BANACH CONVOLUTION ALGEBRAS OF WIENER TYPE

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INTRODUCTION

The purpose of this note is to introduce the concept of Banach spaces of functions (or measures, distributions) on locally compact groups that are defined by means of the global behaviour of certain local properties of their elements (such as $p$-integrability, continuity, smoothness, etc.). Using a quite elementary approach we obtain a family of Banach spaces that contains as special cases a number of spaces that have been discussed by several authors for different purposes. Except a few classical forerunners, such as the Banach algebra used by N. Wiener in the proof of his famous Tauberian theorem for the Fourier Transform on the real line (see [36], [37]), most of these spaces have been treated in papers that have appeared within the last few years (see e.g. [2]-[4], [7], [8], [13]-[15], [18], [20], [22], [23], [29], [31]).

There is, of course, a number of results concerning these spaces in general, as well as for certain special cases of interest (e.g. [15]). These will be published in a series of papers of which this one may be considered as the first one. Here we shall state definitions and basic properties of these Banach spaces "of Wiener type", as well as sufficient conditions for
spaces of this type to be Banach algebras with respect to convolution. Therefore the paper may also be considered as a continuation of earlier works done by the author (cf. [10], [12]). As we shall show at the end of this paper one obtains as application results concerning functions (distributions) with compact spectrum on locally compact abelian groups. As special cases one has inequalities, usually associated with the names of Bernstein, Sobolev or Nikolskij (cf. [35]).

NOTATIONS

Throughout \( G \) will be a locally compact group with left Haar measure \( dx \). We shall mainly be interested in non-discrete, non-compact groups (e.g. \( G = \mathbb{R}^m \)). \( \mathcal{A}(G) \) denotes the space of all continuous, complex-valued functions on \( G \) with compact support (supp), endowed with its natural inductive limit topology. \( (L^p, \| \cdot \|_p) \ (1 \leq p \leq \infty) \), denotes the usual Lebesgue spaces. Given a subset \( M \subseteq G \) we write \( c_M \) for its characteristic function. The space \( L^1_{\text{loc}}(G) \) consists of all (classes of) measurable functions \( f \) on \( G \) such that \( fc_K \in L^1(G) \) for any compact subset \( K \subseteq G \). It is a topological vector space with the family of seminorms \( f \rightarrow \| fc_K \|_1 \). A BF-space on \( G \) is a Banach space \( (B, \| \cdot \|_B) \) which is continuously embedded into \( L^1_{\text{loc}}(G) \). As usual we shall speak of 'functions' in such spaces, identifying two measurable functions in \( B \), if they are equal locally almost everywhere (l.a.e.). A BF-space \( B \) is called solid if any measurable function \( g \), for which there exists \( f \in B \) such that \( |g(x)| \leq |f(x)| \) l.a.e. belongs to \( B \), with \( \|g\|_B \leq \|f\|_B \). A BF-space \( B \) is called left translation invariant (translation invariant) if the left (left and right) translation operators, given by

\[
L_y f(x) := f(y^{-1}x), \quad A_y f(x) := f(xy)
\]

act boundedly on \( B \). Their operator norm is written \( \| \cdot \|_B \).

Corresponding terminology is applied to spaces of measures or distributions, to which the translation operators are extended by transposition. A left invariant BF-space will be called a homogeneous Banach space on \( G \) if \( G \) acts (by left translations) isometrically on \( B \), and if translation is continuous in \( B \), i.e. if \( \lim_{y \rightarrow e} \|L_y f - f\|_B = 0 \) for all \( f \in B \). The
homogeneous Banach spaces which are dense in $L^1(G)$ are exactly the \textit{Segal algebras} in the sense of R e i t e r ([26], [27]). A triple $(B^1, B^2, B^3)$ will be called a \textit{Banach convolution triple} (BCT), if convolution, given by

$$f^1 \ast f^2(x) := \int_G f^1(y^{-1}x)f^2(y) \, dy$$

for $f^i \in \mathcal{X}(G) \cap B^i \quad (i = 1, 2)$,

extends to a bounded, bilinear map from $B^1 \times B^2$ into $B^3$. Clearly $(A, A, A)$ is a BCT for some $A \subseteq L^1(G)$ iff $A$ is a \textit{Banach convolution algebra} (BCA). A Banach space $B$ is a (left) \textit{Banach module} over the Banach algebra $A$ iff $(A, B, B)$ is a BCT, and a (left) \textit{Banach ideal} in $A$ if furthermore $B \subseteq A$. Any homogeneous Banach space is known to be a left $L^1(G)$ Banach convolution module. Finally we mention, that constants without importance will be denoted by $C, C_1, \ldots$.

**GENERAL HYPOTHESIS**

As a standing assumption we suppose throughout this paper that for any Banach space $B$ used below there exists a \textit{homogeneous Banach space} $(A, \| \cdot \|_A)$, continuously embedded into the Banach algebra $(C^b(G), \| \cdot \|_\infty)$, which is a \textit{regular Banach algebra} with respect to pointwise multiplication (i.e. separating points from closed sets), and which is \textit{closed under complex conjugation}, such that $(B, \| \cdot \|_B)$ is continuously embedded into $A'_0 := (A \cap \mathcal{X}(G))'$, as well as a Banach module over $A$ with respect to pointwise multiplication, i.e. $\|hf\|_B \leq \|h\|_A \|f\|_B$ for $h \in A, \ f \in B$.

Here we have used that $A \cap \mathcal{X}(G)$ is a topological vector space with respect to the usual inductive limit topology. It is also a (topological) module over $A$ with respect to pointwise multiplication. $A'_0$, the topological dual of $A_0$ is therefore again a module over $A$. Since the operation is the natural extension of pointwise multiplication on $A_0$, we denote the module operation on $B$, which is inherited from the corresponding action on $A'_0$, again as pointwise multiplication.

In this situation it makes sense to speak of elements of $A'_0$, belonging
locally to $B$: $B_{\text{loc}}$ will denote the space of all elements in $A'_0$ such that $h f \in B$ for all $h \in A_0$. $B_{\text{loc}}$ is a topological vector space with respect to the family of seminorms given by $f \mapsto \|f h\|_B$ ($h \in B$). It is not difficult to verify that $B_{\text{loc}}$ does not depend in the particular Banach algebra $A$ satisfying the above conditions.

In order to show that our assumptions are quite natural let us mention right away that any homogeneous Banach space $B$ which is a Banach module over $A$ with respect to 'ordinary' pointwise multiplication satisfies the above conditions. Below we shall give further examples of Banach spaces $B$ satisfying the general hypotheses, and which are well suited for the description of local properties of distributions:

**Example A.** The spaces $L^p(G)$ ($1 \leq p \leq \infty$) or $C^0(G)$ (of all continuous functions on $G$ which vanish at infinity, with the norm $\| \cdot \|_\infty$) are the simplest examples (take $A = C^0(G)$).

**Example B.** The Fourier algebra $A(G)$ in the sense of Eymard, or more general the Figa Talamanca – Herz algebras $A_p(G)$ (cf. [9]) satisfy the general hypotheses as well (take $A = B$).

**Example C.** If $G$ happens to be an abelian group, one may take $B$ to be $\mathcal{F} L^p(G)$ ($1 \leq p \leq \infty$), the image of $L^p(\hat{G})$ under the (extended) Fourier transform (here $G$ has been identified with $\hat{G}$). It is clear that $\mathcal{F} L^p(G)$ is a Banach module over $A(G)$ and that translation is isometric on $\mathcal{F} L^p$ (even continuous for $1 \leq p \leq \infty$). Thus it remains to check whether $\mathcal{F} L^p$ is contained in $(A(G) \cap \mathcal{X}(G))'$. But this is just the space of quasimeasures $Q(G)$ on $G$ (see [5]). The required assertion can therefore be found in the literature (cf. [17] or [21], Chap. 5). A short proof can be obtained using the invariance of $S'_0(G) \subseteq Q(G)$ under the Fourier transform, since $S'_0(G)$ contains any $L^p$-space (cf. [15]).

**Example D.** Various kinds of generalized Lipschitz spaces and potential spaces satisfy the general hypotheses. Let us consider for simplicity the case $G = \mathbb{R}^m$ (cf. [28], [33], [34]). Then the Besov spaces $B^s_{p,q}$ as well as the Triebel spaces $F^s_{p,q}$ ($s \in \mathbb{R}, 1 \leq p, q \leq \infty$), among them the Sobolev spaces $W^s_p$, satisfy our conditions if one takes $A$ to
be one of the following Banach algebras:

\[ C^0_k(\mathbb{R}^m) := \{ f \mid D^\alpha f \in C^0(\mathbb{R}^m) \text{ for all } \alpha, |\alpha| \leq k \} \]

with their natural norm. Here \( k \) has to be taken sufficiently large, depending on the parameters of the spaces (cf. [34], §2.6.1, or [5]).

Example E. One may also use Banach spaces \( B \) describing the oscillation of their elements, such as the space \( BV(\mathbb{R}) \) with the variation norm, which is in fact a pointwise Banach algebra, or the spaces \( V^p(\mathbb{R}) \) \((1 < p < \infty)\), the functions of bounded \( p \)-variation (cf. [38], [1]). These spaces may be considered as Banach modules over the (pointwise) Banach algebra of all absolutely continuous functions \( \text{Abs}(\mathbb{R}) \), which is a closed subalgebra of \( BV(\mathbb{R}) \) satisfying all requirements (see [25]). There are also \( n \)-dimensional analogs.

Example F. Any Banach dual of a homogeneous Banach space \( B \) containing \( A_0 \) as a dense subspace is also admitted, e.g. the spaces \( M(G) \) or \( P(G) \) of all bounded (or pseudo-) measures on \( G \).

THE WIENER TYPE SPACES \( W(B, C) \)

We are now in the position to define the Wiener type spaces:

Definition. Let \( Q \) be any open subset of \( G \) with compact closure. Then \( W(B, C) \) consist of all elements \( f \in B_{\text{loc}} \) such that the function \( F := F_f : z \rightarrow \| f \|_{B(zQ)} \) belongs to the solid, translation invariant BF-space \( C; \| f \|_{W(B, C)} := \| F \|_C \). \( B \) is called the local, and \( C \) the global component of \( W(B, C) \). \( \| f \|_{B(zQ)} \) denotes the 'restriction norm' of \( f \) over \( zQ \), i.e. \( \| f \|_{B(zQ)} := \inf \{ \| g \|_B, g \in B \} \), \( g \) coincides with \( f \) on \( zQ \), i.e. \( hf = hg \) for all \( h \in A_0 \) with \( \text{supp } h \subseteq zQ \).

It is not difficult to show that spaces \( B \) coinciding 'locally' and spaces \( C \) coinciding 'globally' define the same spaces \( W(B, C) \).

For the sake of completeness let us mention that weighted \( L^p \)-spaces are good examples for global components: If, e.g. \( w \) is a continuous, strictly positive function on \( G \) satisfying \( w(xy) \leq Cw(x)w(y) \) for all \( x, y \in G \), then \( L^p_w(G) := \{ f \mid fw \in L^p(G) \} \), is a translation invariant,
solid BF-space with respect to the norm \( \|f\|_{p,w} := \|fw\|_p \) \((1 \leq p \leq \infty)\). For \( G = \mathbb{R}^m \) the weight functions \( c_{w_{a,d}} \), given by
\[
c_{w_{a,d}}(x) := (1 + |x|)^a \exp(c|x|^d) \quad \text{for} \quad c, a \geq 0, \ d \in [0, 1].
\]
are the most natural ones. As a consequence of our general hypotheses concerning \( B \) and the translation invariance of \( C \) one can prove:

**Theorem 1.** Let \( B, C \) be as in the definition. Then the following properties of \( W(B, C) \) can be shown:

(i) \( W(B, C) \) is a Banach space, continuously embedded in \( B_{\text{loc}} \), and \( B_0 := \{f \in B, \ \text{supp} f \text{ is compact}\} \) is continuously embedded into \( W(B, C) \);

(ii) \( W(B, C) \) does not depend on the particular choice of \( \mathcal{Q} \), i.e. different compact subsets of \( G \) with nonvoid interior define the same space and equivalent norms. It is also independent of the Banach algebra \( A \) used;

(iii) If \( B \) is left invariant, then \( W(B, C) \) is also left invariant, and
\[
\|L_x\|_{W(B, C)} \leq \|L_x\|_{B} \|L_x\|_{C};
\]

(iv) If translation is continuous in \( B \), and if \( \mathcal{F}(G) \cap C \) is a dense subspace of \( C \), then translation is also continuous in \( W(B, C) \);

(v) \( W(B, C) \) is a Banach module over \( W(A, L^\infty) \) with respect to pointwise multiplication.

**Proof.** We do not give a direct proof of (i) and (ii) because these assertions follow without difficulties from Theorem 2. The proof of (v) is left to the reader, and (iii) follows from
\[
F_{L_y,f}(z) = \|L_yf\|_{B(zQ)} \leq \|L_y\|_B \|f\|_{B(y^{-1}zQ)} = \|L_y\|_B \|L_yf\|_{B(zQ)} = \|L_y\|_B \|L_yf\|_f(z).
\]
Let us prove (iv). Given \( \varepsilon > 0 \) and \( f \in W(B, C) \) it is possible to find some \( k \in \mathcal{K}(G) \), \( \text{supp} k := K \), such that \( \|(1 - k)L_yf\|_C < \varepsilon \) for all
in some compact neighbourhood \( U_0 \) of the identity (use the density of \( \mathcal{X}(G) \) in \( C \), the above inequality and the fact that \( y \to \| L_y \|_B \) is locally bounded, cf. [12]). Choose then \( h \in A_0 \) such that \( h(x) \equiv 1 \) on \( U_0 \cdot K \cdot Q \). Then the continuity of translation in \( B \) implies the existence of \( U \subseteq U_0 \) such that

\[
\| L_y(hf) - hf \|_B < \varepsilon \| k \|_C^{-1} \quad \text{for all} \quad y \in U.
\]

Combining these facts one obtains the following estimate:

\[
\| F_{L_yf - f} \|_C \leq \| (1 - k) F_{L_yf} \|_C + \| (1 - k) F_f \|_C + \\
+ \| k \|_C \sup_{z \in \text{supp} \ k} F_{L_yf - f}(z) \leq 2\varepsilon + \| k \|_C \| L_y(hf) - hf \|_B \leq \\
\leq 3\varepsilon \quad \text{for all} \quad y \in U.
\]

Q.E.D.

**Corollary 1.** Let \( B \) a homogeneous Banach space on \( G \). Then \( W(B, L^p) \) is a homogeneous Banach space on \( G \) for \( 1 \leq p < \infty \). In particular, \( W(B, L^1) \) is a Segal algebra on \( G \) (see also [14]), if \( B \cap \mathcal{X}(G) \) is dense in \( \mathcal{X}(G) \).

In order to derive another characterization of the spaces \( W(B, C) \) we introduce certain partitions of unity.

**Definition.** A family \( \Psi = (\psi_i)_{i \in I} \) in \( A \) is called a bounded uniform partition of unity in \( A \) of norm \( M \) and of size \( U \), if there exists a (discrete) family \( Y = (y_i)_{i \in I} \) in \( G \) and a neighbourhood \( U \) of the identity such that the following holds:

\( \sum_{i \in I} \psi_i(x) \equiv 1 \) for all \( x \in G \);

\( \sup_{i \in I} \| \psi_i \|_A \leq M \);

\( \text{supp} \ \psi_i \subseteq y_i U \) for all \( i \in I \);

\( \sup_{x \in G} \# \{ i \mid x \in y_i \cdot K \} \leq C_K < \infty \) for any compact \( K \subseteq G \).

For short we shall write that \( \Psi \) is a BUPU in \( A \).
That it is possible to find BUPU's of prescribed size in any homogeneous Banach space $A$ has been shown in [14]. For many examples $Y$ may be chosen to be a discrete cocompact subgroup of $G$. The following theorem is the key for most results given in this note.

**Theorem 2.** Let $\Psi = (\psi_i)_{i \in I}$ be a BUPU of size $U$ in $A$. Then for any compact set $W \supseteq U$ the norm of the function

$$F_W := \sum_{i \in I} \| f\psi_i \|_B c_{y_i W}$$

in $C$ defines an equivalent norm on $W(B, C)$.

**Proof.** The assertion is proved in several steps:

(a) Suppose that $f$ belongs to $W(B, C)$. We show first that $F_U$ belongs to $C$ for any BUPU $\Psi$ of size $U$, if $U^2 \subseteq Q$. In fact, under this condition one obtains

$$\| f\psi_i \|_B \leq \sup_{i \in I} \| \psi_i \|_A \| f \|_{B(zQ)}$$

if $y_i U \subseteq zQ$,

hence

$$F_U(z) = \sum_{i \in I} \| f\psi_i \|_B c_{y_i U}(z) =$$

$$= \sum_{\{i : y_i \in zU\}} \| f\psi_i \|_B \leq M_{CQ} \| f \|_{B(zQ)},$$

i.e. $\| F_U \|_C \leq C_1 \| f \|_{W(B, C)}$ for all $f \in W(B, C)$.

(b) Next we show that the right invariance of $C$ implies that $F_W$ belongs to $C$ for an arbitrary compact set $W$, if $F_U$ is in $C$. Observing that there exists a finite sequence $(z_i)_{i = 1}^n$ in $G$ such that $W \subseteq \bigcup_{k = 1}^n U z_k^{-1}$, one obtains

$$F_W \leq \sum_{i \in I} \sum_{k = 1}^n \| f\psi_i \|_B c_{y_i U z_k^{-1}} = \sum_{k = 1}^n \sum_{i \in I} \| f\psi_i \|_B A_{z_k} c_{y_i U} =$$

$$= \sum_{k = 1}^n A_{z_k} F_U,$$
\begin{equation}
\|F_W\|_C \leq \left( \sum_{k=1}^{n} \|A_{z_k}\|_C \right) \|F_U\|_C = C_2 \|f\|_{w(B,C)}.
\end{equation}

(c) Let now \( f \in B_{\mathrm{loc}} \) be given such that \( F_W \) belongs to \( C \) for some compact set \( W \) satisfying \( W \supseteq QU^2 \). Choose \( g \in A_0 \), satisfying \( g(z) \equiv 1 \) on \( Q \) and \( \text{supp} \, g \subseteq QU \). Setting \( M_z := \{ i \mid \gamma_i U \cap zQU \neq \emptyset \} \) one obtains

\begin{equation}
\|f\|_{B(zQ)} \leq \|L_z g\|_B \leq \|g\|_A \sum_{i \in M_z} \|f\psi_i\|_B \leq \|g\|_A \sum_{i \in I} \|f\psi_i\|_B c_{\gamma_i}w(z) = \|g\|_A F_W(z),
\end{equation}

hence \( \|f\|_{w(B,C)} \leq C_3 \|F_W\|_C \).

(d) Since we might have chosen another (larger) set \( W \) in step (c) it is clear that the assertion to be prove is true for arbitrary BUPU's without restriction on their size.

**Remark 1.** Essentially as a consequence of step (b) of the above proof one obtains assertion (ii) of Theorem 1. That it need not be true if \( C \) is not right invariant, has been shown by an example due to R. Bürger (private communication).

**Remark 2.** It also follows from the above proof that the norm of \( z \rightarrow \|L_z g\|_B \) in \( C \) defines another equivalent norm on \( W(B,C) \) for any \( g \in A_0 \) taking some constant value \( \equiv 0 \) on some open subset of \( G \). If the algebra \( A \) possesses local inverses (e.g. \( C^0(G), A(G) \)) then this restriction is of course not necessary (for related questions cf. [14]).

**Remark 3.** Using the above characterization it is not difficult to identify most of the spaces considered in the papers mentioned in the introduction with (special cases) of the \( W(L^q, L^p) \)-spaces \( (1 \leq p, q \leq \infty) \). The spaces \( CA(p) \) and \( C(p) \) considered in [8] correspond to spaces of the type \( W(M(G), L^\infty_w) \) and \( W(M(G), C^0_w) \) on \( G = \mathbb{R} \). The spaces \( W(A(G), L^p) \) and \( W(Q(G), L^p) \) have been considered by J.P. Bertrandias for abelian groups (private communication), and spaces of the form \( W(B, L^\infty) \) have appeared occasionally for spaces \( B \) of smooth functions.
(see for example [5], [24], [30]). Moreover, one has $W(L^p, L^p_w) = L^p_w$, $W(B^s_{p,p}, L^p) = B^s_{p,p}$ (see [24], p. 149), $W(L^p_s, L^p) = L^p_s$ (see [30], p. 1049), $W(V^p, L^p) = V^p$.

**Remark 4.** The assertion of Theorem 2 is equivalent to the following one: For each translation invariant space $C$ on $G$ there exists a solid BF-space $C_Y$ on $Y$, such that $(\|f\psi_i\|_B)_{i \in I} \in C_Y$ iff $f \in W(B, C)$. If $C = L^p_w$, then

$$C_Y = \left\{ (x_i)_{i \in I} \mid \sum_i |x_i|^p w(y_i)^p < \infty \right\}.$$  

We come now to our main result. Using Theorem 2 we are able to describe in a simple manner the convolution product of two Wiener type spaces on an [IN]-group (i.e. a group having a compact neighbourhood of identity that is invariant under inner automorphisms).

**Theorem 3.** Let $(B^1, B^2, B^3)$ and $(C^1, C^2, C^3)$ be two Banach convolution triples on an [IN]-group $G$. Then $(W(B^1, C^1), W(B^2, C^2), W(B^3, C^3))$ is a Banach convolution triple.

**Corollary 2.** Let $C$ be a solid Banach convolution algebra on an [IN]-group $G$. Then $W(L^p, C)$ is a Banach ideal in $W(L^1, C)$; in particular, it is a Banach convolution algebra for any $p$ ($1 \leq p \leq \infty$). (Note that $W(L^p, C) \subseteq W(L^1, C)$ for $p \geq 1$!)

**Proof of Theorem 3.** Let $(\psi'_i)_{i \in I^r}$ be BUPU's for $W(B^r, C^r)$, $r = 1, 2$. Let $U = U^{-1}$ be a compact, invariant neighbourhood of the identity, such that $\text{supp } \psi'_i \subseteq y'_i U$. Then there exists $C_1 > 0$ such that

$$\left\| \sum_{i \in I^r} ||f^r \psi'_i||_{B^r C^{-r}_y y'_i U^5} \right\|_C \leq C_1 \|f^r\|_{W(B^r, C^r)}$$

for all $f^r \in W^r$, $r = 1, 2$.

Let us now give an estimate of the norm of $z \rightarrow \|f^1 \ast f^2\|_{B^3(z U)}$ in $C^3$. For $g \in A_0$, with $g(x) = 1$ on $U$ and $\text{supp } g \subseteq U^2$ one has

$$(L_z g)(f^1 \ast f^2) = \sum_{(i,j) \in I_z} (L_z g)(f^1 \psi^1_i \ast f^2 \psi^2_j)$$

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for \( I_z \) being given by \( I_z := \{(i, j) \mid z \supp g \cap y_i^1 U y_j^2 U \neq \emptyset\} \). Using the assumption concerning \((B^1, B^2, B^3)\) we obtain

\[
\|f^1 * f^2\|_{B^3(zU)} \leq \sum_{(i, j) \in I_z} \|g\|_A \|f^1\psi_i^1\|_{B^1} \|f^2\psi_j^2\|_{B^2}.
\]

The invariance of \( U \) now implies \( z \in y_i^1 y_j^2 U^4 \) if \((i, j) \in I_z\). A direct computation reveals that

\[
\|c_U\|_1^{-1}(c_{y_i U^5} * c_{y_j U})(x) \geq 1 \text{ for all } x \in y_i y_j U^4.
\]

This allows us to continue the estimate by

\[
\|f^1 * f^2\|_{B^3(zU)} \leq C_2 \left( \sum_{i \in I^1} \|f^1\psi_i^1\|_{B^1} c_{y_i U^5} \right) * \left( \sum_{j \in I^2} \|f^2\psi_j^2\|_{B^2} c_{y_j U^2} \right)(z) \leq C_2 f^1_{U^5} * f^2_{U^5}(z).
\]

Since \((C^1, C^2, C^3)\) is a BCT, this implies

\[
\|f^1 * f^2\|_{W(B^3, C^3)} \leq C_3 \|f^1\|_{W(B^1, C^1)} \|f^2\|_{W(B^2, C^2)}.
\]

**Remark 5.** By choosing special examples of BCT's one obtains a number of concrete results. For example, the analogue of Young's inequality for \( W(L^q, L^p) \)-spaces (cf. [4], Theorem 4.2). For convolutions between Lipschitz spaces compare e.g. [34], [32]. Details are left to the reader.

**Remark 6.** If \( Y \) is a discrete subgroup then the space \( C_Y \) mentioned in Remark 4 is a Banach convolution algebra over \( Y \) if \( C \) is a BCA on \( G \). Conversely, any BCA \( D_Y \) on \( Y \) induces a family \( W(L^p, D) \) of BCA's on \( G \) \((1 \leq p \leq \infty)\).

For abelian groups it is of interest to identify the maximal ideal space of the Banach algebras \( W(B, C) \):

**Theorem 4.** Let \( B \) be a homogeneous Banach space on \( G \), and suppose that \( C \) is a left invariant Banach convolution algebra containing
$\mathcal{K}(G)$ as a dense subspace and satisfying

\begin{equation}
(\text{BD}') \quad \sum_{n=1}^{\infty} n^{-2} \log \| L_y n \|_c < \infty \quad \text{for any} \quad y \in G.
\end{equation}

Then the maximal ideal space of $W(B, C)$ can be identified with $\hat{G}$, and $W(B, C)$ is a regular Banach algebra on $\hat{G}$.

**Proof.** The proof is essentially the same as that given in [10] for spaces of the form $\Lambda(A, B, X)$.

Let us mention here that the spaces $C = L^p_w(\mathbb{R}^m)$ satisfy all assumptions of Theorem 4 for $1 < p < \infty$, if $w = e^{w_a, d}$, if $c = 0$, $a > \frac{m(p-1)}{p}$, or if $c > 0$, $a \in \mathbb{R}$, $d \in (0, 1)$ (cf. [12]).

Combining Theorems 3 and 4 one obtains for example:

**Theorem 5.** Let a homogeneous Banach space $B$ on a locally compact abelian group $G$ and a compact subset $K$ of $\hat{G}$ be given. Let furthermore $C$ be a solid, translation invariant BF-space satisfying (BD'). Then any $f \in W(L^1, C)$ satisfying $\text{supp } \hat{f} \subseteq K$ belongs to $W(B, C)$, and there exists $C > 0$ such that

\[ \| f \|_{W(B, C)} \leq C \| f \|_{W(L^1, C)} \quad \text{for all such} \quad f \in W(L^1, C). \]

**Proof.** By Theorem 4 there exists $k \in W(B, L^1_w)$ (for a suitable weight function $w$) such that $\hat{k}(x) \equiv 1$ on $K$ (cf. [10], §4). Then $f = k \ast f \in W(B, L^1_w) \ast W(L^1, C) \subseteq W(B, C)$ by Theorem 3, and

\[ \| f \|_{W(B, C)} \leq C_1 \| k \|_{W(B, L^1_w)} \| f \|_{W(L^1, C)}. \]  

Q.E.D.

**Corollary 3.** Given $k \in \mathbb{N}$, $p \in (1, \infty)$, and a compact subset $K$ of $\mathbb{R}^m$ there exists a constant $C > 0$ such that any $f \in W(L^1, L^p)$, with $\text{supp } \hat{f} \subseteq K$ satisfies $\| f \|_{W(C(k), L^p)} \leq C \| f \|_p$.

At least formally this is a strengthened version of Sobolev's inequality. In a similar way one shows for $G = \mathbb{R}$ that such $L^p$-functions belong to $V^p(\mathbb{R})$ (cf. [1]). Related results have been proved in [35] (observe that $L^p_w$ satisfies (BD') if $w \in K(a, C_e)$).
Further results concerning spaces of Wiener's type are given in [14]-[16]. In [14] the spaces $W(B, L^1)$ are characterized by a minimality property. A summary of results concerning $S_0(G) := W(A(G), L^1)$ for abelian groups is given in [15] (see also [23]). Results concerning interpolation of spaces of Wiener's type are given in [16].

Concluding we would like to mention that we have not included into the list of references a list of papers in which special cases of Wiener type spaces are used more or less explicitly. The list of reference given below is therefore in a sense quite incomplete.

REFERENCES


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