BANACH SPACES OF DISTRIBUTIONS OF WIENER'S TYPE
AND INTERPOLATION

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In the parallel paper [9] we have introduced "spaces of Wiener's type", a family of Banach spaces of (classes of) measurable functions, measures or distributions on locally compact groups. The elements of these spaces are characterized by what we call the global behaviour of certain of their local properties. In the present paper it is to be shown that interpolation methods can be applied to these spaces in a very natural way. Using the results on interpolation it is not difficult to extend various theorems of analysis to the setting of Wiener-type spaces. As illustration we present a version of Young's inequality for unimodular groups, as well as generalizations of the Hausdorff-Young inequality for locally compact abelian groups. As a consequence, one obtains a sharpened version of Sobolev's embedding theorem.

1. Definitions and Basic Properties

Throughout \( G \) will be a locally compact group with left Haar measure \( dx \). We shall mainly be interested in non-discrete, non-compact groups (e.g. \( G = \mathbb{R}^m \)). \( \mathcal{K}(G) \) denotes the space of all continuous, complex-valued functions on \( G \) with compact support (supp), endowed with its natural inductive limit topology. \((L^p, \| \cdot \|_p), 1 \leq p \leq \infty\), denotes the usual Lebesgue spaces. Given a subset \( M \subseteq G \) we write \( c_M \) for its characteristic function. The space \( L^1_{\text{loc}}(G) \) consists of all (classes of) measurable functions \( f \) on \( G \) such that \( f \mathcal{K} \in L^1(G) \) for any compact subset \( K \subseteq G \). It is a topological vector space with the family of seminorms \( f \mapsto \| f \mathcal{K} \| \). A BF-space on \( G \) is a Banach space \((B, \| \cdot \|_B)\) which is continuously embedded into \( L^1_{\text{loc}}(G) \). As usual we shall speak of "functions" in such spaces, identifying two measurable functions in \( B \), if they are equal locally almost everywhere (l.a.e.). A BF-space is called solid if any measurable function \( g \), for which there exists \( f \in B \) such that \( |g(x)| \leq |f(x)| \) l.a.e. belongs to \( B \), with \( \|g\|_B \leq \|f\|_B \). A BF-space \( B \) is called left translation
invariant (translation invariant) if the left (left and right) translation operators, given by

\[ L_y f(x) := f(y^{-1}x), \quad A_y f(x) := f(xy) \]

act boundedly on \( B \). Their operator norm is written as \( ||| \cdot |||_B \). Corresponding terminology is applied to spaces of measures or distributions, to which the translation operators are extended by transposition. A left invariant BF-space will be called a homogeneous Banach space on \( G \) if \( G \) acts (by left translations) isometrically on \( B \), and if translation is continuous in \( B \), i.e. if

\[ \lim_{y \to e} ||| L_y f - f |||_B = 0 \]

for all \( f \in B \). The homogeneous Banach spaces which are dense in \( L^1(G) \) are exactly the Segal algebras in the sense of Reiter ([15]).

A triple \((B^1, B^2, B^3)\) will be called a Banach convolution triple (BCT), if convolution, given by

\[ f^1 \ast f^2(x) := \int_G f^1(y^{-1}x)f^2(y)dy \quad \text{for} \quad f^1 \in \mathcal{K}(G) \quad B^i, \quad i = 1,2, \]

extends to a bounded, bilinear map (of norm 1) from \( B^1 \times B^2 \) into \( B^3 \). Clearly \((A, A, A)\) is a BCT for some \( A \subseteq L^1(G) \) iff \( A \) is a Banach convolution algebra.

Any weighted \( L^1 \)-space

\[ L^1_w(G) = \{ f \mid f \omega \in L^1(G) \}, \quad ||f||_{L^1_w} = ||f \omega||_1 \]

is a BCA, called Beurling algebra, if \( \omega \) is a continuous function satisfying \( \omega(x) \leq 1 \), and \( \omega(xy) \leq C \omega(x)w(y) \) for all \( x, y \in G \). (cf. [15]). Such functions are called weight functions. A Banach space \( B \) is a (left) Banach convolution module over the Banach algebra \( A \) iff \( (A, B, B) \) is a BCT, and a (left) Banach ideal in \( A \) if furthermore \( B \subseteq A \). Any homogeneous Banach space is known to be a left \( L^1(G) \) Banach convolution module. Constants without importance will be denoted by \( C, c, \ldots \).

**GENERAL HYPOTHESIS**

As a standing assumption we suppose throughout this paper that for any Banach space \( B \) used below there exists some "nice" Banach algebra \( A \) acting on \( B \) by "pointwise" multiplication.

More precisely, we suppose that there exists a homogeneous Banach space \((A, ||| \cdot |||_A)\), continuously embedded into the Banach algebra with respect to
pointwise multiplication (i.e. separating points from closed sets), and which is closed under complex conjugation, and that B is a Banach module over A with respect to "pointwise" multiplication, i.e.

\[ ||hf||_B \leq ||h||_A ||f||_B \text{ for all } h \in A, f \in B. \]

Here some comment concerning the term "pointwise" multiplication is in order. Of course, there is no problem of interpretation, if B happens to be a BF-space on G (which covers the most important examples). In this case the pointwise product of a continuous function with a locally integrable function is to be taken in the ordinary sense. In order to cover more general situations (which occur naturally in the investigations) we assume in the sequel that the following situation is given:

B is continuously embedded into the topological dual \( A'_0 \) of \( A_0 := A \cap K(G) \) (endowed with its natural inductive limit topology). On \( A'_0 \) an action of A by "pointwise multiplication" is given in a natural way, i.e. by transposition of the operation of A on \( A_0 \) by ordinary multiplication (remember the definition of a "pointwise product" of a test function and a distribution). Since the assumptions imply that \( A_0 \) is always a dense subspace of \( K(G), R(G) \) (the space of all Radon measures on G) and in particular \( L^1_{\text{loc}}(G) \) (identified with the closed subspace of all absolutely continuous measures) is always continuously embedded into \( A'_0 \) in a natural way. Since the action of A on a subspace of \( L^1_{\text{loc}} \) defined in the way just mentioned coincides of course with the natural action mentioned above we gain flexibility in adopting our assumptions concerning the definition of pointwise products. We define \( B_{\text{loc}} \) to be space of all elements \( \sigma \) of \( A'_0 \) such that \( h\sigma \in B \) for all \( h \in A_0 \). (Otherwise we would have to restrict our attention to spaces of locally integrable function, which would sometimes be a quite unnatural restriction).

EXAMPLES. The most important examples of algebras A which are defined for arbitrary locally compact groups are the spaces \( (C^0(G), || \cdot ||_\infty) \) of continuous functions vanishing at infinity, and Eymard's Fourier A(G), which coincides with \( \mathcal{F}L^1(G) := \{ \mathcal{F}f | f \in L^1(G) \} \) if G is a locally compact abelian group with dual group \( \hat{G} \) (\( \hat{G} \) is identified with G). Therefore any solid BF-space B on G, in particular the spaces \( L^p(G), 1 \leq p \leq \infty \), is included in our consideration (considered as \( C^0(G) \)-module), but may take \( B = C^0(G) \) itself. If G is abelian,
one may consider \( H = \mathcal{F} L^P(G), 1 \leq p \leq \infty \) (the Fourier transform being taken in the sense of tempered distributions or as a quasimeasure, cf. \([8]\) or \([12]\)) as a module over \( \Lambda(G) \). As further examples we only mention here the spaces of Besov-Hardy-Sobolev type \( B^s_{p,q} \) and \( L^p \), \( s \in \mathbb{R} \), \( 1 \leq p,q \leq \infty \), as considered by H. Triebel (see \([18]\), \([19]\)) including Lipschitz and Bessel potential spaces (cf. \([16]\)). For further examples cf. \([9]\). The Wiener type spaces \( W(B,C) \) are now defined as follows:

**Definition 1.1.** Let \( B \) satisfy the general hypothesis, and let \( C \) be a solid, translation invariant BF-space on \( G \). Given any open subset \( Q \) of \( G \) with compact closure and \( f \in B^l_{loc} \) we set: \( F := F_f : z \mapsto ||f||_{B(zQ)}, \) with

\[
||f||_{B(zQ)} := \inf \{||g||_B | g \in B, g \text{ coincides with } f \text{ on } zQ, \text{ i.e. } h_f = h_g \text{ for all } h \in A_0 \text{ with supp } h \subseteq zQ \}.
\]

The **Wiener-type space** \( W(B,C) \) with local component \( B \) and global component \( C \) is then defined by

\[
W(B,C) := \{ f | f \in B^l_{loc}, F \in C \}.
\]

The natural norm on \( W(B,C) \) is given by

\[
||f||_{W(B,C)} := ||F||_C.
\]

**Theorem 1.1.** Let \( B,C \) be as in Definition 1.1. Then \( W(B,C) \) is a Banach space, continuously embedded into \( B^l_{loc} \). It does not depend on the particular choice \( Q \), i.e. two different open subsets of \( G \) with compact closure define the same space and equivalent norms.

It should be mentioned here that good examples of solid translation invariant BF-spaces are weighted \( L^p \)-spaces \( L^p_w(G) = \{ f | fw \in L^p(G) \}, ||f||_{p,w} := ||fw||_p \), for \( w \) being a continuous weight function on \( G \).

In the present paper we shall consider mainly spaces of the form \( W(L^p,L^q) \) or \( W(F^p,F^q) \), \( 1 \leq p,q \leq \infty \). Practically all spaces of Wiener's type that have been considered in a number of mostly recent papers (only to mention \([2-4,6-11,13,15,17,20]\)) arise as special cases of the above families, most of them are even of the first kind. In order to give the reader some orientation concerning inclusions among these spaces we state the following lemma:
LEMMA 1.2. Let \(1 \leq p, p_1, p_2, q_1, q_2 \leq \infty\) be given. Then

i) \(W(L^p, L^p) = L^p(G)\);

ii) \(W(L^{p_1}, L^{q_1}) \subseteq W(L^{p_2}, L^{q_2})\) if \(p_1 \geq p_2, q_1 \leq q_2\);

iii) \(W(L^{p_1}, L^{q_1}) \subseteq W(L^{p_2}, L^{q_2})\) if \(p_1 \leq p_2, q_1 \leq q_2\);

iv) \(W(L^p, L^q) \subseteq W(\mathcal{F} L^{p'}, L^q)\) for \(1 \leq p \leq 2\), and

\[W(\mathcal{F} L^{p'}, L^q) \subseteq W(L^p, L^q)\] for \(2 \leq p \leq \infty\), for all \(q, 1 \leq q \leq \infty\)

and \(1/p' = (1 - 1/p)\).

REMARK 1.1. If \(G\) is nondiscrete and noncompact it can be shown that equality holds in ii) and iii) only for \(p_1 = p_2\) and \(q_1 = q_2\), and in iv) only for \(p = 2\).

REMARK 1.2. One also has \(W(M(G), L^q) \subseteq W(\mathcal{F} L^\infty, L^q)\) for all \(q\) (here \(M(G) = (C^0(G))'\) denotes the space of bounded measures on \(G\)). The spaces \(W(M(G), L^q), q > 1\), arise as dual spaces of the spaces \(W(C^0(G), L^{p'})\) (cf. [10, 11, 13, 17]).

2. The Abstract Main Result

The following theorem is the basic result of this paper:

THEOREM 2.1. Let \(A, B, C\) be as in definition 1.1. Assume furthermore that \(C\) is a left Banach convolution module over some Beurling algebra \(L^1_w(G)\). Then \(W(B, C)\) is a retract of the vector-valued function space \(C(B)\), i.e. there exist bounded linear operators \(T: W(B, C) \rightarrow C(B)\) and \(S: C(B) \rightarrow W(B, C)\) such that \(S \circ T = \text{Id}_{W(B, C)}\).

REMARK 2.1. It can be shown that \(C\) satisfies the above condition for a solid translation invariant BF-space containing \(K(G)\) as a dense subspace, or if \(C\) is of the form \(C = L^p_w(G), 1 \leq p \leq \infty\) for some weight function \(w\).

PROOF. The proof is given in four steps.

Step 1. In order to define a mapping \(T\) in a suitable way we choose some \(g \in A_0\)
satisfying \( \int_G g(x^{-1}) \, dx = 1 \), and \( \text{supp } g \subseteq Q \). Then we set:

\[
(2.1) \quad Tf(z) := (L_z g)f, \quad z \in G.
\]

We show first that \( z \mapsto (L_z g)f \) defines a continuous mapping from \( G \) into \( B \), for \( g \) as above and for every \( f \in W(B, C) \subseteq B \). In fact, let \( x \in G \) and some relatively compact neighbourhood \( V \) of \( x \) be given. Then there exists \( h \in A_0 \) such that \( h(x) = 1 \) on \( V(\text{supp } g) \). This implies for \( x, y \in V \):

\[
(2.2) \quad ||(L_y g)f - (L_x g)f||_B = ||(L_y g - L_x g)fh||_B \leq ||L_y g - L_x g||_A ||fh||_B \to 0
\]

for \( y \to x \) (in \( V \)), since translation is continuous in \( A \). It therefore remains to give an estimate of \( z \mapsto ||(L_z g)f||_B \) in the space \( C \). Making use of the following inequality

\[
(2.3) \quad ||(L_z g)f||_B \leq ||L_z g||_A ||f||_B(z, Q) \quad \text{for any } z \in G
\]

we obtain

\[
(2.4) \quad ||Tf||_{C(B)} \leq ||g||_A ||f||_{W(B, C)} \quad \text{for all } f \in W(B, C).
\]

This completes the proof of step 1.

Step 2. Having defined \( T \) as above we are now looking for the corresponding operator \( S: C(B) \to W(B, C) \). Choosing \( g^1 \in A_0 \), satisfying \( g^1(x) \equiv 1 \) on \( \text{supp } g \) (\( g \) as above) we shall define \( SF \) (at first formally) by

\[
(2.5) \quad SF := \int_G (L_z g^1)F(z) \, dz \quad \text{for } f \in C(B).
\]

Before we can verify that \( S \) satisfies all requirements we have to make (2.5) precise: At a first stage we claim that it makes sense to interprete \( SF \) as the element of \( A^!_0 \), given by

\[
(2.6) \quad <SF, h> := \int_G <(L_z g^1)F(z), h> \, dz, \quad h \in A_0.
\]

We have now to verify that the right hand expression is well defined as an element of \( A^!_0 \) (i.e. as a measure, quasimeasure or distribution in our applications). In order to show continuity of the functional defined in (2.6) let some compact subset \( K \) of \( G \), and any \( h \in A_0 \) with \( \text{supp } h \subseteq K \) be given. Writing \( K_1 \) for \( \text{supp } g^1 \) and using the fact that \( (L_z g^1)h = 0 \) for \( z \notin KK_1^{-1} \) first, and the continuous embeddings \( B \to A^!_0 \) and \( C \to L^1_{loc}(G) \) then, we obtain
\[
(2.7) \quad \left| \int_G \langle (L_z g^1) F(z), h \rangle \, dz \right| \leq \int_{K^K_1} \left| \langle (L_z g^1) F(z), h \rangle \right| \, dz \leq C_1 \| h \|_A \int_{K^K_1} \| (L_z g^1) F(z) \|_B \, dz \leq C_2 \| g^1 \|_A \| h \|_A \| F(z) \|_C(B),
\]

where \( C_2 \) denotes a constant depending on the space \( C \) and on \( K \) (and \( K_1 \)) only.

Step 3. We intend to prove now the boundedness of \( S \) as a mapping from \( C(B) \) into \( W(B,C) \). That \( SF \) belongs to \( B_{\text{loc}} \), i.e. that \( h(SF) \in B \) for all \( h \in A_0 \) can be shown as follows: Since multiplication of \( h \in A_0 \) with \( SF \in A_0' \) is to be understood in the usual sense, i.e. as being defined by \( \langle h(SF), h_1 \rangle = \langle SF, hh_1 \rangle \) for all \( h_1 \in A_0' \), one has \( h(SF) = \int_G h(L_z g^1) F(z) \, dz \). But the last integral is convergent in \( B \), since the integrand is an integrable function on \( G \) with values in \( B \) and compact support (recall \( C \ast L^1_{\text{loc}}(G) \)).

In order to show that \( SF \) belongs to \( W(B,C) \) let us look for an estimate for \( y \mapsto \| SF \|_B(zQ) \), for \( F \in C(B) \). Let \( g^2 \in A_0' \) be chosen such that \( g^2(x) \equiv 1 \) on \( Q \). Then one has (as in step 2) for any \( y \in G \):

\[
(2.8) \quad \| SF \|_{B(YQ)} \leq \| (L_y g) \int_G (L_z g^1) F(z) \, dz \|_B \leq \| g^2 \|_A \| g^1 \|_A \| F(z) \|_B \, dz,
\]

if we set \( N := (\text{supp} \, g^2)(\text{supp} \, g^1)^{-1} \). Noting that the function \( \phi : z \mapsto \| F(z) \|_B \) belongs to \( C \), and that \( C \) is a left Banach convolution module over some Beurling algebra \( L^1_{\text{w}}(G) \) we obtain, as a continuation of (2.8)

\[
(2.9) \quad \| SF \|_{B(YQ)} \leq \| g^2 \|_A \| g^1 \|_A \| C_N \|_1, \| \phi \|_C.
\]

Combining (2.8) and (2.9) we arrive at

\[
(2.10) \quad \| SF \|_{W(B,C)} \leq C_3 \| \phi \|_C = C_3 \| F \|_C(B) \quad \text{for all } F \in W(B,C).
\]

Step 4. In this last step it is shown that under the assumptions made the relation \( Sot(f) = f \) holds true for all \( f \in W(B,C) \). Since \( W(B,C) \) is continuously embedded into \( B_{\text{loc}} \), and hence into \( A_0' \), it will be sufficient to verify that this identity holds in \( A_0' \). Given any \( h \in A_0 \) one has (using the identity \( g^1 = g^2 g^1 \) and applying Fubini's theorem):
(2.11) \[ <S(Tf), h> = \int_G (L_z g^2) (L_z g^1) f, h > G = \int_G (L_z g^1) f, h > dz = \]
\[ = \int_G \int_G g^1 (z^{-1} y)f(y)h(y)dydz = \left( \int_G g^1 (x^{-1})dx \right) <f, h> = <f, h>, q.e.d. \]

This completes the proof of theorem 2.1.

REMARK 2.2. There is also a more elementary, but somewhat longer proof showing that the spaces \( W(B, C) \) can be represented as retracts of vector-valued sequence spaces. In this case one makes use of the characterization of \( W(B, C) \) by means of uniform, bounded partitions of unity (cf. [9], Theorem 2).

THEOREM 2.2. Suppose that the same algebra \( A \) acts on \( B^1 \) and \( B^2 \), and assume that \( c^1 \) or \( c^2 \) has absolutely continuous norm (i.e. that \( f_n(x) \to 0 \) for \( n \to \infty \) and \( |f_n(x)| \leq |f(x)| \) a.e. implies \( \|f_n\|_0 \to 0 \)). Then one has for \( \theta \in (0,1) \):

\[ \left( W(B^1, c^1), W(B^2, c^2) \right)[\theta] = W \left( B^1, B^2 \right)[\theta], (c^1, c^2)[\theta] \]

PROOF. As a consequence of Theorem 2.1 and general interpolation principles the interpolation results follow from the corresponding interpolation results for the vector-valued function spaces \( c^1(B^1) \) (cf. [1], § 6.4). The needed "complex" result is then found in § 13/6 of Calderon's paper ([5]).

COROLLARY 2.3. For \( \theta \in (0,1) \), \( 1 \leq p_1, p_2, q_1, q_2 \leq \infty \), \( q_2 < \infty \) one has

\[ \left( W(L_{p_1}^{1}, L_{q_1}) \right), \left( W(L_{p_2}^{2}, L_{q_2}) \right) \left[ \theta \right] = W(L_{p}^{\theta}, L_{q}^{1-\theta}), \]

and

\[ \left( W(L_{p_1}^{1}, L_{q_1}^{1}) \right), \left( W(L_{p_2}^{2}, L_{q_2}^{2}) \right) \left[ \theta \right] = W(L_{p}^{\theta}, L_{q}^{1-\theta}), \]

with \( 1/p = (1-\theta)/p_1 + \theta/p_2 \), \( 1/q = (1-\theta)/q_1 + \theta/q_2 \), \( w = w_1^{1-\theta} w_2^{\theta} \).

REMARK 2.3. There are of course corresponding results for real interpolation spaces, based on the real interpolation results for - say - weighted vector-valued \( L^p \)-spaces (cf. [1], [18]). Since we do not need these results here we leave it to the reader to combine known results to new explicit statements, if they should be useful to him.
3. Applications

As a direct application of the results of section 2 we are able to extend Young's inequality for Wiener-type spaces to unimodular groups (for abelian groups this result has been proved in [2], for [IN]-groups it is Theorem 4.2 of [4], or a corollary to Theorem 3 of [9]).

**THEOREM 3.1.** Let $G$ be a unimodular, locally compact group, and let $p_1, p_2, q_1, q_2 \in [1, \infty]$ be given, such that $1/p := 1/p_1 + 1/p_2 - 1 \geq 0$ and $1/q := 1/q_1 + 1/q_2 - 1 \geq 0$. Then
\[
\left[ W(L^{p_1}_1, L^{q_1}_1), W(L^{p_2}_2, L^{q_2}_2), W(L^p, L^q) \right]
\]
is a Banach convolution triple.

**PROOF.** The result follows - as the usual Young inequality - by means of complex interpolation from the fact that the convolution operator $T^g_f : f \ast f \ast g$ is bounded from $W(L^{p_2}_2, L^{q_2}_2)$ into $W(L^{p_2}_2, L^{q_2}_2)$ (conjugate indices, cf. [4], Proposition 3.7, here use of the unimodularity of $G$ is made into $L^\infty(G) = W(L^\infty, L^\infty)$, and from $L^1(G) = W(L^1, L^1)$ into $W(L^{P_2}_2, L^{Q_2}_2)$, for any $g \in W(L^{P_2}_2, L^{Q_2}_2)$ (note that the translation operators $L_y$ act uniformly bounded on these spaces).

The following result is an extension of the usual Hausdorff-Young inequality:

**THEOREM 3.2.** Let $G$ be a locally compact abelian group. For $1 \leq r \leq p \leq \infty$ the Fourier transform defines a bounded linear mapping from $W(\mathcal{F}L^P, L^r)$ into $W(\mathcal{F}L^r, L^P)$. In particular, $W(\mathcal{F}L^P, L^P)$ on $G$ is mapped onto the corresponding space on $\hat{G}$ by the Fourier transform.

The theorem will follow essentially by means of complex interpolation from the following proposition, which is of interest for itself.

**PROPOSITION 3.3.** For $1 \leq p \leq \infty$ the Fourier transform maps $W(\mathcal{F}L^P, L^1)$ into $W(\mathcal{F}L^1, L^P)$.

**PROOF.** It is known (see [9], Theorem 2, cf. also [7]) that there exists some compact set $K \subseteq G$ and $C > 0$ such that any $f \in W(\mathcal{F}L^P, L^1)$ has a representation of the form $f \sim \sum_1^a \lambda_n \gamma_n f_n$, with $\sum_1^a \lambda_n |a_n| \leq C \|f\|_{W(\mathcal{F}L^P, L^1)}$, $\text{supp } f_n \subseteq K$ and $\|f_n\|_{\mathcal{F}L^P} \leq 1$ for all $n$. 
Applying Theorem 5 of [9] to \( \mathcal{F} f_n \) (take B = \( \mathcal{F} L^1(\hat{G}) \), C = \( L^P(\hat{G}) \) there) one obtains

\[
\|f_n\|_{W(\mathcal{F} L^1(\hat{G}), L^P)} \leq C_1 \|\mathcal{F} f_n\|_{L^P} = C_1 \|f_n\|_{\mathcal{F} L^P}
\]

and

\[
\|f\|_{W(\mathcal{F} L^1(\hat{G}), L^P)} \leq C_1 \sum_{k=1}^{\infty} |a_n| \|I_{Y_n} f_n\|_{\mathcal{F} L^P} \leq C_2 \|f\|_{W(\mathcal{F} L^P, L^1)}.
\]

**PROOF (of Theorem 3.2).** We first consider the case \( r = p \). By Proposition 3.3 \( \mathcal{F} \) (and also \( \mathcal{F}^{-1} \)) map \( W(\mathcal{F} L^1, L^1) \) onto the corresponding space on the dual group (cf. also [8], Theorem A2 i), \( W(\mathcal{F} L^1, L^1) = \mathcal{S}(G) \). By Plancherel's theorem the same assertion is true for \( W(\mathcal{F} L^2, L^2) = L^2(G) \), hence for all \( p \in [1, 2] \) by complex interpolation. For \( p \geq 2 \) it can be proved by transposition (i.e. as in the case of tempered distributions, as we shall prove in detail elsewhere one has \( W(\mathcal{F} L^r, L^s) = W(\mathcal{F} L^r', L^s') \) for \( 1 < r, s < \infty \)). The general case is then derived by means of further complex interpolation between the "diagonal" case and the result of Proposition 3.3.

**REMARK 3.1.** The above result is in various direction best possible. We shall show below that the Fourier transform does not map \( W(\mathcal{F} L^1, L^P) \) (which is contained in \( W(L^r, L^P) \) and \( W(\mathcal{F} L^r, L^P) \) for any \( r \geq 1 \)) into \( W(\mathcal{F} L^q, L^\infty) \) nor into \( W(\mathcal{F} L^\infty, L^q) \) for any \( q < p \). In particular, the assertions of Theorem 3.2 break down for \( r < p \). It also follows therefrom that the Fourier transform is never surjective in Theorem 3.2 for \( r \neq p \).

**REMARK 3.2.** Combining Theorem 3.2 with Lemma 2.2 one obtains the main result of [3], which has been proved by F. Holland for the case \( G = \mathbb{R} \). Theorems 3.4, 3.5 and 4.2 of [17] (cf. Remark 1.2) also arise as consequences of our result.

**PROOF (of Remark 3.1).** It will be sufficient to show that for any \( p < \infty \), and \( q < p \) there is a bounded sequence \((f_n)_{n=1}^{\infty}\) in \( W(\mathcal{F} L^1(G), L^P) \) for which \((\mathcal{F} f_n)_{n=1}^{\infty}\) is unbounded in \( W(\mathcal{F} L^\infty, L^q) \) or \( W(\mathcal{F} L^q, L^\infty) \) respectively. Given any \( f_0 \neq 0 \), \( f_0 \in W(\mathcal{F} L^1, L^1) \) let us consider expressions of the form \( g_n = \sum_{k=1}^{n} y_k M_{t_k} f_0 \) (recall that \( M_{t_k} \) denote the operator of pointwise multiplication with the character \( t_k \)).
Since \( \mathcal{K}(G) \cap W(\mathcal{F} L^1, L^p) \) is a dense subspace of \( W(\mathcal{F} L^1, L^p) \) for \( p < \infty \) it is possible to choose \( (y_{k_n})_{k=1}^n \) ("sufficiently large") such that
\[
\| g_{n} \|_{W(\mathcal{F} L^1, L^p)} \leq 2 n^{1/p} \| f_{o} \|_{W(\mathcal{F} L^1, L^p)} \quad \text{and} \quad \| g_{n} \|_{q} \geq (1/2) n^{1/q} \| f_{o} \|_{q}
\]
for an arbitrary sequence \( (t_{k})_{k=1}^n \subset \hat{G} \). On the other hand one has
\[
\mathcal{F} g_{n} = t_{1} \ldots t_{n} \mathcal{F} f_{o}, \quad \text{which implies} \quad \| \mathcal{F} g_{n} \|_{W(\mathcal{F} L^q, \mathcal{F} L^q)} \geq (1/2) n^{1/q} \| f_{o} \|_{W(\mathcal{F} L^q, \mathcal{F} L^q)}
\]
for an appropriate choice of \( (t_{k})_{k=1}^n \subset \hat{G} \). Hence \( f_{n} := n^{-1/p} g_{n} \) is a suitable sequence for our first assertion. If \( \mathcal{F} f_{o} \) has suitable compact support, then the second assertion follows if \( t_{k} = t_{o} \) for all \( k \), because then
\[
\| \mathcal{F} f_{n} \|_{W(\mathcal{F} L^q, \mathcal{F} L^q)} = \| \mathcal{F} f_{n} \|_{W(\mathcal{F} L^q, \mathcal{F} L^q)} = \| f_{n} \|_{q} \geq (1/2) n^{-1/p} \| f_{o} \|_{q}.
\]

As the last application to be mentioned here we give a version of Sobolev's embedding theorem (cf. [16] Chap. V, § 2.2) for the (fractional) potential spaces \( L^p_s \) in the setting of Wiener type spaces:

**Theorem 3.4.** i) For \( s > m/2 \) one has the following continuous embeddings:
\[
L^2_s(\mathbb{R}^m) \hookrightarrow W(\mathcal{F} L^1, L^2) \hookrightarrow W(C^0, L^2) \hookrightarrow C^0(\mathbb{R}^m).
\]

ii) More generally, one has for \( p \in [1,2] \) and \( s > m(1/q-1/p) \geq 0 \) the embedding
\[
L^p_s(\mathbb{R}^m) \hookrightarrow W(L^q, L^p).
\]

**Proof.** (i) By definition one has \( \mathcal{F} L^2_s = L^2_w(\mathbb{R}^m) := \{ h | \text{w}_s \in L^p_w \} \), with \( \text{w}_s(x) := (1+|x|^2)^{s/2} \). Since \( \text{w}_s^{-1} \in L^2(\mathbb{R}^m) \) for \( s > m/2 \), Hölder's inequality implies \( L^2_w \hookrightarrow W(L^2, L^2) \hookrightarrow W(L^2, L^1) \). Assertion (i) follows now from 3.3.

(ii) We apply complex interpolation to the pair of inclusions given by (i) and \( L^p \hookrightarrow W(\mathcal{F} L^p, L^p) \) (cf. Lemma 2.2). Using the fact that
\[
(L^p_s, L^p_t)_{\theta} = L^p_u \quad \text{for} \quad \theta \in (0,1), 1/s = (1-\theta)/x + \theta/x \quad \text{and} \quad u = (1-\theta)s + \theta t.
\]
(c.f. [14], Chap. 5, Theorem 5).

Further results concerning Wiener-type spaces, in particular on their multiplier spaces, Tauberian theorems, as well as a characterization of the Banach dual of \( W(B,C) \) will be given in subsequent papers.
REFERENCES


