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STRONG ALMOST PERIODICITY AND WIENER TYPE SPACES

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Summary. Not long after the appearance of the classical theory of uniformly almost periodic functions various generalizations of this concept arose, due to Stepanoff, Wiener and later Love, Upton and others. It is the purpose of this note to show how a number of related concepts of strong almost periodicity for spaces of functions or distributions on locally compact Abelian groups can be treated simultaneously by using the concept of Wiener type spaces. Using these spaces of local-global type, introduced in an earlier paper, it is possible to extend many of the classical results (approximability of the almost periodic elements by trigonometric polynomials, existence of an invariant mean) to arbitrary locally compact Abelian groups or to spaces of differentiable functions or distributions on the Euclidean spaces. It will be shown that it is not difficult to derive these results from the results on uniform almost periodicity. In view of the flexibility of the choice of the "local" component B, describing the degree of "smoothness" of the distributions under considerations the approach offered also admits the treatment of new types of almost periodicity, even on the real line.

We assume throughout that G is a locally compact Abelian group. The group operation is written multiplicatively. We are mainly interested in noncompact and nondiscrete groups. $(L^p(G), \| \|_p)$, $1 \leq p \leq \infty$ denotes the usual Lebesgue spaces with respect to the Haar measure dx. We write $C^0(G)$ for the closure of $K(G)$ (space of continuous functions with compact support) in $L^\infty(G)$. Its Banach dual is identified with the space $M(G)$ of all bounded measures on G. As a standing general hypothesis we suppose throughout this paper, that $(B, \| \|_B)$ denotes a strongly translation invariant Banach space of distributions on G carrying both a $L^1(G)$-Banach module structure with respect to convolution and a certain pointwise multiplicative structure. To be precise: We suppose that there exists a homogeneous Banach space $(A, \| \|_A)$ (i. e. $f \in A$ implies $L_y f \in A$ for all $y \in G$, where $L_y f(x) := f(y^{-1}x)$, and $\| L_y f \|_A = \| f \|_A$ and $\lim_{y\to 0} \| L_y f - f \|_A = 0$), which is continuously embedded in $(C^b(G), \| \|_\infty)$, which is also a regular and selfadjoint Banach algebra with respect to pointwise multiplication. Then $A_0 := A \cap K(G)$ is a locally convex topological vector space with its natural inductive limit topology, densely embedded in $K(G)$. It therefore makes sense to consider its continuous dual A_0' as a vector space of distributions. Since A_0 is an ideal in A, one may extend the pointwise multiplication by elements of A to all of A_0' by transposition. Now we suppose: $(B, \| \|_B)$ is continuously embedded in A_0' and a Banach module over A with respect to pointwise multiplication,

i. e. $\|hf\|_B \leq \|h\|_A \|f\|_B$ for all $h \in A$, $f \in B$. Furthermore we suppose $\|L_y f\|_B = \|f\|_B$ for all $f \in B$ and $y \in G$. Finally we suppose that B is a Banach convolution module over $L^1(G)$, i. e. that there is a representation π of $L^1(G)$ on B such that one has $\pi(g)f(x) = \langle L_x \check{g}, f \rangle$ (the duality between A_0 and A_0') for all $g \in A_0$ ($\check{g}(x) := g(x^{-1})$). Because $\pi(g)f$ coincides with $g * f$, whenever the convolution integrals make sense, we shall use by abuse of notation the convolution sign throughout. We mention that the choice $A = C^0(G)$ allows to consider Banach spaces of measures (since $K(G)' = R(G)$, the space of all Radon measures on G), in particular Banach spaces of classes of locally integrable functions such as the spaces $L^p(G)$, $1 \leq p \leq \infty$, or more general rearrangement invariant Banach spaces on G. The choice $A = A(G) := \{\hat{h} : h \in L^1(\hat{G})\}$, i. e. the Fourier algebra on G allows to treat subspaces B of $A_0' = Q(G)$, the space of quasimeasures on G, such as the image of $L^p(G)$ under the inverse Fourier transform (cf. [11]). The special case $p = \infty$ is the space $P(G)$ of all pseudomeasures on G. It is identified with the dual of the Fourier algebra $A(G)$. For $G = \boldsymbol{R}^m$ spaces of differentiable functions or distributions of a certain order, such as Besov- or Bessel-potential spaces are of interest (cf. [9, 12, 23]). These spaces are Banach modules over $C^{(k)}(\boldsymbol{R}^m)$ (for k large enough, depending on the parameters of the spaces under consideration). More generally, it can be shown that it makes sense to use Banach spaces embedded in $\mathfrak{D}'(\boldsymbol{R}^m)$, satisfying $\mathfrak{D}(\boldsymbol{R}^m)$. $B \subseteq B$ in all considerations below. Finally, we mention that further interesting examples arise, if one uses Banach spaces of functions on $\boldsymbol{R}$ (or $\boldsymbol{R}^m$) defined by various forms of variation. For further examples and explanations cf. [12].

Before we can define B-almost periodicity we have to introduce the notion of Wiener type spaces (only the special cases $p = 1$ and $p = \infty$ of the definition below will be used).

D e f i n i t i o n 1. Let B satisfy the general hypothesis. Fix $g \in A_0$ such that $g(z) \geq a > 0$ on some open set in G. Then $W(B, L^p(G))$ is defined as the space of those elements of $B_{\mathrm{loc}} := \{\sigma : \sigma \in A_0', h \cdot \sigma \in B$ for all $h \in A_0\}$, for which $\sigma^{(g)}$, given by $\sigma^{(g)}(z) := \|(L_z g) \cdot \sigma\|_B$, belongs to $L^p(G)$. By $\sigma \to \|\sigma\|_{W(B, L^p)} := \|\sigma^{(g)}\|_p$ a norm is defined on $W(B, L^p)$.

We recall (see [12] for details) that $(W(B, L^p(G)), \| \|_W)$ is a Banach space continuously embedded in A_0'. Different "test functions" define the same space and equivalent norms. The spaces $W(B, L^1)$ satisfy a certain minimality property (cf. [12]). If A possesses local inverses (with respect to multiplication), any $g \in W(A, L^1)$, $g \neq 0$, is admitted in the definition of $W(B, L^p)$. The space $S_0(G) := W(A(G), L^1)$ is treated in detail in [13] (cf. also [11]), where it is shown that the Fourier transform maps $S_0(G)$ onto $S_0(\hat{G})$. In the proofs below we shall use the fact that the Banach dual $S_0'(G)$ of $S_0(G)$ coincides with $W(P(G), L^\infty)$ and that the Fourier transform, extended to $S_0'(G)$ by transposition ($G = (\hat{G})\hat{}$!), maps $S_0'(G)$ onto $S_0'(\hat{G})$. For simplicity the Fourier transform of σ is written as $\hat{\sigma}$ whenever it makes sense.

D e f i n i t i o n 2. $\sigma \in W(B, L^\infty)$ is called *B-almost periodic* if the set $\{L_y \sigma\}_{y \in G}$ is relatively compact in $(W(B, L^\infty), \| \|_W)$. We shall use the symbol B-AP for the space of all B-almost periodic elements in $W(B, L^\infty)$.

It is clear that the choice $B = L^\infty(G)$ gives the concept of uniform almost periodicity and that the choice $B = L^p(G)$, $1 \leq p \leq \infty$, gives the spaces of almost periodic functions on R considered by Stepanoff and Wiener on the real line. By a suitable choice of the "local component" B many other generalizations of these classical spaces, including several spaces to be found in the literature, can be interpreted as special cases of the general definition given above. Details are left to the reader (see also [4, 8]). For the proof of the main results the following facts will be useful.

Proposition 1. a) *$W(A, L^1)$ is a Segal algebra on G, i. e. a homogeneous Banach space, densely and continuously embedded in $L^1(G)$.*

b) *$W(A, L^1)$ is a dense Banach ideal in L^1G and $\Lambda^K := \{f \mid f \in L^1(G)$, $\operatorname{supp} \hat{f}$ compact in $\hat{G}\}$ is a dense ideal in $W(A, L^1)$. Therefore, given any compact subset $Q \subseteq \hat{G}$, there exists $h \in W(A, L^1)$ such that $\hat{h}(y) = 1$ on Q. Consequently there exist L^1-bounded approximate units in $W(A, L^1)$.*

c) *There exists some constant $C > 0$ such that one has $g * f \in L^\infty$ and $\| g * f \|_\infty \leq C \| g \|_{W(A, L^1)} \| f \|_{W(B, L^\infty)}$ for all $g \in W(A, L^1)$ and $f \in W(B, L^\infty)$.*

Proof. That $W(A, L^1)$ is a Segal algebra, has been proved in [11]. The assertions stated in b) are true for arbitrary Segal algebras (cf. [21, Chap. 6. § 2]). Assertion c) follows from Theorem 3 of [12], since $W(B, L^\infty)$ is continuously embedded in $W(A', L^\infty)$.

Lemma 2. *Let $\sigma \in B$-AP be given. Then one has $\sigma \in W(B, L^\infty)_c := \{\sigma : \sigma \in W(B, L^\infty), \lim_{y \to 0} \| L_y f - f \|_{W(B, L^\infty)} = 0\}$. In particular, $\lim_{\alpha \to \infty} u_\alpha * \sigma = \sigma$ in $W(B, L^\infty)$ for any bounded approximate unit $(u_\alpha)_{\alpha \in I}$ in $L^1(G)$.*

Proof. Suppose there exists $\sigma \in B$-AP and a sequence $(y_n)_{n \geq 1}$ in G such that $\lim_{n \to \infty} y_n = 0$, but $\| L_{y_n} f - f \|_{W(B, L^\infty)} > \varepsilon_0 > 0$ for all n. The relative compactness of $\{L_y \sigma\}_{y \in G}$ in $W(B, L^\infty)$ then implies the existence of a subsequence $(y_k)_{k \geq 1}$ such that $\lim_{k \to \infty} L_{y_k} \sigma = \sigma_0$ in $W(B, L^\infty)$ for some $\sigma_0 \in W(B, L^\infty)$. Since B is continuously embedded in A'_0 the same is true for $W(B, L^\infty)$. In particular, $L_{y_k} \sigma$ converges to σ_0 in A'_0, endowed with the topology $\sigma(A'_0, A_0)$. But since A is a homogeneous Banach space, translation is continuous in A'_0 with respect to this weak topology, i. e. $\lim_{k \to \infty} L_{y_k} \sigma = \sigma$ in A'_0. This implies $\sigma_0 = \sigma$ in contradiction to the assumption.

As a consequence one may interpret $k * f$ as the vector-valued integral ${}_G\int (L_y f) k(y)\, dy$ for $k \in L^1(G)$. Using the usual approximate units in $L^1(G)$ (i. e. positive, continuous functions of norm 1 and with small support), one verifies that any $\sigma \in W(B, L^\infty)_c$ can be approximated by elements of the form $k * \sigma$, $k \in K^+(G)$. Since $\lim_\alpha u_\alpha * k = k$ in $L^1(G)$, the second assertion follows.

We are now able to give a characterization of B-AP.

Theorem 3. *For $\sigma \in W(B, L^\infty)$ the following conditions are equivalent:*

(i) *$\{L_y \sigma\}_{y \in G}$ is relatively compact in $W(B, L^\infty)$. i. e. $\sigma \in B$-AP.*

(ii) *σ can be approximated in $W(B, L^\infty)$ by trigonometric polynomials.*

(iii) *$T_\sigma : L^1 \to W(B, L^\infty)$, with $T_\sigma(G) = g * \sigma$ is a compact operator.*

For *$G = R^n$ the following condition is equivalent as well:*

(iv) *For any $\varepsilon > 0$ the ε-translation numbers are relatively dense in G, i. e. there exists a compact subset $K \subseteq G$ such that for any $y \in G$ there exists $x \in y + K$ with $\| L_x f - f \|_{W(B, L^\infty)} < \varepsilon$.*

The proof will be given in several steps. We mention here that it makes use of the fact that the equivalences stated in the theorem hold true in the case $B=L^\infty(G)$ (cf. [16], where also $B=M(G)$ and $B=L^p(G)$ are considered, and [17], Proposition 1.2 as possible references). For brevity we write UAP for L^∞-AP.

i) $\to$ (ii). Let $\sigma \in B$-AP and $\varepsilon>0$ be given. By Lemma 2 and Proposition 1, there exist $u_1, u_2\, W(A, L^1)\subseteq W(B, L^1)$ such that $\| u_1 * u_2 * \sigma - \sigma \|_W < \varepsilon$. Since $T_2 : \sigma \to u_2 * \sigma$ defines a bounded linear operator from $W(B, L^\infty)$ to $L^\infty(G)$ commuting with translations, the set $\{L_y(u_2 * \sigma)\}_{y \in G} = \{T_2(L_y\sigma)\}_{y\in G}$ is relatively compact in $L^\infty(G)$, i. e. $u_2 * \sigma \in$ UAP. Therefore one can find a trigonometric polynomial t such that $\| u_2 * \sigma - t \|_{W(L^1, L^\infty)} \leq C \| u_2 * \sigma - t \|_\infty \leq \|u_1\|^{-1}_{W(B, L^1)}\varepsilon$, hence $\|u_1 * u_2 * \sigma - u_2 * \sigma\|_{W(B, L^\infty)} \leq \| u_1 \|_{W(B, L^1)} \| u_2 * \sigma - t \|_{W(L^1, L^\infty)}$ and $\| \sigma - u_1 * t \|_{W(B, L^\infty)} \leq 2\varepsilon$, where $u_1 * t$ is a trigonometric polynomial.

(ii) $\to$ (iii). This implication is obvious since the convolution operator $T_t : g \to g * t$ is finite dimensional for a trigonometric polynomial t.

(iii) $\to$ (i). It is sufficient to verify that the (compact) closure of $\{f * \sigma \mid f \in L^1(G), \|f\|_1 \leq 1\}$ in $W(B, L^\infty)$ contains $\{L_y\sigma\}_{y\in G}$. In fact, for any approximate unit $(u_\alpha)_{\alpha\in I}$ in A_0 the net $((L_y u_\alpha) * \sigma)_{\alpha \in I}$ is $\sigma(A_0', A_0)$ convergent to $L_y\sigma$. As it contains a norm convergent subnet by the compactness of T_σ the assertion follows.

The proof that iv) is equivalent as well can be given by similar arguments. A slight modification of the method used in the first step yields the following result (cf. [16, Theorem 7, p. 82] for the special case $B^1 = L^\infty(G)$, $B^2 = M(G)$, it goes back to Bochner):

Corollary 4. *Let $B^1 \subseteq B^2$ be two Banach spaces satisfying the general hypothesis. Then* B^1-AP $= B^2$-AP $\cap\, W(B^1, L^\infty)_c$.

Corollary 5. B-AP *coincides with the closure of the set of trigonometric polynomials in* $W(B, L^\infty)$.

Proof. In view of Corollary 4 and Theorem 3 it is sufficient to show that any trigonometric polynomial t is contained in $W(B, L^\infty)_c$: Choose $h \in \Lambda^K \subseteq W(A, L^1) \subseteq W(B, L^1)$ such that $\hat{h}(y) = 1$ on the (finite) set supp $\hat{t}$. Then $h * t = t$ and therefore $t \in W(B, L^1) * W(L^1, L^\infty)_c \subseteq W(B, L^\infty)_c$.

Corollary 6. B-AP *is a Banach module over* A-AP *with respect to pointwise multiplication, i. e.* $h\,.\,\sigma \in B$-AP *and*

$$\| h\sigma \|_{W(B, L^\infty)} \leq \| h \|_{W(A, L^\infty)} \| \sigma \|_{W(B, L^\infty)}$$

for all $h \in W(A, L^\infty)$, $\sigma \in W(B, L^\infty)$.

Proof. Since the general assumptions imply that $W(B, L^\infty)$ is a Banach module over $W(A, L^\infty)$ with respect to pointwise multiplication (cf. [12, Theorem 1.V]), the assertion follows from Theorem 3 and the fact that the space of trigonometric polynomials is closed under pointwise multiplication.

In connection with Corollary 3 the following characterization of the elements of $P(G)$-AP, which we shall call almost periodic quasimeasures, might be of particular interest, because it has a simple formulation.

Theorem 7. i) *A quasimeasure* $\sigma \in W(P(G), L^\infty) = S_0'(G)$ *on G is almost periodic if there exists $k_0 \in S_0(G)$ such that $M_t k_0 * \sigma \in$ UAP for all $t \in \hat{G}$ and* $\lim_{t\to\infty} \| M_t k_0 * \sigma \|_\infty = 0$. *In case k_0 satisfies $\hat{k}_0(t) \neq 0$ for all $t \in \hat{G}$ it is sufficient to suppose $k_0 * \sigma \in$ UAP and* $\lim_{t\to\infty} \| M_t k_0 * \sigma \|_\infty = 0$.

ii) *Conversely, any almost periodic quasimeasure satisfies* $k*\sigma\in$ UAP *and* $\lim_{t\to\infty}\|M_t k*\sigma\|_\infty=0$ *for all* $k\in S_0(G)$.

Proof. i) Let $\sigma\in S_0'(G)$ and $0\neq k_0\in S_0(G)$ be given, satisfying the assumptions. Since $\sup_{t\in\widehat{G}}\|M_t k_0*\sigma\|_\infty<\infty$ and since UAP is a Banach convolution module over $L^1(G)$, the sums $\sum_{n=1}^\infty f_n*M_{t_n}k_0*\sigma$ converge in UAP for any sequence $(f_n)_{n\geq 1}^\infty$ in $L^1(G)$ satisfying $\sum\|f_n\|_1<\infty$. On the other hand, a characterization of $S_0(G)$ (cf. [13, Theorem 2.C]) implies that any $g\in S_0(G)$ can be obtained as a sum $\sum f_n*M_{t_n}k_0$ of this kind. Therefore $g*\sigma\in$ UAP for all $g\in S_0(G)$. Since one has $\|M_t k_0*\sigma\|_{P(G)}=\|(L_t\widehat{k})\cdot\sigma\|_\infty$, it follows that $\widehat{\sigma}$ belongs to the closed subspace $W(P(\widehat{G}), C^0(\widehat{G}))$ of $S_0'(G)$. $K(\widehat{G})$ being a dense subspace of $C^0(\widehat{G})$ it is not difficult to verify that $\widehat{\sigma}$ can be obtained as $\lim_{n\to\infty}\widehat{h}_n\widehat{\sigma}$ in $S_0'(\widehat{G})$ for a suitable sequence $(h_n)_{n\geq 1}$ in $\Lambda^K(G)\subseteq S_0(G)$ (one may think of $\widehat{h}_n$ as trapezoid functions in the Fourier algebra). But $\widehat{h}_n\widehat{\sigma}=(h_n*\sigma)^\wedge$ implies $\sigma=\lim_{n\to\infty}h_n*\sigma$ in $S_0'(G)$, since the Fourier transform establishes an isomorphism between $S_0'(G)$ and $S_0'(\widehat{G})$ ([11, Théorème B.2. i)]). Since $h_n*\sigma$ belongs to UAP for all n, this implies that σ belongs to the norm closure of UAP in $S_0'(G)$, which of course coincides with the closure of the set of trigonometric polynomials in $S_0'(G)$. That $k_0*\sigma\in$ UAP implies $g*\sigma\in$ UAP for all $g\in S_0(G)$ if $\widehat{k}_0(t)\neq 0$ for all $t\in\widehat{G}$ follows from Wiener's approximation theorem: The assumption $\widehat{k}_0(t)\neq 0$ for $t\in\widehat{G}$ implies that the closed linear span (= closed ideal) generated by k_0 in $L^1(G)$ coincides with $L^1(G)$ (cf. [21, Chap. 6, § 1]). The ideal theorem for Segal algebras ([21, Chap. 6, § 2.5]) implies that the closed invariant ideal in $S_0(G)$ generated by k_0 coincides with $S_0(G)$. Since one has $(f*k_0)*\sigma=f*(k_0*\sigma)\in L^1*\text{UAP}\subseteq\text{UAP}$ for all $f\in L^1(G)$ and $\|g*\sigma\|_\infty\leq\|g\|_{S_0}\|\sigma\|_{S_0'}$ for all $g\in S_0(G)$ the assertion $g*\sigma\in$ UAP for all $g\in$ UAP follows, UAP being a closed subspace of $L^\infty(G)$.

ii) Let now some almost periodic quasimeasure σ be given. Since σ is obtained as a limit of trigonometric polynomials t_n in $S_0'(G)$, $\widehat{\sigma}$ is obtained as a limit of finitely supported discrete measures $\mu_n=\widehat{t}_n$ in $S_0'(\widehat{G})$. This implies $\widehat{\sigma}\in W(P(\widehat{G}), C^0(\widehat{G}))$, hence $\lim_{t\to\infty}\|M_t k*\sigma\|_\infty\to 0$ for any $k\in S_0(G)$. On the other hand, this implies $\sigma=\lim_{n\to\infty}h_n*\sigma$ for a suitable sequence $(h_n)_{n\geq 1}$, $h_n\in S_0(G)$ (cf. part i). It also implies $g*\sigma=\lim_{n\to\infty}g*t_n$ in L^∞ for any $g\in S_0(G)$. But $g*t_n$ is again a trigonometric polynomial and therefore $g*\sigma$ must be long to UAP for all $g\in S_0(G)$.

Next we prove the existence of a mean in B-AP. It will be obtained by means of asymptoticly invariant nets in $L^1(G)$.

Definition 3. A net $(h_\gamma)_{\gamma\in J}$ in $L^1(G)$ satisfying $\|h_\gamma\|_1=1$, $h_\gamma\geq 0$ for all $\gamma\in J$ will be called *asymptoticly invariant* if it satisfies

$$\lim_{\gamma\to\infty}\|L_x h_\gamma*g-h_\gamma*g\|_1\to 0\quad\text{for any}\quad g\in L^1(G),\quad x\in G.$$

Theorem 8. *Let $\sigma\in B$-AP be given and suppose that B is continuously embedded in $S_0'(G)$. Then there exists a constant (multiple of the Haar measure) $C_\sigma\in\mathbf{C}$ such that one has for any asymptoticly invariant net $\{h_\gamma\}_{\gamma\in J}$*

$$\lim_{\gamma\to\infty}\|h_\gamma*\sigma-C_\sigma\|_{W(B,L^\infty)}=0.$$

Proof. Starting with any asymptoticly invariant net $\{h_\beta\}_{\beta \in I}$, one observes that $\{h_\beta * \sigma\}_{\beta \in I}$ is relatively compact in $W(B, L^\infty)$. Therefore there exists a convergent subset $\{h_\alpha\}_{\alpha \in J}$, such that for some $\sigma_0 \in W(B, L^\infty)$ $\lim_{\alpha\to\infty} h_\alpha * \sigma = \sigma_0$ in $W(B, L^\infty)$. The asymptotic invariance of $\{h_\beta\}_{\beta \in I}$ implies $L_y\sigma_0 = \sigma_0$ for all $y \in G$. The only translation invariant elements of $S_0'(G)$ being the scalar multiples of the Haar measure (cf. [14]), it is clear that we may identify σ_0 with a constant function. We write C_σ for the value it takes. Let now $\varepsilon > 0$ be given. Then there exists $\alpha_0 \in J'$ such that $\| h_{\alpha_0} * \sigma - \sigma \|_{W(B, L^\infty)} < \varepsilon$. On the other hand, for any asymptoticly invariant net $\{h_\gamma\}_{\gamma \in I}$ there exists γ_0 such that for $\gamma \geqq \gamma_0$ one has $\| h_{\alpha_0} * h_\gamma - h_\gamma \|_1 < \varepsilon \| \sigma \|^{-1}_{W(B, L^\infty)}$. Using the identity $h_\gamma * \sigma_0 = \sigma_0$ one obtains

$$\| h_\gamma * \sigma - \sigma_0 \|_{W(B, L^\infty)} \leqq \| h_\gamma * \sigma - h_{\alpha_0} * h_\gamma * \sigma \|_{W(B, L^\infty)}$$

$$\leqq \| h_{\alpha_0} * h_\gamma - h_\gamma \|_1 \, \| \sigma \|_{W(B, L^\infty)} + \| h_\gamma \|_1 \, \| h_\alpha * \sigma - \sigma_0 \|_{W(B, L^\infty)} \leqq 2\varepsilon \text{ for } \gamma \geqq \gamma_0.$$

This completes the proof.

It is now obvious that it is possible to define a bounded linear functional M on B-AP by setting $M(\sigma) := C_\sigma$. Furthermore, one has $M(L_y\sigma) = M(\sigma)$ for all $\sigma \in B$-AP and $y \in G$. This functional M is called a *mean* on B-AP. Furthermore, one has $M(g * \sigma) = (\int g(y)dy) M(\sigma)$ for any $g \in L^1(G)$.

Corollary 9. *Let $B \subseteq S_0'(G)$ be given and let $\widehat{G}$ be countable at infinity. Then $S_\sigma := \{t \mid t \in \widehat{G},\ M(M_t\sigma) \neq 0\}$ is a countable subset of $\widehat{G}$ for any $\sigma \in B$-AP.*

Proof. By Lemma 2 and Proposition 1 there exists a sequence $(u_n)_{n \geqq 1}$ in $S_0(G)$ such that $\lim_{n\to\infty} \| u_n * \sigma - \sigma \|_{W(B, L^\infty)} = 0$ and $\lim_{n\to\infty} \widehat{u}_n(t) = 1$ uniformly on compact subsets. This implies $M(M_t\sigma) = \lim_{n\to\infty} M(M_t u_n * M_t\sigma)$. Consequently S_σ is a subset of the countable union of the sets $S_{u_n * \sigma}$. Since the spectrum of these UAP functions $u_n * \sigma$ is countable for all $n \geqq 1$ the corollary is proved.

Among various further extensions of classical results to the present setting we have chosen the following one (telling us that the derivative of an almost periodic function is again almost periodic).

Theorem 10. *Let $\sigma \in A_0'$ be given, having compact support. Suppose that T_σ defines a convolution operator from B^1 to B^2 (i. e. that $T_\sigma : g \to \sigma * g$, $\sigma * g(x) = \langle L_x g, \sigma \rangle$, $g \in A_0$, extends to a bounded linear operator from B^1 to B^2). Then T_σ extends to a convolution operator from B^1-AP to B^2-AP.*

Proof. A slight modification of the proof of Theorem 3 of [12] shows that T_σ extends to a bounded operator from $W(B^1, L^\infty)$ to $W(B^2, L^\infty)$. Since the convolution operator induced by $T_\sigma(\sigma_0)$ is compact from $L^1(G)$ to $W(B^2, L^\infty)$, as it coincides with $T_\sigma \circ T_{\sigma_0}$ for any $\sigma_0 \in W(B^1, L^\infty)$, the assertion follows from Theorem 3.

The author is indebted to Professor Gil de Lamadrid for having attracted his interest in the subject and for making preliminary announcements on related (and in a certain direction more general) results (cf. [3, 16]) available. One of the differences is the use of $S_0'(G)$ here instead of "transformable measures" (cf. [2]).

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