = \|f\|_{S_o} + \|\hat{f}\|_{S_o}$) it is an isometry from $S_o(G)'$ onto $S_o(G)'$ as well. Since the inversion theorem is applicable to any $f \in S_o(G)$ it is obvious that $F^{-1}$ is given in the above way.

Remark: It is obvious that $F$ has the usual properties with respect to the elementary operations such as conjugation, translation, and multiplication with characters.

As an immediate consequence of Theorem 3.7 we obtain a decisive extension of Theorem 5.4.2 of [12]:

**Theorem 3.8** Let $B_1, B_2$ be two strongly character invariant, homogeneous Banach spaces on $G$ such that $B_1 \cap K(G)$ is dense in $B_1$.

Then any multiplier $T \in (B_1, B_2)$ has a Fourier transform $\hat{T} \in S_o(\hat{G})'$ such that $F(Tf) = \hat{T}f$ for all $f \in S_o(G)$.

**Proof.** Combine Theorems 3.6 and 3.7.

We shall now prove that the space $\tau_T(G)$ of all transformable measures coincides with a certain subspace of $S_o(G)'$.

**Theorem 3.9**

$$\tau_T(G) = \{\mu | \mu \in R(G) \cap S_o(G)', \hat{\mu} \in T(G)\}$$

(Here $\hat{\mu}$ has to be taken in the sense of $S_o(G)'$).

**Proof.** Let $\mu \in \tau_T(G)$ be given. Then the uniquely determined measure $\hat{\mu}$ (in the sense of [17]) satisfies (we write $F^{-1}f = \hat{f}$)

$$\langle \hat{\mu}, |k|^2 \rangle = \langle \mu, k \ast k \ast \rangle$$

for all $k \in K(G)$.

must be translation bounded ([17], Theorem 2.5). Applying the inverse Fourier transform (in the sense of Theorem 3.7 above) we obtain some $\sigma = F^{-1}(\hat{\mu}) \in S_o(G)'$. As a consequence of the identity
the space $K_2(G)$ (§11, p. 8) coincides with the linear span of $K(G) \ast K(G)$, which is dense in $S_o(G)$ (see §7, Corollary 4). The equality

$$\langle u, f \rangle = \langle \hat{u}, \hat{f} \rangle$$

for all $f \in K_2(G)$

thus implies $u = \pi$, and the two notions of Fourier transform are the same. We have now proved $\hat{u} \in T(\hat{G}) \subset S_o(\hat{G})'$, and therefore $u \in R(G) \cap S_o(G)'$.

Conversely, suppose that $u \in R(G) \cap S_o(G)'$ is given such that $\hat{u}$ belongs to $T(G)$. Since $|k|^2 \in S_o(\hat{G}) \subset W(\hat{G})$ for any $k \in K(G)$, we have $|\hat{k}|^2 \in L^1(\hat{G})$ for any $\hat{u} \in T(\hat{G})$. Thus

$$\int_G k \ast k^*(y) \, du(y) = \langle u, k \ast k^* \rangle = \langle \hat{u}, |\hat{k}|^2 \rangle = \int_{\hat{G}} |\hat{k}|^2 \, d\hat{u}(y)$$

for all $k \in K(G)$, i.e. $u \in \mathbb{M}_T(G)$.

Remark: Since a transformable measure need not be translation bounded (cf. Richards's example, §11, Prop. 7.1), and since therefore the transform $\hat{u}$ of $u \in \mathbb{M}_T(G)$ need not be transformable itself the asymmetry of the notion of transformability given in §11 is apparent. In view of the above characterization there are two natural candidates of spaces of measures which are mapped onto the corresponding space of $\hat{G}$ by the Fourier transform:

$$T_1(G) := \{ u \in T(G), \hat{u} \in T(\hat{G}) \}$$

$$R_1(G) := \{ u \in R(G) \cap S_o(G)', \hat{u} \in R(\hat{G}) \}$$

In both cases the Fourier transform is to be taken in the sense of $S_o(G)$! Therefore $\hat{u} \in S_o(\hat{G})'$ for $u \in R_1(G)$ and the symmetry is perfect in this case, too. There is the following connection between $T_1(G)$ and spaces considered in §11:
Proposition 3.10 In the terminology of [1] we have:

$$T_1(G) = S(G) = \mathfrak{m}_T(G) \cap \mathfrak{m}_B(G).$$

Proof. Since $\mathfrak{m}_B(G) = T(G) \subset S_0(G)$, it follows directly from Theorem 3.9 that $T_1(G)$ coincides with $\mathfrak{g}(G) = \{ u \in \mathfrak{m}_T(G), \hat{u} \in \mathfrak{m}_T(G) \}$ as well as with $\mathfrak{m}_T(G) \cap \mathfrak{m}_B(G)$.

The space $T_1(G)$ is certainly of interest in harmonic analysis because it is a translation invariant Banach space (with the norm $\| u \|_{T_1} := \| u \|_T(G) + \| \hat{u} \|_T(\hat{G})$) which already includes $L^p(G), 1 \leq p \leq 2$, $M(G)$, and $B(G)$ (cf. [17, Theorem 3.5]). If one is interested in positive definite measures (which are "transformable" by Theorem 4.1 of [17], but which need not be translation-bounded (see [17, Proposition 7.1])) one is lead to consider the space $R_1(G)$ on which the inversion theorem (in the sense of $S_0(G)$) is valid.

If one is interested in defining the Fourier transform on $R_1(G)$ without direct reference to the duality $\langle S_0(G)', S_0(G) \rangle$ one may proceed as follows:

Proposition 3.11 Let $(u_\alpha)_{\alpha \in I}$ and $(v_\beta)_{\beta \in J}$ be bounded approximate units for $A(G)$ and $\hat{A}(\hat{G})$ respectively, and suppose further that every $u_\alpha, \alpha \in I$ and $v_\beta, \beta \in J$ has compact support. Then we have for any $u \in R_1(G)$: $\hat{u}$ is the unique measure on $\hat{G}$ satisfying

$$\lim_{\alpha} \int_{\hat{G}} u_\alpha(x) \hat{f}(x) \, d\hat{u}(x) = \lim_{\beta} \int_{\hat{G}} v_\beta(y) \hat{f}(y) \, d\hat{u}(y) \text{ for all } f \in S_0(\hat{G}).$$

Proof. Since $K(G) \cap S_0(G)$ is a dense subspace of $S_0(G)$ any bounded approximate identity $(u_\alpha)_{\alpha \in I}$ for $A(G)$ also satisfies

$$\lim_{\alpha} \| u_\alpha f - f \|_{S_0} = 0 \text{ for all } f \in S_0(G).$$

We thus have

$$\lim_{\alpha} \langle u_\hat{\alpha}, u \rangle = \langle \hat{u}, f \rangle = \lim_{\beta} \langle \hat{u}, v_\beta \hat{f} \rangle \text{ for all } f \in S_0(\hat{G}).$$
Since, by the assumptions, the continuous functions \( u_\alpha \hat{f} \) and \( v_\gamma \hat{f} \) have compact supports for all \( \alpha \in I, \beta \in J \), one obtains the required result by rewriting \( \langle u, u_\alpha \hat{f} \rangle \) and \( \langle \hat{u}, v_\gamma \hat{f} \rangle \) as integrals.

Remark. Of course, it would have been sufficient to assume that the above relation holds true for a total subset of \( S_0(G) \), for example, it follows from the above calculations:

A measure \( \mu \) belongs to \( R_1(G) \) if and only if for some open, relatively compact set \( Q \subset G \) there exists \( C > 0 \) such that

\[
|u(I_y k_\# k^\#)| \leq C \|\hat{f}\|_\infty^2
\]

for all \( k \in C_0(G) \)

and there exists further some \( \hat{\mu} \in \mathcal{R}(\hat{G}) \) such that

\[
\int_G k_\# k^\#(x) d\hat{\mu}(x) = \lim_\gamma \int_G v_\gamma(y) |\hat{k}(y^{-1})|^2 d\hat{\mu}(y)
\]

for all \( k \in \mathcal{K}(G) \).
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