

Weighted L^p -Spaces and the Canonical Mapping $T_H: L^1(G) \rightarrow L^1(G/H)$.

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Sunto. - Dato un sottogruppo normale chiuso H di un gruppo localmente compatto G , T_H denota l'applicazione lineare canonica da $L^1(G)$ su $L^1(G/H)$. Utilizzando l'interpolazione complessa si dimostra che sotto alcune condizioni sulla funzione peso w , T_H applica uno spazio L^p con peso, $L^p_w(G)$, su $L^p_{w\langle p \rangle}(G/H)$ per $1 \leq p \leq \infty$. Inoltre si deducono risultati analoghi per gli spazi $A(L^1, L^p, l^a_{a,b})$ e $F(L^p, l^a_{a,b})$ per $G = \mathbf{R}^m$, $H = \mathbf{R}^k$, $k < m$.

1. - Introduction.

Let a closed, normal subgroup H of a locally compact group G be given. Denote the Haar measures on G , H , and G/H by dx , $d\xi$, and $d\dot{x}$ respectively, and write π_H for the canonical homomorphism from G onto G/H . If $d\dot{x}$ is appropriately normalized the linear mapping T_H , given by

$$T_H f(\dot{x}) = \int_H f(x\xi) d\xi, \quad f \in K(G), \quad \dot{x} = \pi_H(x),$$

extends to a linear contraction from $L^1(G)$ onto $L^1(G/H)$ (cf. [8], III, 4.4). Furthermore, it is known that T_H maps a Beurling algebra $L^1_w(G)$ onto another Beurling algebra $L^1_{w\langle p \rangle}(G/H)$ (see [8], III, 7.4). In the present paper it is proved by means of complex interpolation that under natural restrictions on w T_H maps a weighted L^p -space $L^p_w(G)$ onto another weighted L^p -space $L^p_{w\langle p \rangle}(G/H)$, for $1 < p < \infty$. Besides, for arbitrary weight functions w $T_H(L^1 \cap L^p_w)$, $1 < p < \infty$, and $T_H(L^1 \cap C^a_w(G))$ are characterized. In the second part of this note concrete examples on $G = \mathbf{R}^m$, with $H = \mathbf{R}^k$, $k < m$ are considered. By means of real interpolation it is possible to extend these characterizations to the spaces $A(L^1, L^p, l^a_{a,b})$ and $F(L^p, l^a_{a,b})$ that have been treated in earlier papers by the author ([3], [4]).

Notations concerning harmonic analysis are taken from Reiter's book ([8]). For the sake of shortness we take up the convention that a reference of the form III, 4.5 refers to Chap. 3, § 4.5 of [8].

In particular, for $y \in G$ the translation operators L_y and A_y are defined by

$$L_y f(x) := f(y^{-1}x), \quad A_y f(x) = f(xy) \Delta_a(y),$$

Δ_a being the Haar module on G .

$K(G)$ denotes the space of continuous functions with compact support (supp), $C^0(G)$ the space of continuous functions vanishing at infinity, endowed with the sup-norm. The Lebesgue spaces with respect to the Haar measure are denoted by $(L^p(G), \| \cdot \|_p)$ or simply L^p . The numbers p and p' will always be related by $1/p + 1/p' = 1$. In the sequel w denotes a continuous weight function (cf. III, 7.1), i.e. a continuous function on G satisfying

$$w(x) \geq 1, \quad \text{and} \quad w(xy) \leq w(x)w(y) \quad \text{for all } x, y \in G.$$

For $B = L^p(G)$, $1 < p < \infty$, or $B = C^0(G)$ B_w is to denote the space $\{f | fw \in B\}$. Endowed with the norm $\|f\|_{B,w} := \|fw\|_B$ B_w is a translation invariant Banach space. It is easily verified that $B_{w_1} = B_{w_2}$ if and only if $w_1 \sim w_2$, i.e. iff

$$\exists C > 0 \quad \text{such that} \quad C^{-1}w_1(x) \leq w_2(x) \leq Cw_1(x) \quad \text{for all } x \in G.$$

Thus there is no loss of generality in assuming that w is continuous, since any weight function as defined in III, 7.1 is equivalent to a continuous one (see [9], Proposition III, 1.3).

An equation of the form $T_H B = \dot{B} (T_H B \cong \dot{B})$ has to be interpreted as a short formula for the following statement: T_H is a linear mapping from B onto \dot{B} , and the image $T_H B = B/\text{Ker } T_H$, endowed with the quotient norm is (isometrically) isomorphic to \dot{B} as a Banach space.

2. - Identifying $T_H L_w^p(G)$.

In this section the image under T_H of the spaces $L_w^p(G)$ and $L^1 \cap L_w^p(G)$ respectively is characterized for $1 < p < \infty$. The general result will be derived from the special cases $p = 1$ and $p = \infty$ by means of complex interpolation.

Although it is not necessary to restrict the consideration to subspaces of $L^1(G)$ it is nevertheless necessary to impose certain conditions on w because for $p > 1$ T_H cannot be applied to $L_w^p(G)$

for an arbitrary weight function (e.g. $w \equiv 1$). The following condition on w is to be imposed occasionally:

$$(W_p) \quad w^{-1}|_H \in L^{p'}(H).$$

If (W_p) is satisfied, $T_H f$ is well defined for all $f \in L_w^p(G)$. In fact, by III, 4.5 we have for any $f \in L_w^p(G)$: there exists a set $A_0 \subseteq G/H$ of measure zero such that

$$L_{w^{-1}}(fw)|_H \in L^p(H) \quad \text{for all } x \in \pi_H^{-1}(G \setminus A_0).$$

On the other hand the submultiplicativity of w implies

$$L_{w^{-1}}w^{-1}|_H \in L^{p'}(H) \quad \text{for all } x \in G$$

Hölder's inequality gives now

$$L_{w^{-1}}f|_H \in L^1(H) \quad \text{for all } x \in \pi_H^{-1}(G \setminus A_0),$$

and therefore the integral defining $T_H f(\dot{x})$ exists almost everywhere on G/H .

We begin by recalling the known result for $p = 1$ (III, 7.4):

PROPOSITION 2.1. - $T_H L_w^1(G) \cong L_w^1(G/H)$, with

$$\dot{w}(\dot{x}) := \inf \{w(x\xi), \xi \in H\}, \quad \dot{x} = \pi_H(x).$$

For the case $p = \infty$ the following result is available:

PROPOSITION 2.2. - Let G be countable at infinity, and suppose that w satisfies (W_∞) . Then

$$T_H L_w^\infty(G) \cong L_{\dot{w}}^\infty(G/H), \quad \text{with } \dot{w} := (T_H w^{-1})^{-1}.$$

PROOF. - As explained above (W_∞) implies that T_H is applicable to $L_w^\infty(G)$, in particular, \dot{w} is well defined. Let now $f \in L_w^\infty(G)$ be given. Then $f = hw^{-1}$ for some $h \in L^\infty(G)$, and $\|f\|_{\infty, w} = \|h\|_\infty$.

Therefore

$$T_H f(\dot{x}) = \int_H h(x\xi) w^{-1}(x\xi) d\xi \leq \dot{w}^{-1}(\dot{x}) \text{ess sup}_\xi |h(x\xi)| \quad \text{for } \dot{x} = \pi_H(x).$$

Because G is countable at infinity we may apply III, 4.9:

$$\begin{aligned} \|T_H f\|_{\infty, w} &= \text{ess sup}_{x \in G/H} |T_H(x) \bar{w}(x)| \\ &\leq \text{ess sup}_{x \in G/H} (\text{ess sup}_{\xi \in H} |h(x\xi)|) \\ &= \text{ess sup}_{x \in G} |h(x)| = \|h\|_{\infty} = \|f\|_{\infty, w}. \end{aligned}$$

In order to show that T_H is surjective we define a right inverse $S_{H, w}$ to T_H by the formula

$$S_{H, w} f := [(f\bar{w}) \circ \pi_H] w^{-1}, \quad f \in L_w^{\infty}(G/H).$$

Since $f\bar{w}$ belongs to $L^{\infty}(G/H)$ III, 4.9 implies that $S_{H, w} f$ belongs to $L_w^{\infty}(G)$. Moreover $\|S_{H, w} f\|_{\infty, w} = \|f\|_{\infty, \bar{w}}$, and $T_H S_{H, w} f = f$. This implies that T_H is surjective and that the quotient norm coincides with $\|T_H f\|_{\infty, \bar{w}}$.

Our first main result can now be derived by interpolation.

THEOREM 2.3. — *Let G be countable at infinity and let w be a weight function on G satisfying (W_p) for some p , $1 < p < \infty$. Then*

$$T_H L_w^p(G) = L_{w\langle p \rangle}^p(G/H),$$

with

$$w\langle p \rangle(x) := \|L_{w^{-1}} w^{-1}\|_H^{-1/p}, \quad x = \pi_H(x),$$

$$\text{(i.e. } w\langle p \rangle = (T_H w^{-p'})^{-1/p'} \text{ for } 1 < p < \infty).$$

PROOF. — For $p = 1$ or $p = \infty$ the assertions are just those of Proposition 2.1 and 2.2 respectively, since any weight function satisfies (W_1) , and w coincides with $w\langle 1 \rangle$, and \bar{w} is the same as $w\langle \infty \rangle$ as defined now.

For $1 < p < \infty$ the result is obtained by means of complex interpolation. If we write v instead of $w^{p'}$ we have (cf. [2], 13.6 or [1], 5.8.1):

$$L_{w\langle p \rangle}^p = (L^1)^{1/p} (L_v^{\infty})^{1/p'} = (L^1, L_v^{\infty})_{(1/p, 1/p')}.$$

Hence T_H defines a contraction from L_w^p into

$$(L^1)^{1/p} (L_v^{\infty})^{1/p'} = L_{w\langle p \rangle}^p.$$

Again we have to prove that T_H is surjective. As we have seen in the proof of Proposition 2.2, $S_{H, v}$ is an isometry from L_v^{∞} into L_v^{∞} .

Besides, we have $S_{H,v}|f| = |S_{H,v}f|$ and thus

$$\|S_{H,v}f\|_1 = \|T_H S_{H,v}f\|_1 = \|f\|_1 \quad \text{for all } f \in L^1(G/H)$$

by Weil's formula (III, 4.5). Therefore, again by complex interpolation, $S_{H,v}$ maps $L_{w\langle p \rangle}^p$ into L_w^p and $\|S_{H,v}f\|_{p,w} \leq \|f\|_{p,w\langle p \rangle}$ for all $f \in L_{w\langle p \rangle}^p(G/H)$. The proof is now complete.

As a direct consequence of the proof we have the following result:

COROLLARY 2.4. - *Let $G, w, p, w\langle p \rangle$ be as in 2.3. Then*

$$T_H(L^1 \cap L_w^p(G)) \cong L^1 \cap L_{w\langle p \rangle}^p(G/H).$$

COROLLARY 2.5. - *Let G, w, \bar{w} be as in 2.2. Then*

$$T_H C_w^0(G) \cong C_{\bar{w}}^0(G/H).$$

PROOF. - Since T_H maps $K(G)$ onto $K(G/H)$ (see III, 4.2) it is clear from 2.2 that T_H defines a contraction from $C_w^0(G)$ into $C_{\bar{w}}^0(G/H)$, $C_{\bar{w}}^0(G/H)$ being the closure of $K(G/H)$ in $L_{\bar{w}}^\infty(G/H)$. In order to prove surjectivity we cannot use $S_{H,w}$ anymore because it does not map $K(G/H)$ into $C_w^0(G)$. We proceed as follows. The argument used in the proof of the open mapping theorem shows that it is enough to prove the following result:

for any $f \in C_w^0(G/H)$ and $\varepsilon > 0$, there exists $f \in C_w^0(G)$ such that

$$\|T_H f - f\|_{\infty, \bar{w}} < \varepsilon, \quad \text{and} \quad \|f\|_{\infty, w} < \|f\|_{\infty, \bar{w}}.$$

This condition is satisfied by a function f of the form $h(S_{H,w}f)$, where $h \in C^0(G)^+$, $\|h\|_\infty = 1$, and h is identically one on a sufficiently large compact set.

For $1 < p < \infty$ the result of Theorem 2.3 can be extended to arbitrary locally compact groups (cf. III, 7.8, 7.9 for this problem). After having proved that the kernel of T_H in $L_w^p(G)$ coincides the closed linear span of $\{A_\eta k - k, k \in K(G), \eta \in H\}$ in $L_w^p(G)$ (cf. III, 6.4) one can apply the machinery used in the case of Beurling algebras (see III, 7.8, 7.9). Since no essential new idea is required we state the result without detailed proof.

THEOREM 2.6. - *For $1 < p < \infty$ the assertions of Theorem 2.3 are true for an arbitrary locally compact group G .*

We conclude this section with a companion of Corollary 2.4.

Since T_H is defined on $L^1 \cap L_w^p(G)$ without any restriction on w one may ask what happens if (W_p) is not satisfied. The answer to this question is contained in the following results.

PROPOSITION 2.7. - $T_H(L^1 \cap C_w^0(G)) \cong L^1(G/H)$ if $w^{-1}|_H \notin L^1(H)$.

PROOF. - It will be sufficient to show that given $\check{k} \in K(G/H)$, $\varepsilon > 0$ there exists $k \in K(G)$ such that $T_H k = \check{k}$, $\|k\|_1 = \|\check{k}\|_1$, and $\|k\|_{\infty, w} \leq \varepsilon$. We begin by proving the following assertion:

given a compact set $K \subseteq G$, $\delta > 0$ there exists $h \in K^+(G)$ satisfying $\|h\|_\infty = 1$, and $\int_H h(x\xi)w^{-1}(x\xi) d\xi \geq \delta^{-1}$ for all $x \in K$.

PROOF. - Since a weight function is bounded on compact sets there exists $C_K > 0$ such that

$$C_K^{-1}w(x\xi) \leq w(x_0\xi) \leq C_K w(x\xi) \quad \text{for all } x, x_0 \in K, \xi \in H.$$

Moreover, the assumption implies $L_x w^{-1}|_H \notin L^1(H)$ for all $x \in G$. If we fix any $x_0 \in K$ there exists a compact set $F \subseteq H$ such that

$$\int_F w^{-1}(x_0\xi) d\xi \geq C_K \delta^{-1}.$$

Hence

$$\int_F w^{-1}(x\xi) d\xi \geq \delta^{-1} \quad \text{for all } x \in K.$$

Choose now $h \in K^+(G)$ such that $\|h\|_\infty = 1$ and $h \equiv 1$ on KF . Then h has the required properties.

Let us now return to the proof of the Proposition. Let $\check{k} \in K(G/H)$, $\varepsilon > 0$ be given with $\text{supp } \check{k} = \check{K}$. According to III, 1.8. iii) there exists a compact set $K \subseteq G$ with $\pi_H(K) = \check{K}$. For this set $K \subseteq G$ and some $\delta < \varepsilon \|\check{k}\|_\infty^{-1}$ we set

$$S_{H,\delta} \check{k} := (\check{k} \circ \pi_H) h w^{-1} / T_H(h w^{-1}) \circ \pi_H,$$

h being as in the above assertion. Since $T_H(h w^{-1})(\dot{x}) > 0$ for all $\dot{x} \in \check{K}$ we have $T_H S_{H,\delta} \check{k} = \check{k}$, and also $\|S_{H,\delta} \check{k}\|_1 = \|\check{k}\|_1$ by Weil's formula. Moreover, $\|S_{H,\delta} \check{k}\|_{\infty, w} \leq \|\check{k}\|_\infty \delta < \varepsilon$ by the choice of h . Hence we may take $k = S_{H,\delta} \check{k}$, and the proof is complete.

THEOREM 2.8. - Let $1 < p < \infty$ and w be given, such that w does not satisfy (W_p) . Then

$$T_H(L^1 \cap L_w^p(G)) \cong L^1(G/H).$$

PROOF. — The case $p = \infty$ is a trivial consequence of Proposition 2.7. Let now $1 < p < \infty$ be given. Again it will be sufficient to show that for $\dot{k} \in K(G/H)$, $\varepsilon > 0$ there exists $k \in K(G)$ such that $T_H k = \dot{k}$, $\|\dot{k}\|_1 = \|k\|_1$, and $\|k\|_{p,w} < \varepsilon$. To this end observe that $S_{H,\delta}$ (with w replaced by $w^p = v$) can be considered as a bounded linear operator from the closed subspace $L_{\dot{k}}^\infty(G/H) = \{f | f \in L^\infty(G/H), f \text{ vanishes outside } \dot{K}\}$ of $L^\infty(G/H)$ into $L_v^\infty(G)$ of norm less than δ . Since $\|S_{H,\delta} f\|_1 = \|f\|_1$ for $f \in L_{\dot{k}}^1(G/H)$ we obtain by the use of complex interpolation: $S_{H,\delta}$ defines a bounded linear operator from $L_{\dot{k}}^p(G/H)$ into $L_v^p(G)$ satisfying

$$\|S_{H,\delta} \dot{k}\|_{p,w} < \delta^{1/p} \|\dot{k}\|_p \quad \text{for all } \dot{k} \in K(G/H) \text{ with } \text{supp } \dot{k} \subseteq \dot{K}.$$

For $\delta < \varepsilon^p \|\dot{k}\|_p^{-1}$ the choice $k := S_{H,\delta} \dot{k}$ completes the proof.

3. — Applications to spaces on \mathbf{R}^m and \mathbf{Z}^m .

In this section the results of section 2 are applied to weighted L^p -spaces on $G = \mathbf{R}^m$ or \mathbf{Z}^m . Furthermore, they are extended to the most typical examples of the spaces introduced in [3] and [4], including the so called Beurling-Herz spaces. We shall only state the results for $H = \mathbf{R}^k$, $G = \mathbf{R}^{m-k} \times \mathbf{R}^k$, $k < m$. The same results are of course true for \mathbf{Z}^k and \mathbf{Z}^m .

In order to simplify the statements of our results we use the following notation:

$$L_{a,b}^p(\mathbf{R}^m) := L_{w_{a,b}}(\mathbf{R}^m),$$

$$w_{a,b}(x) := (1 + |x|)^a \log^b(1 + |x|), \quad \text{with} \quad |x| := \left(\sum_{i=1}^m x_i^2 \right)^{\frac{1}{2}}, \quad x \in \mathbf{R}^m.$$

Note that we obtain an equivalent weight function generating the same spaces B_w if we replace the Euclidean norm by any other norm on \mathbf{R}^m .

For the definition of the spaces $\mathcal{A}(A, B, X)$ and $\mathcal{F}(B, X)$ the reader is referred to [3] and [4]. We assume throughout that these spaces are built up over the system $(B_n)_{n=0}^\infty$, with $B_n = \{x | |x| \leq 2^n\}$, $n \geq 1$, $B_0 = \emptyset$. We only mention here that $L^1 \cap \mathcal{F}(L^p, l_{a,b}^p) = \mathcal{A}(L^1, L^p, l_{a,b}^p)$ for $a > 0$, and that these space is a Banach algebras with respect to convolution ([4], Theorem 5.1). Besides, $L_{a,b}^p$ coincides with $\mathcal{F}(L^p, l_{a,b}^p)$.

In the sequel the value $k(1 - 1/p) = k/p'$ will play a significant role. We write α for this « critical index ».

THEOREM 3.1. - Let $1 \leq p \leq \infty$, $k < m$ be given. Then

- i) $T_{\mathbf{R}^k}(L_{a,b}^p(\mathbf{R}^m)) = L_{a-\alpha,b}^p(\mathbf{R}^{m-k})$ and
 $T_{\mathbf{R}^k}(L^1 \cap L_{a,b}^p(\mathbf{R}^m)) = L^1 \cap L_{a-\alpha,b}^p(\mathbf{R}^{m-k})$ for $a > \alpha$;
- ii) $T_{\mathbf{R}^k}(L_{a,b}^p(\mathbf{R}^m)) = L_{0,b-1/p'}^p(\mathbf{R}^{m-k})$ for $a = \alpha$, $b > 1/p'$;
- iii) $T_{\mathbf{R}^k}(L^1 \cap L_{a,b}^p(\mathbf{R}^m)) \cong L^1(\mathbf{R}^{m-k})$ for $a < \alpha$, or $a = \alpha$, $b \leq 1/p'$.

These results remain true if L^∞ is replaced by C^0 .

PROOF. - Since $w_{a,b}$ satisfies (W_p) in the given situation iff the conditions stated in i) or ii) are satisfied, it is clear that i) has to be derived from 2.3-2.5, and iii) from 2.7 and 2.8. Since the general result follows from the case $k=1$ by induction the problem can be reduced to the proof of the following statement: for $G = \mathbf{R}^m$, $H = \mathbf{R}$ the relations

$$a) \overline{w_{a,b}} \sim w_{a-1,b}, \quad a > 1, b \in \mathbf{R}, \text{ and}$$

$$b) \overline{w_{1,b}} \sim w_{0,b-1}, \quad b > 1$$

are satisfied. In order to prove a) we replace $w_{a,b}$ by the equivalent weight function w_1 ,

$$w_1(x) := (1 + |y| + |x_m|)^a \log^b(1 + |y| + |x_m|), \quad x = (y, x_m) \in \mathbf{R}^m.$$

Then

$$\overline{w_1}^{-1}(y) \sim \int_0^{|y|^2} w_1^{-1}(y, x_m) dx_m + \int_{|y|^2}^{\infty} w_1^{-1}(y, x_m) dx_m.$$

Since there is some $C > 0$ such that

$$\log^b(1 + |y|) \leq \log^b(1 + |y| + |x_m|) \leq C \log^b(1 + |y|)$$

for all x_m with $|x_m| \leq |y|^2$ we may continue in the following way:

$$\begin{aligned} \overline{w_1}^{-1}(y) &\sim [\log^{-b}(1 + |y|)] \int_0^{|y|^2} (1 + |y| + |x_m|)^{-a} dx_m + \\ &+ O\left(\int_{|y|^2}^{\infty} (1 + |y| + |x_m|)^{-a+s} dx_m\right) \sim [\log^{-b}(1 + |y|)] (1 + |y|)^{-a+1} + \\ &+ O(1 + |y| + |y|^2)^{-a+s+1} \sim w_{a-1,b}^{-1}(y) \end{aligned}$$

for any ε , $0 < \varepsilon < (a-1)/2$. This implies *a*). Using essentially the same method *b*) can be derived, using partial integration.

The following result from real interpolation theory will be needed in the sequel.

PROPOSITION 3.2. - For $a_1, a_2 > 0$, $a = (1-\theta)a_1 + \theta a_2$, $0 < \theta < 1$, $1 < p, q < \infty$

$$F(L^p, l_{a,b}^q) = (L_{a_1,b}^p, L_{a_2,b}^p)_{\theta,q}.$$

PROOF. - Without real loss of generality we may assume $b = 0$. Using Theorem 3.7. i) of [6], with $r = 2$ we find that $F(L^p, l_a^q) = (L^p, L_{a_3}^p)_{\theta,q}$ with $\theta = a/a_3$ for any $a_3 > a$. If we choose $a_3 > \max(a_1, a)$ we have $L_{a_i}^p = (L^p, L_{a_3}^p)_{\theta_i,p}$ with $\theta_i = a_i/a_3 < 1$, $i = 1, 2$ by Theorem 5.4.1 of [1]. An application of the reiteration theorem ([1], 3.11.5) gives the result.

Our results up to this point are still insufficient in order to derive a characterization of $T_{\mathbf{R}}(F(L^p, l_{a,b}^q, \mathbf{R}^m))$. The problem to be solved arises from the fact that we cannot use the operators S_{H, w^p} anymore because they depend not only on p . Fortunately we can find a universal right inverse $S_{\mathbf{R}}$ for $T_{\mathbf{R}}: L^1(\mathbf{R}^m) \rightarrow L^1(\mathbf{R}^{m-1})$, suitable for our purposes. We write again $x = (y, x_m)$ for $x \in \mathbf{R}^m$ and set $\bar{x} := \max(1, |y|)^{-1}$. Then we define for $f \in L_{a,b}^p(\mathbf{R}^{m-1})$, $1 < p < \infty$,

$$S_{\mathbf{R}}f(y, x_m) := \bar{x}f(y)h(\bar{x}x_m),$$

h being any function in $K^+(\mathbf{R})$ with $\|h\|_1 = 1$. A right inverse $S_{\mathbf{R}}h$ for $T_{\mathbf{R}} = T_{\mathbf{R}} \circ \dots \circ T_{\mathbf{R}}$ can be defined inductively. The properties of $S_{\mathbf{R}^k}$, $k < m$ follow immediately from the corresponding results for $S_{\mathbf{R}}$.

LEMMA 3.3. - Let $a > 0$, $b \in \mathbf{R}$, $1 < p < \infty$ be given. Then the following is true for all $f \in L_{a,b}^p(\mathbf{R}^{m-1})$:

- i) $T_{\mathbf{R}}S_{\mathbf{R}}f = f$; $|S_{\mathbf{R}}f| = S_{\mathbf{R}}|f|$;
 - ii) $\|S_{\mathbf{R}}f\|_1 = \|f\|_1$ if $f \in L^1(\mathbf{R}^{m-1})$;
 - iii) $\text{supp } S_{\mathbf{R}}f \subseteq \{(y, x_m), y \in \text{supp } f, |x_m| < \max(1, |y|)\}$,
- and therefore $S_{\mathbf{R}}(K(\mathbf{R}^{m-1})) \subseteq K(\mathbf{R}^m)$;

- iv) there exists $C > 0$ such that $\|S_{\mathbf{R}}f\|_{p, a+1/p', b} \leq C\|f\|_{p, a, b}$.

PROOF. - i) - iii) are easily verified and iv) can be derived from the extreme cases $p = 1$ and $p = \infty$ by complex interpolation, or by direct calculation.

We are now ready to derive the second main result of this section. Recall that α denotes the critical index k/p' .

THEOREM 3.4. — *Let $1 \leq p, q < \infty$, $k, m \in \mathbf{N}$, $k < m$, $a > 0$, $b \in \mathbf{R}$ be given. Then the following relations hold true for $G = \mathbf{R}^m$ and $H = \mathbf{R}^k$:*

- i) $T_{\mathbf{R}^k}(F(L^p, l_{a,b}^a)) = F(L^p, l_{a-\alpha,b}^a)$ for $a > \alpha$;
- ii) $T_{\mathbf{R}^k}(\Lambda(L^1, L^p, l_{a,b}^a)) = \Lambda(L^1, L^p, l_{a-\alpha,b}^a)$ for $a > \alpha$;
- iii) $T_{\mathbf{R}^k}(F(L^1 \cap L^p, l_{b,a}^a)) = F(L^1, l_{a,b}^a) \cap F(L^p, l_{a-\alpha,b}^a)$ for $a > \alpha$;
- iv) $T_{\mathbf{R}^k}(\Lambda(L^1, L^p, l_{a,b}^a)) \cong L^1(\mathbf{R}^{m-k})$ for $a < \alpha$.

PROOF. — Relation i) follows from 3.1–3.3 and the principles of real interpolation theory, applied to the operators $T_{\mathbf{R}^k}$ and $S_{\mathbf{R}^k}$ (see [1], Theorem 3.1.2). Note that one may assume that the value α_1 in 3.2 may be chosen such that $\alpha < \alpha_1 < a$ holds. Since the operator $S_{\mathbf{R}^k}$ is independent of the spaces considered the image of the intersection of two spaces mentioned in this section is just the intersection of their images. Relations ii) and iii) are typical illustrations for this observation. Relation iv) follows from the inclusion $L^1 \cap L_\alpha^p \subseteq \Lambda(L^1, L^p, l_{a,b}^a) \subseteq L^1$.

4. — Concluding remarks.

1) With appropriate modifications the results of section 2 can be extended to arbitrary closed subgroups H of G . In this more general situation one considers the linear contraction $T_{H,q}$ from $L^1(G)$ onto the space $L^1(G/H)$ of all integrable functions on G/H with respect the quasi-invariant measure $d_q \hat{x}$. The arguments given in section 2 go through essentially unchanged if T_H is replaced by $T_{H,q}$ and w by $q^{1/p'} w$. In particular, (W_p) becomes:

$$q^{-1} w^{-p'}|_H \in L^1(H).$$

For details the reader is referred to Chapter 8 of [8], and Lemmata 1.1. and 1.2 of [7].

2) Another extension is possible concerning the function w . It is not necessary to suppose submultiplicativity of w . Instead, it would be sufficient to assume that w is a «moderate» function or equivalently, that $L_w^p(G)$ is a translation invariant space (see [5]).

3) As in the case of $L^1(G)$ (cf. III, 6.4) the kernel of T_H (or $T_{H,q}$) in the spaces B considered in this section can be characterized

as the closed linear span of the set $\{A_\eta k - k | \eta \in H, k \in K(G)\}$ if $K(G)$ is dense in B , e.g. if we have $1 < p, q < \infty$.

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