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A FRESH APPROACH TO HARMONIC ANALYSIS: NUHAG, WINTER-TERM 08/09

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ABSTRACT. The results are non-trivial for the real line $\mathbb{R}$, but are formulated for the $d-$dimensional Euclidean space $\mathbb{R}^d$, viewed as a prototype for a locally compact Abelian group (with the usual addition of vectors and the topology provided by the Euclidian metric). [1] We start with the description of $\boldsymbol{M}_b(G)$, the space of bounded linear measures, as the dual space of $\boldsymbol{C}_0$, which is naturally endowed with a convolution structure.

sec:intro

1. INTRODUCTION

We start with a few symbols. First note that we will permanently make use of the fact that $\mathbb{R}^d$ is a locally compact Abelian group with respect to addition (dilation will come in only as a convenient but not crucial side aspect).

First we define the most simple algebra (pointwise, later on with respect to convolution) of continuous *"test functions"*. [1] Because of the *local compactness* of $\mathbb{R}^d$ the following object is a non-trivial (not just the zero-space) linear space of functions:

Definition 1.

$$\boldsymbol{C}_c(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{C}, \text{ continuous and with compact support}\} \quad ^{2,3}$$

Here we make use of the standard definition of the support of a function:

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[1]They can be defined on any topological group, and the space is interesting and non-trivial for any locally compact group. So much of the material given below extends without difficulty to the setting of locally compact, or at least locally compact Abelian group, except for the statements which involve dilations

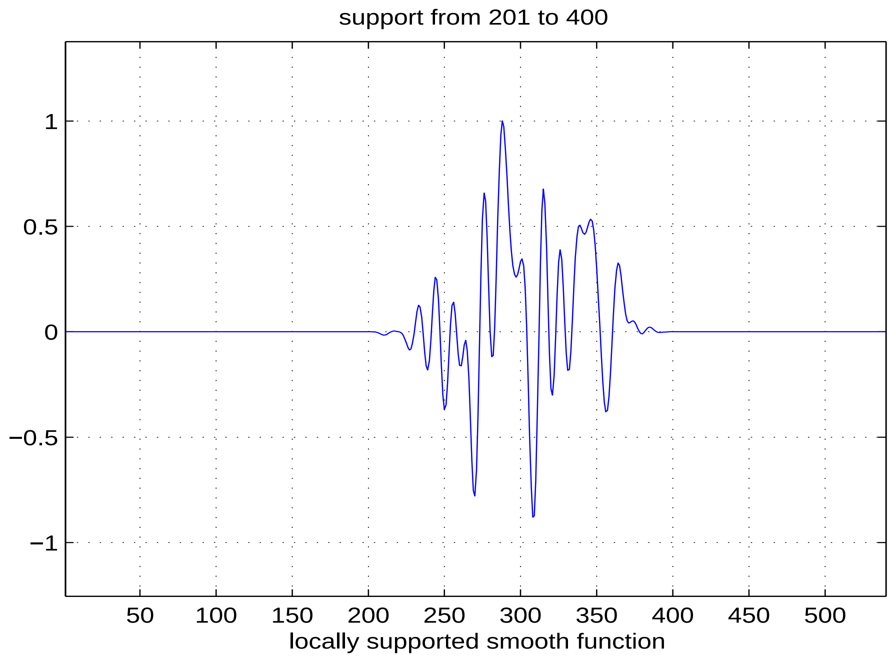
[2]Why one takes the closure in the above definition will become more clear later on, when the support of a generalized function or distribution will be defined.

[3]The capital "C" stands for *continuous*, the subscript for *compact* support

Ccdef **Definition 2.** The *support* of a continuous (!) function is defined as the closure of the set of "relevant points": [4]

$$\mathrm{supp}(f) := \{x \mid f(x) \neq 0\}^{-}$$

Lemma 1. *(Ex to Def.* Ccdef *2) A continuous, complex-valued function on $\mathbb{R}^d$ is in $\boldsymbol{C}_c(\mathbb{R}^d)$ if and only if there exists $R = R(f) > 0$ such that $f(x) = 0$ for all x with $|x| \geq R$.*



A list of symbols: Operators, names of operators, and their definitions:

Definition 3. (1) translation $T_x : T_x f(z) = [T_x f](z) = f(z - x)$
(the graph is preserved but moved by the vector x to another position);
(2) dilation D_ρ (value preserving) and St_ρ (mass preserving);
(3) involution $f \mapsto \check{f}$ with $\check{f}(z) = f(-z)$;
(4) modulation M_ω: Multiplication with the character $x \mapsto exp(2\pi i \omega x)$, i.e. $[M_\omega f](z) := e^{2\pi i \omega \cdot z} f(z)$
(5) Fourier transform $\mathcal{F}, \mathcal{F}^{-1}$, to be discussed only later: our normalization will be that given for $f \in \boldsymbol{C}_c(\mathbb{R}^d)$ by the integral

our-intdef (1)
$$\mathcal{F} \colon f \mapsto \hat{f} : \quad \hat{f}(s) = \int_{\mathbb{R}^d} f(t) e^{2\pi i s \cdot t} dt$$

Cbdef **Definition 4.**

$\boldsymbol{C}_b(\mathbb{R}^d) := \{f : \mathbb{R}^d \mapsto \mathbb{C}$, continuous and bounded with the norm $\|f\|_\infty = sup_{x \in \mathbb{R}^d} |f(x)|$ $\}$

The spaces $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ and $\boldsymbol{C}_0(\mathbb{R}^d)$ are defined as the subspaces of $\boldsymbol{C}_b(\mathbb{R}^d)$ consisting of functions which are *uniformly continuous* resp. *decaying at infinity*, i.e.,

$$f \in \boldsymbol{C}_0(\mathbb{R}^d) \quad \text{if and only if} \quad \lim_{|x| \to \infty} |f(x)| = 0.$$

[4]The superscript bar stands for "closure" of a set. Hence the supp(f) is by definition a closed set.

Lemma 2. *Characterization of $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ within $\boldsymbol{C}_b(\mathbb{R}^d)$ (or even $\boldsymbol{L}^\infty(\mathbb{R}^d)$).*

$$\|T_x f - f\|_\infty \to 0 \quad \textit{for} \quad x \to 0.$$

if and only if $f \in \boldsymbol{C}_{ub}(\mathbb{R}^d)$ (characterization within $\boldsymbol{C}_b(\mathbb{R}^d)$).

We will use the symbol $\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ for the Banach space of all bounded and linear operators on the Banach space $\boldsymbol{C}_0(\mathbb{R}^d)$, endowed with the operator norm $\|T\|_\infty := sup_{\|f\|_\infty \le 1} \|Tf\|_\infty$.

2. Banach algebras of bounded and continuous functions

banalg-de **Definition 5.** (Banach algebra) A Banach algebra is a Banach space $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ with a bilinear multiplication $(a, b) \to a \bullet b$ (or simply $a \cdot b$ or just ab, this means that it is also assocative and distributive, with the extra property that for some constant $C > 0$).[5]

Banalgsubm (2)
$$\|a \bullet b\|_{\boldsymbol{A}} \le C \|a\|_{\boldsymbol{A}} \|b\|_{\boldsymbol{A}} \quad \forall a, b \in \boldsymbol{A}$$

Theorem 1. *(Banach algebras of continuous functions)*

1. $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$ *is a Banach algebra with respect to pointwise multiplication, even a B^*-algebra, with involution $f \mapsto \bar{f}$ (i.e., $\|\bar{f}\|_\infty = \|f\|_\infty$ and $\bar{\bar{f}} = f$).*
2. $\big(\boldsymbol{C}_{ub}(\mathbb{R}^d), \|\cdot\|_\infty\big)$ *is a closed subalgebra of $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$.*
3. $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ *is a closed ideal within $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$.*

Proof. First we show that $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ is a closed subspace of $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$. In fact, let (f_n) be a uniformly convergent sequence in $\boldsymbol{C}_{ub}(\mathbb{R}^d)$, convergent to $f \in \boldsymbol{C}_b(\mathbb{R}^d)$. Then for given $\varepsilon > 0$ there exists n_0 such that $\|f - f_{n_0}\|_\infty < \varepsilon/3$. Since $f_{n_0} \in \boldsymbol{C}_{ub}(\mathbb{R}^d)$ we can find $\delta > 0$ such that $\|T_z f_{n_0} - f_{n_0}\|_\infty < \varepsilon/3$ for $|z| < \delta$. This implies of course for $|z| < \delta$:

if-uniflim (3)
$$\|T_z f - f\|_\infty \le \|T_z(f - f_{n_0})\|_\infty + \|T_z f_{n_0} - f_{n_0}\|_\infty + \|f - f_{n_0}\|_\infty < 3\varepsilon/3 = \varepsilon.$$

We have to estimate $\|T_x(gf) - gf\|_\infty$, for $|x| \to 0$. If we choose $\delta > 0$ such that both $\|T_x f - f\|_\infty \le \varepsilon'$ and $\|T_x g - g\|_\infty \le \varepsilon'$ for $|x| \le \delta$ we have

$$\|T_x(g \cdot f) - g \cdot f\|_\infty \le \|T_x g \cdot (T_x f - f)\|_\infty + \|(T_x g - g) \cdot f\|_\infty \le 2(\|f\|_\infty + \|g\|_\infty)\varepsilon'.$$

□

Lemma 3. *Characterization of $\boldsymbol{C}_0(\mathbb{R}^d)$ within $\boldsymbol{C}_b(\mathbb{R}^d)$: $\boldsymbol{C}_0(\mathbb{R}^d)$ coincides with the closure of $\boldsymbol{C}_c(\mathbb{R}^d)$ within $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$.*

Definition 6. A directed family (a net or sequence) $(h_\alpha)_{\alpha \in I}$ in a Banach algebra $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called a **BAI** (= bounded approximate identity or "approximate unit" for $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$) if

$$\lim_\alpha \|h_\alpha \cdot h - h\|_{\boldsymbol{B}} = 0 \quad \forall h \in \boldsymbol{B}.$$

We need the following definition:

Definition 7. Definition of value preserving **dilation operators**:

Drho1 (4)
$$D_\rho f(z) = f(\rho \cdot z), \; \rho > 0, \; z \in \mathbb{R}^d$$

[5]Without loss of generality one can assume $C = 1$, because for the case that $C \ge 1$ one moves on to the equivalent norm $\|a\|'_{\boldsymbol{A}} := C \cdot \|a\|_{\boldsymbol{A}}$.

It is easy to verify that

Drho2
$$\|D_\rho f\|_\infty = \|f\|_\infty \quad \text{and} \quad D_\rho(f \cdot g) = D_\rho(f) \cdot D_\rho(g) \tag{5}$$

For later use let us mention that the dilation and translation operators satisfy the following commutation relation:

trans-comm
$$D_\rho \circ T_x \ = \ T_{x/\rho} \circ D_\rho, \quad x \in \mathbb{R}^d, \rho > 0. \tag{6}$$

Proof. For any $f \in \boldsymbol{C}_b(\mathbb{R}^d)$ one has, for any $z \in \mathbb{R}^d$:

$$TEXTFORPROOF$$

□

Theorem 2. *$\big(\boldsymbol{C}_0(\mathbb{R}^d), \, \|\cdot\|_\infty\big)$ is a Banach algebra with bounded approximate units. In fact, any family of functions $(h_\alpha)_{\alpha \in I}$ which is uniformly bounded, i.e. with*

$$|h_\alpha(x)| \leq C < \infty \qquad \forall x \in \mathbb{R}^d, \forall \alpha {\in} I$$

and satisfies

$$\lim_\alpha h_\alpha(x) = 1 \quad \textit{uniformly over compact sets}$$

constitutes a BAI (the converse is true as well).

In particular, one obtains a BAI by stretching any function $h_0 \in \boldsymbol{C}_0(\mathbb{R}^d)$ with $h_0(0) = 1$, i.e. by considering the family $D_\rho h_0(g) := h_0(\rho \cdot t)$, for $\rho \to 0$.

The following elementary lemma is quite standard and could in principle be left to the reader. Probably it should be placed in the appendix. However, it is quite typical for arguments to be used repeatedly throughout these notes and therefore we state it explicitly.

Lemma 4. *Assume that one has a bounded net $(T_\alpha)_{\alpha \in I}$ of operators from a Banach space $(\boldsymbol{B}^1, \, \|\cdot\|^{(1)})$ to another normed space $(\boldsymbol{B}^2, \, \|\cdot\|^{(2)})$, such that $T_\alpha \to T_0$ strongly, i.e. for any finite set $F \subset \boldsymbol{B}^{(1)}$ and $\varepsilon > 0$ there exists an index α_0 such that for $\alpha \succeq \alpha_0$ one has $\|T_\alpha f - T_0 f\|_{\boldsymbol{B}^2}$, then one has uniform convergence over compact subsets $M \subset \boldsymbol{B}^1$.*

Proof. The argument is based on the usual compactness argument. Assuming that $|||T_\alpha||| \leq C_1$ for all $\alpha \in I$ we can find some finite set $f_1, \dots f_K \in M$ such that balls of radius $\delta = \varepsilon/(3C_1)$ around these points cover M. According to the assumption (and the general properties of nets) one finds α_0 such that

$$\|T_\alpha f_j - T_0 f_j\|_{\boldsymbol{B}^{(2)}} < \varepsilon/3, \quad \text{for } j = 1, 2, \dots K.$$

Consequently on has for any $f \in M$ and suitable chosen index j (with $\|f - f_j\|_{\boldsymbol{B}^{(1)}} < \delta$):

comp-convg
$$\|T_\alpha f - T_0 f\|_{\boldsymbol{B}^{(2)}} < \|T_\alpha(f - f_j)\|_{\boldsymbol{B}^{(2)}} + \|T_\alpha f_j - T_0 f_j\|_{\boldsymbol{B}^{(2)}} + \|T_0(f_j - f)\|_{\boldsymbol{B}^{(2)}}. \tag{7}$$

Since the operator norm of T_0 is also not larger than C_1 (! easy exercise) this implies

omp-convg2
$$\|T_\alpha f - T_0 f\|_{\boldsymbol{B}^{(2)}} \leq 2C_1 \|f - f_j\|_{\boldsymbol{B}^{(1)}} + \|T_\alpha f_j - T_0 f_j\|_{\boldsymbol{B}^{(2)}} \leq 2C_1 \delta + \varepsilon/3 < \varepsilon. \tag{8}$$

□

otal-test1 *Remark* 1. In fact, a similar argument can be used to verify that the w^*-convergence of a bounded net of operators by verifying the convergence only for f from a total subset within its domain. In fact, using linearity implies that convergence is true for linear combinations of the elements for such a set, and by going to a limit one obtains convergence for arbitrary elements in $\boldsymbol{B}^{(1)}$.

Obviously the above argument applies to nets of bounded linear functionals as well (choose $\boldsymbol{B}^{(2)} = \mathbb{C}$).

Lemma 5. *The family of operators $(D_\rho)_{\rho>0}$ is a family of isometric isomorphisms on $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$. Moreover, the mapping $\rho \mapsto D_\rho, \mathbb{R}^+ \to \mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ is a group homomorphism from the multiplicative group of positive reals into the isometric linear operators on $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$.*

Lemma 6. *Let I be the family of all compact subsets of $\mathbb{R}^d$, and define $K \succeq K'$ if $K \supseteq K'$. If we choose for every such $K \subset\subset \mathbb{R}^d$* [6] *a plateau function p_K such that $0 \leq p(x) \leq 1$ on $\mathbb{R}^d$ and $p_K(x) \equiv 1$ on K. Then $(p_K)_{K\in I}$ constitutes a BAI for $\boldsymbol{C}_0(\mathbb{R}^d)$.*

Proof. First we have to show that the "direction" of I is reflexive ($K \supseteq K$ and transitive, i.e. $K_1 \supseteq K_2$ and $K_2 \supseteq K_3$ obviously implies $K_1 \supseteq K_3$. Finally the key property for the index set of a net is easily verified: Given two "indices" K_1, K_2 the set $K_0 := K_1 \cup K_2$ is a element of I, i.e. a compact set, with $K_0 \succeq K_i$ for $i = 1, 2$. Hence $(p_K)_{K\in I}$ is in fact a net.

In order to verify the BAI property let $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ and $\varepsilon > 0$ be given. Then - by definition of $\boldsymbol{C}_0(\mathbb{R}^d)$ there exists some $R > 0$ such that $|x| > R$ implies $|f(x)| \leq \varepsilon$. Then one has $|f(x) \cdot p(x) - f(x)| = |(1 - p(x))||f(x)| \leq |f(x)| \leq \varepsilon$ for $|x| \geq R$. On the other hand $|f(x) \cdot p(x) - f(x)| = |f(x) \cdot 1 - f(x)| = 0$ for $|x| \leq R$, as long as p is a plateau function equal to 1 on the compact set $K_0 = B_R(0)$ [7], i.e. as long as $p = p_K$ has an index with $K \succeq K_0$. Altogether $\|f \cdot p - f\|_\infty \leq \varepsilon$ $\square$

Lemma 7. *Another characterization of $\boldsymbol{C}_0(\mathbb{R}^d)$ within $\boldsymbol{C}_b(\mathbb{R}^d)$:*
$h \in \boldsymbol{C}_b(\mathbb{R}^d)$ belongs to $\boldsymbol{C}_0(\mathbb{R}^d)$ if and only if

char-CO1 (9)
$$\lim_\alpha \|h_\alpha \cdot h - h\|_\infty = 0$$

for one (hence all) BAIs for $\boldsymbol{C}_0(\mathbb{R}^d)$ (as described above).

Remark 2. It is a good exercise that such a BAI acts *uniformly* on compact subsets, i.e. for any relatively compact set M in $\big(\boldsymbol{C}_0(G), \|\cdot\|_\infty\big)$ one finds: Given $\varepsilon > 0$ one can find some α_0 such that for $\alpha \succeq \alpha_0$ implies

char-CO2 (10)
$$\|h_\alpha \cdot h - h\|_{\boldsymbol{B}} \leq \varepsilon \quad \forall h \in M.$$

The following simple lemma will be useful later on (characterization of LTIS):

[6] we write $K \subset\subset \mathbb{R}^d$ to indicate that K is a *compact* subset of $\mathbb{R}^d$.

[7] $B_R(0)$ denotes the ball of radius R around 0.

homog-BF1 **Lemma 8.** *A function* $f \in \boldsymbol{C}_b(\mathbb{R}^d)$ *belongs to* $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ *if and only if the (non-linear) mapping* $z \mapsto T_z f$ *is continuous from* $\mathbb{R}^d$ *into* $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$. *In fact, such a mapping is continuous at zero if and only if it is uniformly continuous.*

Proof. It is clear that continuity is a necessary condition (and we have already seen that it is equivalent to uniform continuity for $f \in \boldsymbol{C}_b(\mathbb{R}^d)$). Conversely assume continuity at zero, i.e. $\|T_z f - f\|_\infty < \varepsilon$ for sufficiently small z ($|z| < \delta$). Then one derives continuity at x as follows:

$$\|T_{x+z} f - T_x f\|_\infty = \|T_x(T_z f - f)\|_\infty = \|T_z f - f\|_\infty < \varepsilon$$

implying uniform continuity of the discussed mapping. [8] $\square$

Remark 3. On can say, that the mapping $x \mapsto T_x$ from $\mathbb{R}^d$ into $\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ is a representation of the Abelian group $\mathbb{R}^d$ on the Banach space $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$, which by definition means that the mapping is a homomorphism between the additive group $\mathbb{R}^d$ and the group of invertible (even isometric) operators on $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$. The additional property that $z \mapsto T_z f$ is continuous for $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ is referred to as the *strong continuity* of this representation. The same mapping into the larger space $\mathcal{L}(\boldsymbol{C}_b(\mathbb{R}^d))$ would not be strongly continuous, because for $f \in \boldsymbol{C}_b(\mathbb{R}^d) \setminus \boldsymbol{C}_{ub}(\mathbb{R}^d)$ this mapping fails to be continuous (cf. below)

Remark 4. One can derive from the above definition that the mapping $(x, f) \to T_x f$, which maps $\mathbb{R}^d \times \boldsymbol{C}_0(\mathbb{R}^d)$ into $\boldsymbol{C}_0(\mathbb{R}^d)$ is continuous with respect to the product topology (Exercise).

Definition 8. We denote the dual space of $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ with $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. Sometimes the symbol $\boldsymbol{M}_b(\mathbb{R}^d)$ is used in order to emphasize that one has "bounded" (regular Borel) measures.

The Riesz representation theorem provides the justification for this definition and establishes the link to the concept explained in measure theory. In that case the action of μ on the test function is of course written in the form

meas-act (11) $$\mu(f) = \int_{\mathbb{R}^d} f(t)\, d\mu(x).$$

In the classical case of functionals on $\boldsymbol{C}(I)$, where $I = [a, b]$ is some interval, on can describe these integrals using Riemann-Stieltjes integrals. They make use of functions F of bounded variation. The distribution function [9] F is connected with the *measure* μ

[8] The proof gives an argument, that a homomorphism from an Abelian topological group G ($\mathbb{R}^d$ in our case) into a group of operators on a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ (here $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)\big)$, is strongly continuous, i.e. satisfies the discussed continuity property analogue to $f \mapsto T_z f$, is continuous from G into $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ if and only it is continuous at zero (the identity element of G).

[9] This use of the word distribution is quite different from the use of distributions in the sense of generalized functions, as it is used in the rest of these notes.

(defined on the σ-algebra of Borel sets in I via

$$(12)\quad dist - measF(x) = \int_a^x d\mu(x); \quad \int_I f(x)d\mu(x) = lim_{\delta\to 0}\sum_i f(\xi_i)[F(x_i) - F(x_{i-1})])$$

EXAMPLES: point measures $\delta_0 : f \mapsto f(0)$, $\delta_x : f \mapsto f(x)$, or integrals over bounded sets: $f \mapsto \int_a^b f(x)dx$, or more generally $f \mapsto \int_{\mathbb{R}^d} f(x)k(x)dx$, where integration is taken in the sense of Riemannian (or Lebesgue) integrals, for $k \in \boldsymbol{C}_c(\mathbb{R}^d)$. For the setting of locally compact groups $\mathscr{G}$ one will use the *Haar measure* for the definition of such integrals. Note: the norm of $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ is of course just the functional norm, i.e.

$$\|\mu\|_{\boldsymbol{M}} := \sup_{\|f\|_\infty \le 1} |\mu(f)| = sup_{\|f\|_\infty = 1} |\mu(f)|$$

EXERCISE: $\|\delta_t\| = 1$, and more generally:

Theorem 3. *For any finite linear combinations of Dirac-measures, i.e. for* $\mu = \sum_{k\in F} c_k \delta_{t_k}$ *(where F is some finite index set and we assume that one has the natural representation, with $t_k \neq t_{k'}$) one has* $\|\mu\|_{\boldsymbol{M}_b} = \sum_{k\in F} |c_k|$

A simple lemma helps us to identify the closed linear subspace of $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ generated from the (linear) subspace of finite-discrete measures.

Lemma 9. *Let V be a linear subspace of a normed space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. Then the closure of V coincides with the space obtained by taking the absolutely convergent sequences in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ with elements from V:*

s-convser1

$$(13)\qquad Abs(\boldsymbol{B}) := \{x = \sum_n v_n, \ with \quad \sum_n \|v_n\|_{\boldsymbol{B}} < \infty\}$$

Proof. It is obvious that $Abs(\boldsymbol{B}) \subseteq V^-$, the closure of V in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. Conversely, any element $x = lim_{k\to\infty} v_k$ can also be written as a limit of a sequence with $\|x - v_k\|_{\boldsymbol{B}} < 2^{-n}$ and may therefore be rewritten as a telescope sum, with $y_1 = v_1$, $y_{n+1} = v_{n+1} - v_n$. . . . . ∎

conv-dense

Remark 5. A simple but powerful variant of the above lemma is obtained if one allows the elements v_n being only taken from a dense subset of V (density of course in the sense of the $\boldsymbol{B}-$norm. Since such a set has the same closure this is an immediate corollary from the above lemma.

As a consequence (of the last two results) one finds that the closed linear subspace generated by the finite discrete measures coincides with absolutely convergent series of Dirac measures:

Definition 9.

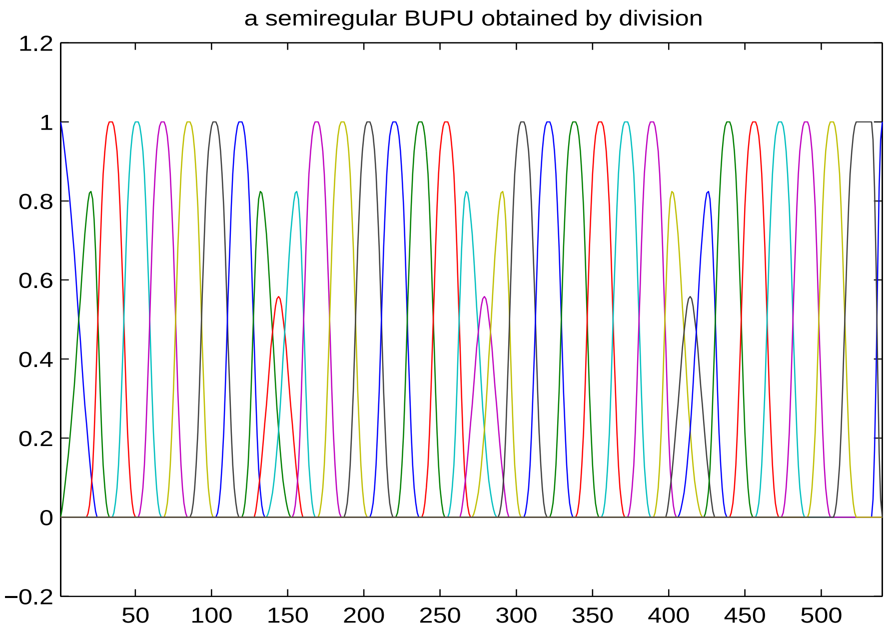
$$\boldsymbol{M}_d(\mathbb{R}^d) = \{\mu \in \boldsymbol{M}(\mathbb{R}^d) : \mu = \sum_{k=1}^\infty c_k \delta_{t_k} \ s.t. \sum_{k=1}^\infty |c_k| < \infty\}$$

The elements of $\boldsymbol{M}_d(\mathbb{R}^d)$ are called the discrete measures, and thus we claim that they form a (proper) subspace of $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$.

We claim, however, that the finite discrete measures form a w^*-dense subspace of $\boldsymbol{M}(\mathbb{R}^d)$. For this reason we will introduce a simple tool, the so-called BUPUs, the "bounded uniform partitions of unity". For simplicity we only consider the regular case, i.e. BUPUs which are obtained as translates of a single function:

Definition 10. A sequence $\Phi = (T_\lambda \varphi)_{\lambda\in\Lambda}$, where φ is a compactly supported function (i.e. $\varphi \in \boldsymbol{C}_c(\mathbb{R}^d)$), and $\Lambda = A(\mathbb{Z}^d)$ a lattice in $\mathbb{R}^d$ (for some non-singular $d \times d$-matrix) is called a **regular BUPU** if

$$\sum_\lambda \varphi(x-\lambda) \equiv 1.$$



The regular BUPUs are sufficient for our purposes. They are a special case for a more general concept of (unrestricted) BUPUs:

BUPU-gen0

Definition 11. A BUPU, a so-called **b**ounded **u**niform **p**artition of **u**nity in some Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ of continuous functions on $\mathcal{G}$ is a family $\Psi = (\psi_i)_{i\in I}$ of non-negative functions on $\mathcal{G}$, if the following set of conditions is satisfied:

(1) There exists some neighborhood U of the identity element of the group $\mathcal{G}$ that for each $i \in I$ there exists $x_i \in \mathcal{G}$ such that $\mathrm{supp}(\psi_i) \subseteq x_i + U$ for all $i \in I$;
(2) The family Ψ is bounded in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$, i.e. there exists $C_A > 0$ such that $\|\psi\|_{\boldsymbol{A}} \leq C_A$ for all $i \in I$;
(3) The family of supports $(x_i + U)_{i\in I}$ is *relatively separated*, i.e. for each $i \in I$ the number of intersecting neighbors is uniformly bounded in the following sense

$$\#\{j \,|\, (x_i + U) \cap (x_j + U) \neq \varnothing\} \leq C_0;$$

(4) $\sum_{i\in I} \psi_i(x) \equiv 1$.

Occasionally we will refer to U as the *size of the* BUPU. The constant C_A is the norm of the family Ψ in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$, and C_0 is a kind of *overlapping constant* of the family.

Let $\Psi = (\psi_i)_{i\in I}$ be a BUPU, i.e. a bounded uniform partition of unity.

Theorem 4. *Then* $\|\mu\|_{\boldsymbol{M}} = \sum_{i\in I} \|\mu\psi_i\|_{\boldsymbol{M}}$, *i.e.* $\mu = \sum_{i\in I} \mu\psi_i$ *is absolutely convergent for* $\mu \in \boldsymbol{M}(\mathbb{R}^d)$.

Proof. The estimate $\|\mu\|_{\boldsymbol{M}} \leq \sum_{i\in I} \|\mu\psi_i\|_{\boldsymbol{M}}$ is obvious, by the triangular inequality of the norm and the completeness of $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. In order to prove the opposite inequality (and in fact the finiteness of the series on the right hand side) we can argue as follows. Given $\varepsilon > 0$ we can choose a sequence $\varepsilon_i > 0$ such that $\sum_{i\in I} \varepsilon_i < \varepsilon$. By the definition of $\|\mu\psi_i\|_{\boldsymbol{M}}$ we can find $f_i \in \boldsymbol{C}_0(\mathbb{R}^d)$ with $\|f_i\|_\infty$, such that $|\mu\psi_i(f_i)| = |\mu(\psi_i f_i)| > \|\mu\psi_i\|_{\boldsymbol{M}} - \varepsilon_i$. Without loss of generality (by changing the phase of f_i if necessary) we can assume that $\mu(\psi_i f_i)$ is real-valued and in fact non-negative, i.e. absolute values can be omitted.

Putting together the function $f := \sum_{i\in I} f_i\psi_i$ we have $|f(t)| \leq \sum_{i\in I} \|f_i\|_\infty |\psi_i(t)| \leq \sum_{i\in I} \psi_i(t) = 1$, and

$$\mu(f) \geq \sum_{i\in I} |\mu(\psi_i f)| = \sum_{i\in I} \mu(\psi_i f) > \sum_{i\in I} (\|\mu\psi_i\|_{\boldsymbol{M}} - \varepsilon_i) \geq \sum_{i\in I} \|\mu\psi_i\|_{\boldsymbol{M}} - \varepsilon,$$

thus completing our argument. □

Corollary 1. *Every measure μ is a limit of its finite partial sums. Hence the compactly supported measures are dense in* $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. *In particular,* $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ *is an essential Banach module over* $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ *with respect to pointwise multiplications. One possible approximate unit consists of the families Ψ_J, where the index J is running through the finite subsets of I and is given as* $\Psi_J = \sum_{i\in J} \psi_i$.

For every BUPU $\Psi = (\psi_i)_{i\in I}$ we can define two operators, the spline-approximation operator $\mathrm{Sp}_\Psi : f \to \mathrm{Sp}_\Psi f$, given by $\mathrm{Sp}_\Psi(f) := \sum f(x_i)\psi_i$, where $(x_i)_{i\in I}$ is a family of points with $x_i \in \mathrm{supp}(\psi_i)$, for $i \in I$, and its adjoint operator, which we will call *discretization* operator, which maps bounded measures into discrete measures, denoted by D_Ψ.

Lemma 10. *The operators* Sp_Ψ *are uniformly bounded on* $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ *or* $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$ *respectively. Moreover* $\|\mathrm{Sp}_\Psi(f) - f\|_\infty \to 0$ *for* $|U| \to 0$.

The adjoint operator can be obtained from the following reasoning:

Ppsi-dual0

$$\mathrm{Sp}_\Psi^*(\mu)(f) = \mu(\mathrm{Sp}_\Psi(f)) = \mu(\sum_{i\in I} f(x_i)\psi_i) = \sum_{i\in I} \mu(\psi_i) f(x_i) = \sum_{i\in I} \mu(\psi_i)\delta_{x_i}(f), \tag{14}$$

hence we can make the following definition

DPsi-def1 **Definition 12.** $D_\Psi(\mu)$ *(or)* $D_\Psi\mu = \sum_{i\in I} \mu(\psi_i)\delta_{x_i}$.

Definition 13. For any BUPU Φ the *Spline-type Quasi-Interpolation operator Sp_Φ* is given by:

$$f \mapsto Sp_\Phi(f) := \sum_{\lambda\in\Lambda} f(\lambda)\phi_\lambda$$

PERHAPS BETTER/MORE CONSISTENT:
For any BUPU Ψ the *Spline-type Quasi-Interpolation operator* Sp_Ψ is given by:

$$f \mapsto \mathrm{Sp}_\Psi f := \sum_{i \in I} f(x_i)\psi_i$$

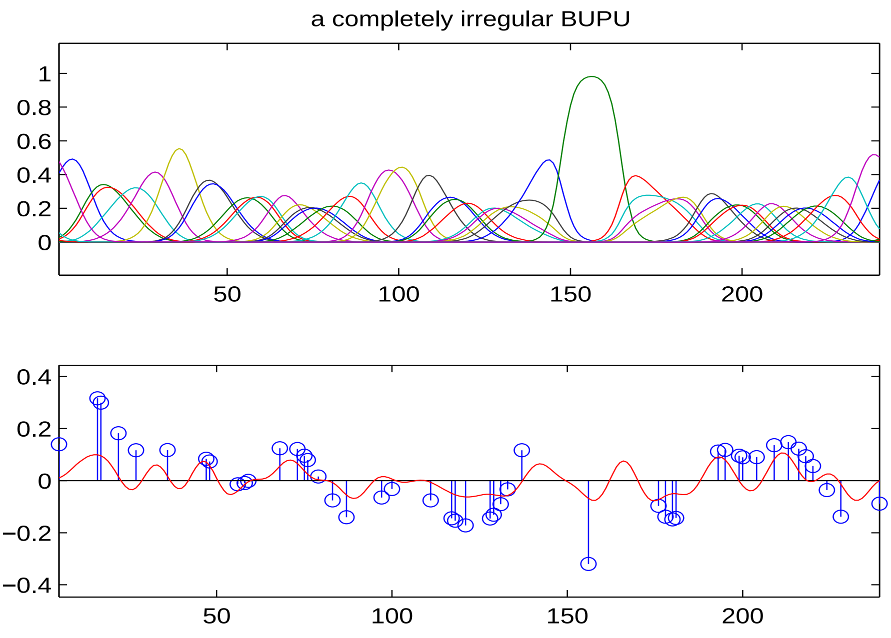
It is a good exercise to verify the following statements:

Lemma 11. *For $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ the sum defining $Sp_\Phi(f)$ is (unconditionally) norm convergent in $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ (even finite at each point), and $\|Sp_\Phi(f)\|_\infty \le \|f\|_\infty$, i.e. Sp_Φ is a linear and non-expansive mapping on $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$. In particular, the family of operators $(Sp_\Phi)_\Phi$, where Φ is running through the family of all (regular) BUPUs of uniform size (that means that the support size of ϕ with $\|\phi\|_\infty$ is limited) is uniformly bounded on $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$.* [10] *Moreover, these spline-type quasi-interpolants are norm convergent to f as $|\Phi|$ (the maximal diameter of members of Φ) tends to zero. In other words, we claim: For every $\varepsilon > 0$ and any finite subset $F \subset \boldsymbol{C}_0(\mathbb{R}^d)$ there exists $\delta > 0$ such that $\|Sp_\Phi f - f\|_\infty < \varepsilon$ if only $|\Phi| \le \delta_0$.*

[11]

For every BUPU Φ we denote the adjoint mapping to Sp_Φ by D_Φ: Discretization operator to the partition of unity Φ. It maps $\boldsymbol{M}(\mathbb{R}^d)$ into itself (in a linear way), and since Sp_Φ is non-expansive the same is true for D_Φ

AGAIN: ONE MIGHT PREFER TO WRITE D_Ψ !



[10]these statements have to be simplified

[11]In some cases it might be of interest to look at BUPUs which preserve uniform continuity, i.e. which have the property that $SpPsif \in \boldsymbol{C}_{ub}(G)$ for any $f \in CbG$. This is certainly the case if one has a regular BUPU, i.e. a BUPU which is generated from translate of a single, or perhaps a finite collection of "building blocks".

Definition 14. Definition of **dilation operators**:

$$D_\rho f(z) = f(\rho \cdot z),\ \rho > 0,\ z \in \mathbb{R}^d$$

We will allow to apply operators of this kind of operators also to families, i.e. we will shortly write $D_\rho \Phi$ for the family $(D_\rho(T_\lambda \varphi))_{\lambda \in \Lambda}$. Since D_ρ preserves values of functions (it moves the values via stretching or compression to "other places"), hence $D_\rho oneRd = oneRd$. Consequently Φ is a (regular) BUPU if and only if $D_\rho \Phi$ is a BUPU (for some, hence all) $\rho > 0$.

Remark 6. REMARK

In order to understand the engineering terminology of an "impulse response" uniquely describing the bevaviour of a linear time-*invariant* system let us take a quick look at the situation over the group $G = \mathbb{Z}_n$, the cyclic group (of complex unit roots of order n). Obviously $c_0(\mathbb{Z}_n)$ is just $\mathbb{C}^n$, and the (group-) translation is just cyclic index shift (mod n).

Cn-TILS **Lemma 12.** *A matrix A represents a "translation-invariant" linear mapping on $\mathbb{C}^n$ if and only if it is circulant, i.e. if it is constant along side-diagonals (we will also call such matrices "convolution matrices");*

Proof. We can start with the "first unit vector", which is mapped onto some column vector in $\mathbb{C}^n$ by the linear mapping $x \mapsto A * x$. Since we can interpret all further vectors as the image of the other unit vectors, but these are obtained from the first unit vector by cyclic shift, we see immediately that the columns of the matrix are obtained by cyclic shift of the first column, hence the matrix A has to be circulant (in the cyclic sense).

Usually engineers call the first column of this matrix, which is the output corresponding to an "impulse like" input (the first unit vector) the *impulse response of the translation invariant linear system A*.

Since the converse is easily verified we leave it to the interested reader. In fact, it may be interesting to verify the translation invariance by deriving first for a general matrix action A the action of $T_{-1} \circ A \circ T_1$ and to observe subsequently that this operation does not change circulant matrices. □

For the "continuous domain" (i.e., for linear systems over $\mathbb{R}$ resp. over $\mathbb{R}^d$) one has to invoke functionals in order to be able to "represent" the translation invariant systems.

—————————————————- IDEA (to be worked out):
One can first show that the translation invariant systems on $\boldsymbol{C}_0(\mathbb{R}^d)$ are exactly given by "convolution" with bounded measures. Given this, it follows however that the adjoint mapping has to map $\boldsymbol{M}(\mathbb{R}^d)$ into $\boldsymbol{M}(\mathbb{R}^d)$, and has to commute with the adjoint action of translation operators (which can be interpreted as a translation of measures, since it acts on point measures in the expected way: $T_z(\delta_x) = \delta_{x+y}$.

It is also not obvious ?? whether the adjoint action will give a Banach algebra (well, it should!)

================= END of IDEA =======================

mpact-CORd **Theorem 5.** *A bounded and closed subset $M \subset \boldsymbol{C}_0(\mathbb{R}^d)$ is compact if and only if it is uniformly tight and equicontinuous, i.e. if the following conditions are satisfied:*

- *for $\varepsilon > 0$ there exists δ_0 such that $|y| \leq \delta \Rightarrow \|T_y f - f\|_\infty \leq \varepsilon \quad \forall f \in M$;*
- *for $\varepsilon > 0$ there exists some $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ such that $\|f - kf\|_\infty \leq \varepsilon, \quad \forall f \in M$;*

Proof. It is obvious that finite subsets $M \subset \boldsymbol{C}_0(\mathbb{R}^d)$ have these two properties, and it is easy to derive them for compact sets by the usual approximation argument.

So we have to show the converse. We observe first that $\|pf - f\|_\infty \to 0$, if p is a sufficiently large plateau-function. The set $\{pf \mid f \in M\}$ is still equicontinuous (cf. the proof, that $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ is a Banach algebra with respect to pointwise multiplication). We may assume that p has compact support. Then we apply a (sufficiently) fine BUPU to ensure that $\|pf - Sp_\Psi(pf)\|_\infty \leq \varepsilon$. Since p has compact support only finitely many terms make up $Sp_\Psi(pf)$, i.e. one can approximate by finite linear combinations of the elements of Ψ, and the proof is complete (?more details)? $\square$

f-BIBOTLIS **Definition 15.** The Banach space of all "translation invariant linear systems" on $\boldsymbol{C}_0(\mathbb{R}^d)$ is given by [12]

$$\mathcal{H}_G(\boldsymbol{C}_0(\mathbb{R}^d)) = \{T : \boldsymbol{C}_0(\mathbb{R}^d) \to \boldsymbol{C}_0(\mathbb{R}^d), \text{ bounded, linear} : T \circ T_z = T_z \circ T, \forall z \in \mathbb{R}^d\}$$

Remark 7. [13] It is easy to show that $\mathcal{H}_G(\boldsymbol{C}_0(\mathbb{R}^d))$ is a closed subalgebra of the Banach algebra of $\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ (in fact it is even closed with respect to the strong operator topology), hence it is a Banach algebra of its own right (with respect to composition as multiplication). We will see later that it is in fact a commutative Banach algebra.

Definition 16. recall the notion of a FLIP operator: $\check{f}(z) = f(-z)$

Given $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ we define the *convolution operator* C_μ by: $C_\mu(f)(z) := \mu(T_z\check{f})$.

The reverse mapping R recovers a measure $\mu = \mu_T$ from a given translation invariant system T via $\mu(f) := T(\check{f})(0)$.

ation-TLIS **Theorem 6.** *[Characterization of LTISs on $\boldsymbol{C}_0(\mathbb{R}^d)$]*
It is possible to identify the Banach space $\mathcal{H}_G(\boldsymbol{C}_0(\mathbb{R}^d))$ isometrically with $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$, the dual of $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$, by means of the following pair of mappings:

1. *Given a bounded measure $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ we define the operator C_μ (to be called convolution operator with convolution kernel μ later on) via:*
$$C_\mu f(x) = \mu(T_x\check{f}).$$
2. *Conversely we define $T \in \mathcal{H}_G(\boldsymbol{C}_0(\mathbb{R}^d))$ the linear functional $\mu = \mu_T$ by*
$$\mu_T(f) = [T\check{f}](0).$$

The claim is that both of these mappings: $C : \mu \mapsto C_\mu$ and the mapping $T \mapsto \mu_T$ are linear and non-expansive, and inverse to each other, and consequently they constitute an isometric isomorphism between the two Banach spaces.

[12] The letter $\mathcal{H}$ in the definition refers to *homomorphism* [between normed spaces], while the subscript G in the symbol refers to "commuting with the action of the underlying group $G = \mathbb{R}^d$ realized by the so-called regular representation, i.e. via ordinary translations

[13] Sometimes we will write $[T, T_z] \equiv 0$ in order to express the commutation formula using the commutator symbol, and the "$\equiv$"–symbol to express that this relation holds true $\forall z \in \mathbb{R}^d$.

Proof. The proof has a number of partial steps carried out in the course.

First of all it is easy to check that the definition of C_μ really defines an operator on $\boldsymbol{C}_0(\mathbb{R}^d)$, with the output being a bounded function. Since

conv-bd1
$$|C_\mu(f)(x)| \leq \|\mu\|_{\boldsymbol{M}} \|T_x \check{f}\|_\infty = \|\mu\|_{\boldsymbol{M}} \|f\|_\infty, \tag{15}$$

and hence the operator norm of C_μ [14] satisfies $\|C_\mu\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))} \leq \|\mu\|_{\boldsymbol{M}(\mathbb{R}^d)}$.

Furthermore it commutes with translations, since $D_{-1}(T_z f) = T_{-z} D_{-1} f = T_{-z}\check{f}$

nv-commut1
$$C_\mu(T_z f)(x) = \mu(T_x T_{-z}\check{f}) = \mu(T_{x-z}\check{f}) = C_\mu(f)(x-z) = T_z C_\mu f(x). \tag{16}$$

It is also easy to check - using this fact - that (uniform) continuity is preserved, since

conv-cont
$$\|T_z(C_\mu(f)) - C_\mu(f)\|_\infty = \|C\mu(T_z f - f)\| \leq \|\mu\|_{\boldsymbol{M}} \|T_z f - f\|_\infty \to 0 \quad \text{for} \quad |z| \to 0. \tag{17}$$

$\square$

Remark 8. The convolution of two measures (for now the order matters) can be written more or less directly in the following way: Givin $\mu_1, \mu_2 \in \boldsymbol{M}_b(\mathbb{R}^d)$ the action of $\mu_1 * \mu_2$ on $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ is given, in the usual format, by

conv-meas1
$$\mu_1 * \mu_2(f) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y+x) d\mu_2(y) d\mu_1(x). \tag{18}$$

Proof. First recall the definition, i.e. that $\mu 12$ is the linear functional corresponding to $T := C_{\mu_1} \circ C_{\mu_2} \in \mathcal{H}_G(\boldsymbol{C}_0(\mathbb{R}^d))$

$$\mu_1 * \mu_2(f) = T(\check{f})(0) = [(C_{\mu_1} \circ C_{\mu_2})(\check{f})](0) = [C_{\mu_1}(C_{\mu_2}(\check{f})](0) = \mu_1(g)$$

with $g(x) = [C_{\mu_2}(\check{f})]\check{}(x) = C_{\mu_2}(\check{f})(-x) = \mu_2(T_x f)$. Putting everything into the standard notation we have $g(x) = \int_{\mathbb{R}^d} f(y+x) d\mu_2(y)$ which implies the result stated above. $\square$

Although the above result, combined with Fubini's theorem indicates that convolution is *commutative* we do not make use of this fact, because we do not want to invoke results from measures theory at this point. In fact, using an approximation argument (using discrete measures) we can get the result. Looking back we can say however that this is also more or less they way how we prove Fubini, using suitable resummation of suitable Riemannian sums.

The previous characterization allows to introduce in a natural way a Banach algebra structure on $\boldsymbol{M}(\mathbb{R}^d)$. In fact, given μ_1 and μ_2 the translation invariant system $C_{\mu_1} \circ C_{\mu_2}$ is represented by a bounded measure μ. In other words, we can define a new (so-called) *convolution product* $\mu = \mu_1 * \mu_2$ of the two bounded measures such that the relation (completely characterizing the measure $\mu_1 * \mu_2$)

CONV01
$$C_{\mu_1 * \mu_2} = C_{\mu_1} \circ C_{\mu_2} \tag{19}$$

It is immediately clear from this definition that $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ is a Banach algebra with respect to convolution.

[14]Of course we think of the "convolution by the measure μ, in conventional terms, if we write the symbol C.

The translation operators themselves, i.e. T_z are elements of $\mathcal{H}_G(\boldsymbol{C}_0(\mathbb{R}^d))$, which correspond exactly to the Dirac measures $\delta_z, z \in \mathbb{R}^d$.

Since the algebra $\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ is *not* commutative it is not at all clear from this definition why $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_M, *)$ should be a commutative Banach algebra, which is in fact true.

In order to prepare for this statement we have to provide a few more statements.

The compatibility of the (isometric) dilation operators D_ρ with translations, i.e. the rule

dil-trans
$$D_\rho T_z = T_{z/\rho} D_\rho \tag{20}$$

makes it possible to define another norm preserving *automorphism* for the Banach algebra $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_M, *)$, i.e. one has CORRECTION!, BETTER St_ρ

Strhodef **Definition 17.** The *adjoint* action of the group $\mathbb{R}^+$ on $\boldsymbol{M}(\mathbb{R}^d)$ is defined as the family of adjoint operators on $\boldsymbol{M}(\mathbb{R}^d)$ via:

def-Stroh
$$St_\rho \mu(f) := \mu(D_\rho f), \quad \forall f \in \boldsymbol{C}_0(\mathbb{R}^d),\ \rho > 0. \tag{21}$$

Being defined as adjoint operators each of the operators St_ρ is not only isometric on $\boldsymbol{M}(\mathbb{R}^d)$, but also w^*-w^*-continuous on $\boldsymbol{M}(\mathbb{R}^d)$.

Definition 18. ALTERNATIVE DESCRIPTION (replaced later >> theorem): Given $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ the uniquely determined measure corresponding to the operator $D_\rho^{-1} \circ C_\mu \circ D_\rho$ will be denoted by $St_\rho \mu$.

We collect the basic facts for this new mapping:

conv-strho
$$St_\rho \mu_1 * St_\rho \mu_2 = St_\rho(\mu_1 * \mu_2) \tag{22}$$

Proof. To be provided later. It shows that the two alternative definitions above are indeed equivalent!! □

Strho-isom
$$\|St_\rho \mu\|_{\boldsymbol{M}} = \|\mu\|_{\boldsymbol{M}}. \tag{23}$$

$$St_\rho \mu * D_{1/\rho} f = D_{1/\rho}(\mu * f)$$

Stroh-map **Lemma 13.** *(?? Ex. to Def. Strho-isom 23) The mapping $\rho \mapsto St_\rho$ is continuous from $\mathbb{R}^+$ into $\mathcal{L}(\boldsymbol{M}(\mathbb{R}^d))$, endowed with the strong operator topology. It is also true that the mapping $(f, \rho) \mapsto C_{St_\rho \mu}(f)$ is continuous from $\boldsymbol{C}_0(\mathbb{R}^d) \times \mathbb{R}^+$ into $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$.*

Proof. Proof to be typed later on. □

rho-deltas
$$St_\rho \delta_x = \delta_{\rho x}, \tag{24}$$

Remark 9. Due to the w^*-density of finite discrete measures in $\boldsymbol{M}(\mathbb{R}^d)$ one can characterize St_ρ as the uniquely determined norm-to-norm continuous *and* $w^* - w^*$-continuous mapping which is isometric on $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ and satisfies formula (Strho-deltas 24).

A series of lemmata (lemmas) making use of density.

Lemma 14. *Assume a bounded linear mapping between two Banach spaces is given on a dense subset only, then it can be extended in a unique way to a bounded linear operator of equal norm on the full space.*

Proof. It is enough to know a bounded, linear mapping T on a dense subset D of a normed space, in order to observe that for any element v in the domain of T one has a convergent sequence $(d_n)_{n\in\mathbb{N}}$ in D with limit x. Hence $(d_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, so that the boundedness of T implies that $(Td_n)_{n\in\mathbb{N}}$ is a again a Cauchy sequence in the range, hence (by completeness) has a limit. The uniqueness of this extension and the fact that this extension has the same norm, i.e. that

$$|||T||| := \|T\|_{Op} = sup_{\|d\|\leq 1}\|Td\|$$

is also an immediate consequence. □

Lemma 15. Test of BAI on a dense subspace:
A bounded family $(e_\alpha)_{\alpha\in I}$ in a Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is a BAI for $\boldsymbol{A}$ if (only) for some total subset $D \subseteq \boldsymbol{A}$ one has:

$$\|e_\alpha \cdot d - d\|_{\boldsymbol{A}} \to 0 \quad \forall d \in D.$$

Proof. It is easy to derive - using the properties of a net - that one has (uniform) convergence for finite linear combinations of elements from D, and then by a density argument one verifies convergence for all elements. [15] □

doubnetlim **Lemma 16.** *A statement about iterated bounded, strongly convergent nets of operators.*

Assume that two bounded nets of operators between normed spaces, $(T_alpha)_{\alpha\in I}$ and $(S_\beta)_{\beta\in J}$ are strongly convergent to some limit operators T_0 and S_0 respectively. Then the iterated limit of **any order** *exists and the two limits are equal.*

In fact, the index set $(\alpha,\beta) \in I \times J$ with the natural order [16] is turning $(S_\beta \circ T_\alpha)$ into a strongly convergent net.

Proof. . . . Without loss of generality we may assume $T_0 = 0$ and $S_0 = 0$ (otherwise treat $T_\alpha - T_0$ etc.). THE REST OF THE PROOF is LEFT TO THE READER.

□

Lemma 17. One can choose the BAI elements from a dense SUBSPACE .
Let $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ be a Banach algebra with a bounded approximate unit, and D some dense subset of $\boldsymbol{A}$. Then there exists also approximate units $(d_\alpha)_{\alpha\in I}$ in D (if D is a dense subspace the new family $(d_\alpha)_{\alpha\in I}$ can even be chosen to be of equal norm).

Proof. One just has to choose d_α close enough to e_α, especially for α large enough (to be expressed properly). By the density of D one can do this. If D is a subspace (not just a subset) one can renormalize the new elements so that they have the same norm in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ as the original elements $(e_\alpha)_{\alpha\in I}$. □

[15] Of course this is more or less a statement about strong operator convergence for a net of bounded operators.

[16] explanation: the natural ordering is given by the fact that $(\alpha,\beta) \succeq (\alpha_0,\beta_0)$ if $\alpha \succeq \alpha_0$ and $\beta \succeq \beta_0$.

Lemma 18. Uniform action of BAIs on compact subsets
By the definition a family $(e_\alpha)_{\alpha\in I}$ acts pointwise like an identity in the limit case, for each "point". However, the action is even uniformly over finite sets, and hence over compact sets, by approximation.

Proof. That one has uniform convergence over finite subsets is easily verified, using the property of a net. The inductive step is based on the following argument: Assume that one has found α_0 such that

$$\|e_\alpha \cdot a_i - a_i\| \leq \varepsilon \quad \forall \alpha \succ \alpha_0,\, 1 \leq i \leq m.$$

Since $\|e_\alpha \cdot a_{m+1} - a_{m+1}\| \leq \varepsilon$ for all $\alpha \succ \alpha_1$ we just have to choose some index α_2 with $\alpha_2 \succ \alpha_0$ *and* $\alpha_2 \succ \alpha_1$ (which is possible due to the definition of directed sets). Obviously

$$\|e_\alpha \cdot a_i - a_i\| \leq \varepsilon \quad \forall \alpha \succ \alpha_2,\, 1 \leq i \leq m+1.$$

In order to come up with uniform convergence over compact sets, we use again a typical approximation argument. Given any compact set $M \subseteq \boldsymbol{C}_0(\mathbb{R}^d)$ and $\varepsilon > 0$ we have to find some index α_3 such that

$$\|e_\alpha \cdot a - a\| \leq \varepsilon \quad \forall a \in M.$$

Recalling that $(e_\alpha)_{\alpha\in I}$ is bounded, i.e. $\|e_\alpha\| \leq C$ for some $C \geq 1$ for all $\alpha \in I$, we may choose some finite subset $F \subseteq M$ such that for any $a \in M$ there exists $a_i \in F$ with $\|a_i - a\| \leq \varepsilon/(3C) > 0$ (which is just another positive constant, known once we know C and ε). Hence for any given $a \in M$ one can use one of such elements $a_i \in F$ in order to argue that the triangular inequality implies (adding and subtracting the term $e_\alpha \cdot a_i$).

$$\|e_\alpha \cdot a - a\| \leq \|e_\alpha \cdot (a - a_i)\| + \|e_\alpha \cdot a_i - a_i\| + \|a_i - a\|.$$

If we choose now α_3 such that $\|e_\alpha \cdot a_i - a_i\| \leq \varepsilon/3 \quad \forall \alpha \succ \alpha_3$ and $a_i \in F$ we obtain altogether (more details are left to the reader, ...),

$$\|e_\alpha \cdot a - a\| \leq \varepsilon$$

□

Lemma 19. *The following properties are equivalent.*
- *there is a bounded approximate identity in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$;*
- *there exists $C > 0$ such that for every finite subset $F \in \boldsymbol{A}$ and $\varepsilon > 0$ there exists element $h \in \boldsymbol{A}$ with $\|h\| \leq C$, such that*

$$\|h \cdot a - a\| \leq \varepsilon \quad \forall a \in F.$$

The argument to turn this family into a bounded net is the obvious one. One just has to set $\alpha := (F, \varepsilon)$, and defining such a pair "stronger than another pair $\alpha_1 := (F_1, \varepsilon_1)$ if $F_1 \supseteq F$ and $\varepsilon_1 \leq \varepsilon$. It is easy to verify that this defines a directed set, and that the choice $e_\alpha = h$ (corresponding to the pair (K, ε) as describe above) turn $(e_\alpha)_{\alpha\in I}$ into a bounded and convergent net, hence constitutes a BAI for $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$.

Note: In words: The existence of a BAI is equivalent to the existence of a bounded net in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ such that the action of "pointwise multiplication" (each element e_α is identified with the left algebra multiplication operator $a \mapsto e_\alpha \cdot a$) is convergent to the identity operator in the strong operator norm topology (which is just the pointwise convergence of operators).

Note: Sometimes one observes that one has unbounded approximate identities, where the "cost" (i.e. the norm of e_α grows as the required quality of approximation, expressed by the smallness of ε), tends to the ideal limit zero. It makes sense to think of limited costs (given by $C > 0$) for arbitrary good quality of approximation which makes the BAI so useful.

Theorem 7. *(The Cohen-Hewing factorization theorem, without proof, see* hero70 *[13])*
Let $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ be a Banach algebra with some BAI, then the algebra factorizes, which means that for every $a \in \boldsymbol{A}$ there exists a pair $a', h' \in \boldsymbol{A}$ such that $a = h' \cdot a'$, in short: $\boldsymbol{A} = \boldsymbol{A} \cdot \boldsymbol{A}$. In fact, one can even choose $\|a - a'\| \leq \varepsilon$ and $\|h'\| \leq C$.

There is a more general result, involving the terminology of **Banach modules**.

BanMod **Definition 19.** A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a *Banach module* over a Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ if one has a bilinear mapping $(a, b) \mapsto a \bullet b$, from $\boldsymbol{A} \times \boldsymbol{B}$ into $\boldsymbol{B}$ with

$$\|a \bullet b\|_{\boldsymbol{B}} \leq \|a\|_{\boldsymbol{A}} \|b\|_{\boldsymbol{B}} \quad \forall\, a \in \boldsymbol{A}, b \in \boldsymbol{B}$$

which behaves like an ordinary multiplication, i.e. is associative, distributive, etc.:

$$a_1 \bullet (a_2 \bullet b) = (a_1 \cdot a_2) \bullet b \quad \forall a_1, a_2 \in \boldsymbol{A}, b \in \boldsymbol{B}.$$

Ex1Banmod **Lemma 20.** *(Ex. to Def.* BanMod *19) A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is an (abstract) Banach module over a Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$* [17] *if and only if there is a non-expansive (hence continuous) linear algebra homomorphism J from $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ into $\mathcal{L}(\boldsymbol{B})$.*

Proof. For a Banach module the mapping: $a \mapsto J(a) : [b \mapsto a \bullet b]$ defines a linear mapping from $\boldsymbol{A}$ into $\mathcal{L}(\boldsymbol{B})$.

Conversely, one can define a $\boldsymbol{A}$-Banach module structure on $\boldsymbol{B}$ by the definition: $a \bullet b := J(a)(b)$.

Without going into all necessary details let as recall that the associativity law

$$a_1 \bullet (a_2 \bullet b) = (a_1 \cdot a_2) \bullet b \quad \forall a_1, a_2 \in \boldsymbol{A}, b \in \boldsymbol{B}.$$

It is a consequence of the homomorphism property of J: The left hand side $a_1 \bullet (a_2 \bullet b)$ translates into $J(a_1)[J(a_2)b]$, while the right hand side equals $J(a_1 \cdot a_2)(b)$. □

ess-Banmod **Lemma 21.** *A Banach module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ over some Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is called essential if it coincides with the closed linear span of $\boldsymbol{A} \bullet \boldsymbol{B} = \{a \bullet b \,|\, a \in \boldsymbol{A}, b \in \boldsymbol{B}\}$.*

For a general Banach module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ the closed linear span of $\boldsymbol{A} \bullet \boldsymbol{B}$ is denoted by $\boldsymbol{B}_e$ or $\boldsymbol{B}_{\boldsymbol{A}}$ (especially if there are different Banach algebras acting on the same space).

The notion of "essential Banach modules" is of course trivial in case the Banach algebra $\boldsymbol{A}$ has a unit element which is mapped into the identity operator, i.e. if there exists $u \in \boldsymbol{A}$ such that $u \cdot a = a \,\forall a \in \boldsymbol{A}$ and also $u \bullet b = b$, $\forall b \in \boldsymbol{B}$.

[17] $\boldsymbol{A}$ may be commutative or non-commutative, with our without unit.

Lemma 22. *Let $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ be a Banach algebra with BAIs $(e_\alpha)_{\alpha\in I}$. Then a Banach module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is essential if and only if*

essmod-BAI (25)
$$\|e_\alpha \bullet b - b\|_{\boldsymbol{B}} \to 0 \quad \forall b \in \boldsymbol{B}$$

In particular, relation essmod-BAI (25) holds true for one such BAI if and only if it is true for every BAI in $\boldsymbol{A}$.

dual-alg **Definition 20.** For any Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ the dual space can be turned naturally into a $\boldsymbol{A}$-Banach module via the action

dual-alg1 (26)
$$[a_1 \bullet \sigma](a) := \sigma(a_1 \cdot a), \quad \forall a, a_1 \in \boldsymbol{A}, \sigma \in \boldsymbol{A}'.$$

Lemma 23. *(Ex. to Def. dual-alg (20) $h\delta_x = h(x)\delta_x$, i.e. multiplication of a Dirac measure is realized as scalar multiplication of this Dirac measure by the point value of the continuous function h at that point.*

For *convenience* we will write $h \cdot \mu$ instead of the abstract symbol $\bullet$ in order to indicate pointwise multiplication between functionals μ and functions h.

Theorem 8. *The Banach space $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ is an essential Banach module over $\boldsymbol{C}_0(\mathbb{R}^d)$ with respect to the natural (dual) action of pointwise multiplication.*

Proof. In fact we will show much more: for any BUPU Φ one has

$$\mu = \sum_{i\in I} \phi_i \mu$$

as absolutely convergent sum in $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. $\square$

Lemma 24. *Let $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ and $\varepsilon > 0$ be given. Then there exists $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ with $\|k\|_\infty \le 1$ such that $\mu(k) \ge 0$ and $\mu(k) \ge \|\mu\|_{\boldsymbol{M}}(1-\varepsilon)$.*

Proof. $\square$

Corollary 2. *Every $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ is the limit of "compactly supported" measures of the form $\sum_{i\in F}(\phi_i \cdot \mu)$, where F is running through the finite subsets of I (which is typically countable).*

In the terminology of Banach modules we can restate the last corollary in the form:

Corollary 3. *The dual space to $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$, i.e. $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$, is an essential Banach module over the Banach algebra $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ with respect to the action of pointwise multiplication.*

We will derive therefrom that one can also "integrate" arbitrary elements $h \in \boldsymbol{C}_b(\mathbb{R}^d)$.

Lemma 25. Integration of bounded functions against bounded measures
For any $h \in \boldsymbol{C}_b(\mathbb{R}^d)$ and $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ the net $\mu(p_K \cdot h) = p_K \cdot \mu(h)$ is a Cauchy-net in $\mathbb{C}$. Therefore it makes sense to define $\mu(h) = \lim_K \mu(p_K \cdot h)$. It is clear that in this way μ extends in a unique way to a bounded linear functional on $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$, and that the norm of this extension equals $\|\mu\|_{\boldsymbol{M}}$.

Remark 10. If the net of bounded measures is (bounded and) tight, then it is an exercise to show, that it is *vaguely convergent* [18] if and only if it is w^*-convergent, resp. if and only if $\mu_\alpha(h) \to \mu_0(h)$ $\forall h \in \boldsymbol{C}_b(\mathbb{R}^d)$. It is a bit more work (but still an exercise) to check out that this "pointwise convergence" even takes place *uniformly over compact subsets* of $\boldsymbol{C}_b(\mathbb{R}^d)$.

Remark 11. (MAYBE MISPLACED REMARK)
Recall that the "mass-preserving" stretching/compression operator St_ρ can be extended to $\boldsymbol{M}_b(\mathbb{R}^d)$ by the definition

$$\mathrm{St}_\rho\,\mu(f) := \mu(D_\rho f).$$

Check that one has $\mathrm{St}_\rho\,\delta_x = \delta_{\rho x}$, and that $lim_{\rho\to 0}\,\mathrm{St}_\rho\,\mu = \mu(1)\delta_0$. In fact, for each $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ one has: $D_\rho f$ is uniformly bounded with respect the sup-norm for $\rho \to 0$ one has: $f(x) \to f(0)$, uniformly over compact sets. Since we can approximate the measure μ (by localizing it) to a measure with compact support its action is defined on sufficiently large compact sets, where $D_\rho f$ is like the constant function $f(0) = \delta_0(f)$, while the action on $Const \equiv 1$ is denoted by $\mu(1)$.

One can use this fact to find out that it is even possible to extend the convolution operators $f \mapsto C_\mu f$ to all of $\boldsymbol{C}_b(\mathbb{R}^d)$ (still with the equality of operator norm on $\big(\boldsymbol{C}_b(\mathbb{R}^d),\,\|\cdot\|_\infty\big)$ with the functional norm of μ), and with the property that the operators arising in such a way commute with translations. However, by means of the Hahn-Banach theorem one can construct translation invariant *means* on $\big(\boldsymbol{C}_b(\mathbb{R}^d),\,\|\cdot\|_\infty\big)$ which in turn allow to construct bounded linear operators on $\boldsymbol{C}_b(\mathbb{R}^d)$ which commute with all the translation operators without being of the form of a convolution by some bounded measure. In fact, those operators are non-zero operators on $\boldsymbol{C}_b(\mathbb{R}^d)$, but they map all of $\boldsymbol{C}_0(\mathbb{R}^d)$ onto the zero function. It is also not much more than a simple exercise to find out (using the characterization of $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ within $\boldsymbol{C}_b(\mathbb{R}^d)$ given early on) to check that any operator on $\boldsymbol{C}_b(\mathbb{R}^d)$ commuting with translations will map $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ into itself (in fact, this argument was used at the beginning of the identification theorem.)

We are now in the position to define the **Fourier transform** of a bounded measure. In the classical literature this is often referred to as the Fourier-Stieltjes transform of a measure, because it can be carried out technically over $\mathbb{R}$ using Riemann-Stieltjes integrals. Such a R-St-integral is the difference of two R-St-integrals with respect to bounded, non-decreasing "distribution" functions F. So in such a definition the ordinary Riemanian sum is replaced by an sum of the same form, but instead of the "natural length" of the interval $[a, b]$, which is $|b - a|$, one uses the length in the sense of F which is $F(b) - F(a)$.

Definition 21. A *character* is a continuous function from a topological group into the torus group $\mathbb{T} = \{z \in \mathbb{C} \,|\, |z| = 1\}$. In other words, χ is a character if $\chi(x+y) = \chi(x)\cdot\chi(y)$ for all $x, y \in G$. Moreover, since $|\chi(x)| = 1$ one has $\overline{\chi(x)} = 1/\chi(x)$ for all $x \in G$.

[18]This means that it is convergent in the $\sigma(\boldsymbol{M}_b(\mathbb{R}^d), \boldsymbol{C}_c(\mathbb{R}^d))$-topology, resp. $\mu_\alpha(k) \to \mu_0(k)$ $\forall k \in \boldsymbol{C}_c(\mathbb{R}^d)$.

Definition 22. The set of all character is called the *dual group*, because those characters form an Abelian group under pointwise multiplication (!Exercise!). We write $\hat{G}$ for the dual group corresponding to G (the group law written as addition in this case).

Theorem 9. *In the case of* $(\mathbb{R}^d, +)$ *the dual group consists of the characters* χ *of the form* $\chi_s : t \mapsto exp(2\pi i s \cdot t)$, *with* $s \in \mathbb{R}^d$. *Due to the exponential law the pointwise multiplication of characters turns into addition of the parameters* s *(describing the "frequency content" of* χ_s*).*

Definition 23. The Fourier transform of $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ is defined by

eas-FT-def
$$\hat{\mu}(s) = \mu(\chi_s) \tag{27}$$

Lemma 26. *The Fourier transform is a linear and non-expansive mapping from* $\boldsymbol{M}_b(\mathbb{R}^d)$ *into* $\big(\boldsymbol{C}_{ub}(\mathbb{R}^d), \|\cdot\|_\infty\big)$.

Proof. The uniform continuity results from the essential concentration of bounded measures over compact sets, the "usual rule" $T_s \mathcal{F}(\mu) = \mathcal{F}(M_s\mu)$., and the properties of characters.

In fact, given $\varepsilon > 0$ (alternative typing: $\varepsilon > 0$, and $\chi_0 \in \hat{G}$, we can find a compact subset $Q \subseteq G$ such that $\|\mu - \psi_Q\mu\|_{\boldsymbol{M}} < \varepsilon/3$. Hence there is a neighborhood W of the identity in $\hat{G}$ such that for all $\chi \in \chi_0 + W$ one has $|\chi(y) - \chi_0(y)| = |\chi(y)/chi_0 - 1| < \varepsilon/3$ for all $y \in Q$, by the definition of neighborhoods in $\hat{G}$ (compact open topology), and the fact that the quotient χ/χ_0 belongs to W. Expressed differently we have $\|\psi_Q(\chi_0 - \chi)\|_\infty < \varepsilon/3$. Altogether we have

$$|\hat{\mu}(\chi_0) - \hat{\mu}(\chi)| \leq |\,\mathcal{F}(\mu - \mu\psi_Q)(\chi_0)| + |\,\mathcal{F}(\mu\psi_Q)(\chi_0 - \chi)| + |\,\mathcal{F}(\mu\psi_Q - \mu)(\chi)|$$

and consequently

$$|\hat{\mu}(\chi_0) - \hat{\mu}(\chi)| \leq 2\|\mu - \mu\psi_Q\|_{\boldsymbol{M}} + \varepsilon/3 \leq \varepsilon.$$

□

Although little can be said about the connection between w^*-convergence (in $\boldsymbol{M}(\mathbb{R}^d)$) and pointwise convergence "on the Fourier transform side" in general one has the following useful fact:

wsttoFOUR1 **Lemma 27.** *Let* (μ_α) *be a* w^**-convergent and tight net in* $\boldsymbol{M}_b(\mathcal{G})$, *with* $\mu_0 = w^* - lim_\alpha \mu_\alpha$. *Then we have* $\widehat{\mu_\alpha}(s) \to \widehat{\mu_0}(s)$, *uniformly over compact subsets of* $\widehat{G}$.

Proof. Pointwise convergence of the Fourier transform is a consequence of □

XXXX

Theorem 10. *The FT on* $\boldsymbol{M}_b(\mathbb{R}^d)$ *is injective, and turns convolution into pointwise multiplication, i.e. in fact, it is a homomorphism of Banach algebras. This also implies that convolution is commutative (because obviously pointwise multiplication is a commutative operation).*

Proof. Since $\mu \mapsto \hat{\mu}$ is a linear mapping it is enough to show that $\hat{\mu}(s) = 0$ for all $s \in \mathbb{R}^d$ implies $\mu = 0$. If $\mu(s) \equiv 0$ we can conclude (by linearity) that μ applied to any trigonometric polynomial $p(t) = \sum_k c_k \chi_{s_k}(t)$ is equal to zero, i.e. $\mu(p) = 0$. Etc.

Assume that $\mu \neq 0$. Then there exists some $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ such that $\mu(k) = 1$ (without loss of generality). Since k has compact support we can write $k = h \cdot k$, for some other $h \in \boldsymbol{C}_c(\mathbb{R}^d)$, or equivalently to claim that $h\mu(k) = 1$.

Since $\mathrm{supp}(h)$ is compact, it follows from the Theorem of (Stone)-Weierstrass (density of trigonometric polynomials, with respect to the sup-norm, over a compact set [19]) we can find a trigonometric polynomial such that

$$\|h \cdot p - h \cdot k\|_\infty < \delta,$$

and $\|p\|_\infty \leq 2\|k\|_\infty$. But this implies

$$1 = |(h\mu)(k)| \leq |(h\mu)(p)| + |\mu(hk) - \mu(hp)| \leq \|h\mu - \mu\|_{\boldsymbol{M}}\|p\|_\infty + |\mu(p)| + \delta\|\mu\|_{\boldsymbol{M}} < \varepsilon'$$

for any given $\varepsilon' > 0$, since $\mu(p) = 0$ and because (by an appropriate choice of h) one can make $\|h\mu - \mu\|_{\boldsymbol{M}}$ arbitrarily small.

ETCC.

The compatibility with convolution is an easy exercise for discrete measures, and can be transferred to the general case using a weak-star argument. Recall again that w^*-convergence of bounded nets of measures implies pointwise convergence of their Fourier transforms. □

There is an alternative way of proving commutativity of convolution. It is easy to see that the convolution of (finite) discrete measures is commutative, and the general case follows from this (by approximation in the strong operator topology).

Material on Banach Modules

The Banach module is called "true" if the mapping J described above is injective.

If one only has a continuous (but not necessarily non-expansive algebra homomorphism J) one can replace the norm on $\boldsymbol{A}$ by another equivalent norm (just some constant multiple of the original one) in order to ensure this (harmless) extra property.

Recall the notions of *weak topology* on any Banach space (such as $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$, and the *$w^*$-topology*) on any dual space, such as $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$.

Theorem 11. *A sequence (or indeed a bounded net) of functions in* $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ *is weakly convergent if and only if it is pointwise convergent (while in contrast norm-convergence means uniform convergence over* $\mathbb{R}^d$*).*

Proof. Since the Dirac measures are specific linear functionals on $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ weak convergence of a sequence (f_n) in $\boldsymbol{C}_0(\mathbb{R}^d)$ implies $f_n(x) = \delta_x(f_n) \to \delta_x(f_0) = f_0(x)$ for any $x \in \mathbb{R}^d$. Conversely, the possibility of approximating a general measure in a bounded way by linear combinations of Dirac measures implies that pointwise convergence indeed implies weak convergence. If one goes into the details of the proof the boundedness of

[19] This is a consequence of the fact that the trigonometric polynomials (restricted to any compact set) form an algebra of continuous functions, which is closed under complex conjugation, contains the constants, and is point separating, i.e. for $x_1 \neq x_2$ there exist trig. polynomials $p(x)$ such that $p(x_1) \neq p(x_2)$.

the set of approximating measures as well as the boundedness of the sequence (resp. a net (f_α)

□

Remark 12. For *equicontinuous* families one can show that weak (or pointwise) convergence is equivalent to "uniform convergence over compact set". A (?bounded) net (f_α) is weakly convergent if and only if it is UCOCS, etc. dots

Remark 13. How can we characterize w^*-convergence in $\boldsymbol{M}(\mathbb{R}^d)$?

Alternative description of "multipliers" on $\boldsymbol{C}_0(\mathbb{R}^d)$ resp. translation invariant BIBOS (bounded input bounded output systems, with the property of mapping $\boldsymbol{C}_0(\mathbb{R}^d)$ into itself).

terizeHGCO **Theorem 12.** *Let H be any w^*- total subset of $\boldsymbol{M}(\mathbb{R}^d)$, and assume that a bounded linear operator $T \in \mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ commutes with the action of H, i.e., that the commutators $[C_h, T] \equiv 0$ for all $h \in H$. Then $T \in \mathcal{H}_G(\boldsymbol{C}_0(\mathbb{R}^d))$.*

Note that in the original definition the set H was just the set of convolution operators by Dirac measures $\delta_x, x \in \mathbb{R}^d$ (or at least from some dense subset).

UPCOMING MATERIAL:

Embedding of test functions into $\boldsymbol{M}(\mathbb{R}^d)$ (over groups this requires the use of the [invariant] Haar measure, which indeed is a linear functional on $\boldsymbol{C}_c(G)$). Compatibility of operators which are now available on both the functions and the measures (resp. functionals). E.g. we can now do an internal convolution of functions (viewed as bounded measures) or an external action (one is acting as a bounded measure, the other is consider as the $\boldsymbol{C}_0(\mathbb{R}^d)$ element on which the action takes place). Associativity of convolution in the most general situation (also of course **commutativity**, etc.).

Further notes:

$\boldsymbol{C}_{ub}(\mathbb{R}^d)$ is a (closed) subspace of the dual of $\boldsymbol{L}^1(\mathbb{R}^d)$. Hence it carries a $\sigma(\boldsymbol{C}_{ub}(\mathbb{R}^d), \boldsymbol{L}^1(\mathbb{R}^d))$ topology which can be shown to be equivalent (at least on bounded sets!?, or more) to the uniform convergence over compact sets (?true).

STATEMENT: Every $f \in \boldsymbol{C}_b(\mathbb{R}^d)$ is a limit (in the sense of uniform limit over compact sets) of a *bounded* sequence of functions from $\boldsymbol{C}_0(\mathbb{R}^d)$ resp. even from $\boldsymbol{C}_c(\mathbb{R}^d)$. In fact, on can take the sequence $p_n \cdot f$, where (p_n) is a BAI for $\boldsymbol{C}_0(\mathbb{R}^d)$ consisting of (increasing) plateau functions).

EXTENSION PRINCIPLE. Let (p_n) be as above, and $f \in \boldsymbol{C}_b(\mathbb{R}^d)$ and $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ be given. Then the sequence $\mu(p_n \cdot f)$ is a Cauchy sequence, hence convergent in $\mathbb{C}$. In fact, the limit is the same for any other BAI in $\boldsymbol{C}_0(\mathbb{R}^d)$. Therefore it makes sense to define $\mu(f) := lim_{n\to\infty} \mu(p_n f)$.

REMARK: this will be important to define the Fourier Stieltjes transforms for bounded measures, i.e. for $\hat{\mu}(s) = \mu(\chi_{-s})$ later on!

Lemma 28. *The convolution operators form a (commutative) Banach algebra of operators. It turns out that the characters can be identified with the joint eigenvectors for this whole class of operators. Indeed, we have $C_\mu(\chi_s) = \hat{\mu}(s)\chi_s$ (RICHTIG?) for any $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ and any character χ_s on $\mathbb{R}^d$.*

3. Identifying "ordinary functions with functionals"

There is a natural way to identify "ordinary functions" (say $k \in \boldsymbol{C}_c(\mathbb{R}^d)$) with linear functions $\mu \in \boldsymbol{M}(\mathbb{R}^d)$, by the following trick: Given

ttomeasure

$$\mu = \mu_k, \qquad \text{resp.} \qquad \mu(f) = \int_{\mathbb{R}^d} f(x)k(x)dx \tag{28}$$

This is also possible over general locally compact Abelian groups, but requires the existence of the Haar measure (we will not go into this direction, a good explanation is given in Deitmar's book).

Lemma 29. *The mapping $k \to \mu_k$ described above defines an isometric embedding from $(\boldsymbol{C}_c(\mathbb{R}^d), \|\cdot\|_1)$ into $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. Hence we may identify the closure of $\boldsymbol{M}_{C_c} = \{\mu_k \,|\, k \in \boldsymbol{C}_c(\mathbb{R}^d)\}$ with the completion of the normed space $(\boldsymbol{C}_c(\mathbb{R}^d), \|\cdot\|_1)$.*

Proposition 1. *There is a natural,* isometric *embedding of $(\boldsymbol{C}_c(G), \|\cdot\|_1)$ into $\boldsymbol{M}_b(G)$, given by*

$$k \mapsto \mu_k : \mu_k(f) = \int_G f(x)k(x)dx.$$ [20]

Proof. It is obvious that each μ_k is in fact a bounded linear functional on $\boldsymbol{C}_0'(G)$ and that the mapping $k \mapsto \mu_k$ is linear and nonexpansive, since evidently for each $f \in \boldsymbol{C}_0(G), k \in \boldsymbol{C}_c(G)$ one has:

$$|\mu_k(f)| = |\int_G f(x)k(x)dx| \le \int_G |f(x)||k(x)|dx \le \|f\|_\infty \|k\|_1.$$

The converse is a bit more involved. In principle one would choose, for the case of a real-valued function k a function $f \in \boldsymbol{C}_0(G)$ which is a minimal (but continuous!) modification of the signum function, which turns (when integrated against k) the negative parts into positive parts, thus turning $\mu_k(f)$ in a good approximation of $\|k\|_1$.

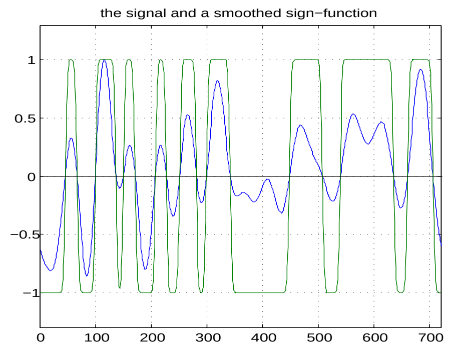
To be more formal let us consider $f \in \boldsymbol{C}_c(G)$, let us consider for any $\eta > 0$ the "essential" support $K_\eta := \{z \in G \,|\, |k(z)| \ge \eta\}$. Then K_η is a compact set and we can find continuous function [21] h_η with values in $[0,1]$ such that $h_\eta(z) = 1$ on K_η and with support of h_η within (the interior) of $K_{\eta/2}$. The function $f_\eta(x) := h_\eta(x)|k(x)|/k(x)$ is then well defined (because $k(x) \neq 0$ for any point in the support of h_η, and $\|f_\eta\|_\infty \le 1$). We observe that

$$\int_G f_\eta(x)k(x)dx = \int_G |k(x)|h_\eta(x)dx.$$

It remains to verify that this tends to $\|k\|_1$ for $\eta \to 0$.

[20] The integration is with respect to the Haar measure on the group G.

[21] The existence of h_η is guaranteed by Tietze's theorem, on of the important theorems concerning locally compact, hence completely regular topological spaces. It helps to avoid the potential problem of a phase discontinuity, i.e. problems with the continuity of $k(x)/|k(x)|$ near the zeros of k.



Writing K_0 for supp(k) this is a direct consequence of the following estimate

$$\int_G |k(x)(1-h_\eta(x))| \leq \int_{K_0} |k(x)|(1-h_\eta(x))dx \leq Vol(K_0)^{22} \cdot \|k(1-h_\eta)\|_\infty \to 0.$$

□

Li-Def **Definition 24.** We define $\boldsymbol{L}^1(\mathbb{R}^d)$ as the closure of $\boldsymbol{M}_{C_c}$ within $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$.

Of course one has to justify this definition, by recalling that the usual definition of $\boldsymbol{L}^1(\mathbb{R}^d)$ based on Lebesgue's integrability criterion provides us with a Banach space (of equivalence classes of measurable functions, identifying two functions if they are equal almost everywhere), which contains $\boldsymbol{C}_c(\mathbb{R}^d)$ as a dense subspace (cf. more or less any book on measure theory for details on this matter: in fact, it is sufficient to approximate - in the $\boldsymbol{L}^1$-norm - indicator functions of parallel-epipeds by continuous functions with compact support, i.e. something like a trapezoidal function sufficiently close to a "box-car"-function in the 1-dimensional case).

It is also of interest to introduce **the concept of a support to measures**, in a way which is compatible with the notion supp(k) for $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ given above:

[22] $Vol(K_0)$ stands for the Haar measure of the set K_0, but the $\|\cdot\|_1$ of a plateau-function $p(x)$ with p with $p(x) \cdot k(x) = k(x)$ would do. In fact, the "measure of K_0, i.e. $Vol(K_0) = \mu(K_0)$ can be shown to be equal to the infimum over all those $\|\cdot\|_1$-norms.

as-support **Definition 25.** A point x does *not* belongs to the support of a measure $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ if there exists some $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ with $k(x) = 1$, but nevertheless $k \cdot \mu \neq 0$. The complement of this set is denoted by $\mathrm{supp}(\mu)$.

Lemma 30. .

- $\mathrm{supp}(\mu)$ *is a closed subset of* $\mathbb{R}^d$,
- *there is consistency with the concept already defined for* $k \in \boldsymbol{C}_c(\mathbb{R}^d)$, *in other words:* $\mathrm{supp}(k)$ *(in the old sense) coincides with* $\mathrm{supp}(\mu_k)$, *just defined.*
- *For a discrete measure the support is given by the closure of the union of all points involved, i.e., for* $\mu = \sum_{k=1}^{\infty} c_k \delta_{t_k}$ *we have*[23] $\mathrm{supp}(\mu) = (\bigcup_k t_k)^-$.
- *The notion of support is compatible with pointwise products: i.e., for any* $h \in \boldsymbol{C}_b(\mathbb{R}^d)$ *on has* $\mathrm{supp}(h\mu) \subseteq \mathrm{supp}(h) \cap \mathrm{supp}(\mu)$ *(as for functions).*

We have the following equivalent description of the $\mathrm{supp}(\mu)$ (which is actually the usual definition):

as-charact **Lemma 31.** *The following properties are equivalent:*

- $z \in \mathrm{supp}(\mu)$;
- *for any* $\varepsilon > 0$ *there exists some* $h \in \boldsymbol{C}_c(\mathbb{R}^d)$ *with* $\mathrm{supp}(h) \subseteq B_\epsilon(z)$ *with* $\mu(h) \neq 0$.
- $\mathrm{supp}(\mu)$ *coincides with the intersection of all supports of [plateau-]functions p such that* $p\mu = \mu$.

The following results indicates the w^*-continuity of the concept of a support.

Lemma 32. *Assume that* $\mu_0 = w^* - lim_\alpha \mu_\alpha$, *then* $\mathrm{supp}(\mu_0) \subseteq \bigcap_\alpha \mathrm{supp}(\mu_\alpha)$

The notion of support is also compatible with convolution: [24]

nv-support **Lemma 33.** *For* $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ *and* $f \in \boldsymbol{C}_b(\mathbb{R}^d)$ *one has*

conv-supp
$$\mathrm{supp}(\mu * f) \subseteq \mathrm{supp}(\mu) + \mathrm{supp}(f). \tag{29}$$

Lemma 34. *Assume that* $\mu_0 = lim_{w^*} \mu_\alpha$. *Then also for any BUPU* Ψ *the family* $D_\Phi \mu_\alpha$ *is* w^**-convergent to* $D_\Phi \mu_0$. *Even more, the family* $D_\Phi \mu_\alpha$ *is uniformly tight and* w^**-convergent to* μ_0 *as* $|\Phi| \to 0$.

Finally we claim that the family $D_\Phi(p_K \mu_\alpha)$, *where* p_K *runs through the family of all plateau functions (with* $K \to \mathbb{R}^d$*), satisfies the same relation. Note that the resulting measures are in fact FINITE discrete measures.*[25]

[23] Assuming of course the canonical representation of μ, with $t_k \neq t_l$, for $k \neq l$.

[24] We probably still have to take care of the notion of support for the case that the measure does not have compact support.

[25] Alternatively one could use only finite subfamilies from the partition of unity, or put p_k on the outside, i.e. write $p_k \cdot D_\Phi \mu_\alpha$. The consequences remain the same for all of these variants!

4. Basic properties of $\left(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\right)$

We have defined $\boldsymbol{L}^1(\mathbb{R}^d)$ as the closure of $\boldsymbol{C}_c(\mathbb{R}^d)$ (identified via $k \mapsto \mu_k$ with a subspace of $\boldsymbol{M}(\mathbb{R}^d)$) in $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. Since this is a Banach space, it is a Banach space itself, and identical with the (abstract) completion of $\boldsymbol{C}_c(\mathbb{R}^d)$ in $\boldsymbol{M}(\mathbb{R}^d)$.

The next theorem gives us some more information about the containment of $\boldsymbol{L}^1(\mathbb{R}^d)$ in $\boldsymbol{M}(\mathbb{R}^d)$:

L1-Basic **Theorem 13.** *(Basic properties of $\boldsymbol{L}^1(\mathbb{R}^d)$)*

- $\left(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\right)$ *is a closed ideal within $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. It is a Banach algebra with a BAI (so-called Dirac sequences).*
- $\boldsymbol{L}^1(\mathbb{R}^d)$ *can be characterized as the closed subspace of with continuous shift, i.e. a bounded measure μ is of the form $\mu = \mu_g$, resp. $\mu(f) = \int_{\mathbb{R}^d} f(x)g(x)dx$ if and only if $\|T_x\mu - \mu\|_{\boldsymbol{M}} \to 0$ for $x \to 0$.*
- $\boldsymbol{L}^1(\mathbb{R}^d)$ *is w^*-dense in $\boldsymbol{M}(\mathbb{R}^d)$, in fact, for every $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ there exists a tight (hence bounded) sequence (f_k) in $\boldsymbol{L}^1(\mathbb{R}^d)$, with f_k resp. $\mu_{f_k} \to \mu$ in the w^*-topology.*

Proof. The main arguments are the identification of the "internal convolution" within $\boldsymbol{M}(\mathbb{R}^d)$ with the usual convolution formula

conf-CcRd
$$f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy, \quad \text{for} \quad f, g \in \boldsymbol{C}_c(\mathbb{R}^d). \tag{30}$$

but also the external action of $\boldsymbol{M}(\mathbb{R}^d)$ on the homogeneous Banach space

$$\boldsymbol{M}(\mathbb{R}^d)_e = \{\mu \,|\, \|T_x\mu - \mu\|_{\boldsymbol{M}} \to 0 \text{ for } x \to 0.\}$$

The typical bounded approximate units are of the form $(St_\rho g)_{\rho>0}$, for an arbitrary $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ with $\hat{g}(0) = \int_{\mathbb{R}^d} g(t)dt = 1$.

It is easy to verify that this net is tight and tends to δ_0 in the w^*-topology. In a similar way one can approximate a finite and discrete measure by a linear combination of such Dirac sequences. Since the Dirac measures form a total subset in $\boldsymbol{M}(\mathbb{R}^d)$ with respect to the w^*-topology the w^*-density of $\boldsymbol{L}^1(\mathbb{R}^d)$ in $\boldsymbol{M}(\mathbb{R}^d)$ is established. $\square$

Lesson of May 29th: Bounded measures operate not only on $\boldsymbol{C}_0(\mathbb{R}^d)$ but also on any homogeneous Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. The proof of this fact is essentially based on the idea that it is enough to establish this fact for bounded discrete measures (which is easy) and then show that for any sequence of discretizations of a given measure (where the diameter of the support of the corresponding BUPUs shrinks to zero) generates a sequence μ_k which is uniformly tight and bounded, but also produces a Cauchy sequence in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ of the form $(\mu_k * f)$, for any given $f \in \boldsymbol{B}$. Obviously it makes sense to define the limit (which does not depend on the choice of discretizations via BUPUs) by $\mu * f$ (although it is formally a new operation, and the "star" just introduced should be distinguished for a little while from the "known" star which denotes convolution within $\boldsymbol{M}(\mathbb{R}^d)$): $\mu * f \in \boldsymbol{B}$ and NORMS

Dirac-Conc **Lemma 35.** *A bounded net of functions $(h_\alpha)_{\alpha\in I}$ is a BAI for $\left(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\right)$ if the following property is satisfied: For every $\varepsilon > 0$ there exists some α_0 such that for $\alpha \succ \alpha_0$*

one has:

Dirac-eps (31)
$$|\int_{B_\varepsilon(0)} h_\alpha(t)dt - 1| \leq \varepsilon \quad \text{and} \quad \int_{|x|\geq\varepsilon} |h_\alpha(t)|dt \leq \varepsilon.$$

Proof. Argument: Due to the density of $\boldsymbol{C}_c(\mathbb{R}^d)$ in $\boldsymbol{L}^1(\mathbb{R}^d)$ one can reduce the discussion to functions $k \in \boldsymbol{C}_c(\mathbb{R}^d)$, i.e., it is enough to show that $h_\alpha * k \mapsto k$ for any $k \in \boldsymbol{C}_c(\mathbb{R}^d)$. The second condition allows to restrict the attention to a net with common compact support K. Consequently one has $h_\alpha * k(x) \neq 0$ only for $x \in K + \mathrm{supp}(k)$. Furthermore we obtain $h_\alpha * k(x) = \int_{\mathbb{R}^d} h_\alpha(y)k(x-y)dy \mapsto$ MORE TO BE DONE TOMORROW!! □

Remark 14. Of course one can also consider $\boldsymbol{M}(\mathbb{R}^d)$ as a Banach module over the Banach algebra $\boldsymbol{L}^1(\mathbb{R}^d)$ (with respect to convolution). Then the $\boldsymbol{L}^1(\mathbb{R}^d)$-essential part of $\boldsymbol{M}(\mathbb{R}^d)$ is equal to $\boldsymbol{L}^1(\mathbb{R}^d)$ itself.

On the other hand we can consider $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ as a Banach module over $\boldsymbol{L}^1(\mathbb{R}^d)$ (again with respect to convolution), and then this is an essential Banach module.

5. Tight subsets

A given $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ is of course "essentially concentrated" on a compact set (and uniformly small outside a sufficiently large compact set, by definition). We also have shown that a functional $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ is having most of its "mass" sitting within a compact set, while its action outside of this compact set is small. Indeed, since for any BUPU Φ we have $\mu = \sum_{iinI} \phi\mu$ as absolutely convergent sum the tails μ minus a large partial sum is small in the $\boldsymbol{M}(\mathbb{R}^d)$-sense.

Next we want to extend this "concentration over compact sets" concept to general bounded subsets of $\boldsymbol{M}(\mathbb{R}^d)$ (and later other functional spaces):

Definition 26. A bounded subset $H \subset \boldsymbol{M}(\mathbb{R}^d)$ is called (uniformly) tight if for every $\varepsilon > 0$ there exists $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ such that $\|\mu - k\cdot\mu\|_{\boldsymbol{M}} \leq \varepsilon$ for al $\mu \in H$.

In a similar way we define tightness in $\boldsymbol{C}_0(\mathbb{R}^d)$:

Definition 27. A bounded subset $H \subset \boldsymbol{C}_0(\mathbb{R}^d)$ is called (uniformly) tight if for every $\varepsilon > 0$ there exists $h \in \boldsymbol{C}_c(\mathbb{R}^d)$ such that $\|h - k\cdot h\|_\infty \leq \varepsilon$ for al $h \in H$.

Note: For a general $\boldsymbol{C}_0(\mathbb{R}^d)$ module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ one can define tightness as follows:

Definition 28. A bounded subset $H \subset (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called (uniformly) tight if for every $\varepsilon > 0$ there exists $h \in \boldsymbol{C}_c(\mathbb{R}^d)$ such that $\|h - k\cdot h\|_\infty \leq \varepsilon$ for al $h \in H$.

The concept of *tightness* plays a big role in the characterization of *relatively compact subsets* (hence compact operators)

Theorem 14. *Assume that W is a tight set within $\boldsymbol{M}(\mathbb{R}^d)$ and that H is a tight subset within $\boldsymbol{C}_0(\mathbb{R}^d)$. Then $W * H = \{\mu * h \mid \mu \in W, h \in H\}$ is a tight subset in $\boldsymbol{C}_0(\mathbb{R}^d)$.*

cf. the "compactness paper" p.307 (bottom):
`http://univie.ac.at/nuhag-php/bibtex/open_files/fe84_compdist.pdf`

Indeed, for any plateau-function τ which satisfies $\tau(x) \equiv 1$ on $\mathrm{supp}(k^1) + \mathrm{supp}(k^2)$, hence the following estimate holds:

$$(1-\tau)(f^1 * f^2) = (1-\tau)(f^1 * f^2 - f^1k^1 * f^2k^2)$$

$$(1-\tau)(\mu * f) = (1-\tau)(\mu * f - \mu k^1 * fk^2)$$

Applying norms to both sides and using the triangle inequality we obtain the following estimate in the sup-norm:

$$\begin{gathered}\|(1-\tau)(\mu * f)\| = \|(1-\tau)(\mu * f - \mu k^1 * fk^2)\| \leq \\ \|(1-\tau)\| \|(1-k^1)\mu * f\| + \|k^1\mu * (1-k^2)f\| \\ \leq \|(1-\tau)\| \|(1-k^1)\mu\|_{\boldsymbol{M}} \|f\| + \|k^1\| \|\mu\|_{\boldsymbol{M}} \|(1-k^2)f\|\end{gathered}$$

6. The Fourier transform for $\boldsymbol{L}^1(\mathbb{R}^d)$

The Fourier transform maps $\boldsymbol{M}(\mathbb{R}^d)$ into $\boldsymbol{C}_{ub}(\mathbb{R}^d)$. It will be seen as a non-expansive Banach algebra homomorphism from the closed ideal $\boldsymbol{L}^1(\mathbb{R}^d)$ into the closed ideal $\boldsymbol{C}_0(\mathbb{R}^d)$ of $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ (this result is usually known as Riemann-Lebesgue Lemma).

The range of the Fourier transform is a dense subalgebra (closed under complex conjugation), due to the "locally compact version" of the Stone-Weierstrass theorem.

Recall the standard version of the Stone-Weierstrass theorem:

ierstr-thm **Theorem 15.** *Let $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ be a Banach algebra within $\boldsymbol{C}(X)$, where X is some compact topological space. Then $\boldsymbol{A}$ is dense with respect to the uniform norm if $\boldsymbol{A}$ contains the constant functions, is closed under conjugation, and separates points, i.e., , if for any pair of points $x_1, x_2 \in X$, with $x_1 \neq x_2$ there exists some $f \in \boldsymbol{A}$ such that $f(x_1) \neq f(x_2)$.*

Since $\mathcal{F}\boldsymbol{L}^1$ does not have a unit (for pointwise multiplication), due to the fact that $\boldsymbol{L}^1(\mathbb{R}^d)$ does not contain a unit (the unit with respect to convolution is the Dirac measure δ_0, which do *not* cannot be approximated in $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ from within $\boldsymbol{C}_c(\mathbb{R}^d)$), one has to modify the above result to the locally compact case, by "adding" the constant functions, and replacing $\mathbb{R}^d$ by its Alexandroff (one-point) compactification X of $\mathbb{R}^d$. Indeed, $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ can be identified with a closed subalgebra of all continuous functions vanishing "at infinity". In fact, if $C_0 + h$ in $\boldsymbol{C}(X)$ is approximated by a sequence of the form $C_n + f_n$, then $|C_n - C_0| \to 0$ for $n \to \infty$, so $\|f_n - h\|_\infty$ for $n \to \infty$ (details left to the reader).

For $\mathbb{R}^d$ one can of course use alternative arguments, e.g. by observing that certain functions, such as the Schwartz functions, belong to the Fourier algebra, so there is enough richness in the function space $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ in order to show the density of $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ in $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$, but the above argument applies the general LCA groups.

On of the central statements concerning the Fourier transform is *Plancherel's theorem*, stating that the Fourier transform can be considered as a unitary linear automorphism of the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$ onto itself. This is in complete analogy to the statement that the for the case of $\mathbb{C}^n$ the Discrete Fourier transform (often realized in the form of the FFT) is a change of base from the orthonormal basis of unit vectors to the orthogonal system of pure frequencies. Since the vectors representing the pure frequencies (which are exactly the joint eigenvalues to all the translation operators) are of absolute value one, they are all of norm $\sqrt{n}$ the inverse FFT is essentially the conjugate (transpose) of the Fourier matrix, with a compensating factor of the form $1/n$. The advantage of our normalization in the continuous case (with the factor 2π as part of the exponent) has the advantage that the inverse Fourier transform will come in the form

nv-Fourdef

$$h(t) = \int_{\mathbb{R}^d} \hat{f}(s) e^{2\pi i t \cdot s} dt \tag{32}$$

Since the integral definition of the FT of its inverse do not apply to general functions $\hat{f} \in \mathcal{F}\boldsymbol{L}^1$, part of the discussion of the **Fourier-Plancherel Theorem** is concerned with technical questions around problems of this kind (how to overcome lack of integrability, e.g., by applying so-called *summability* methods, which are a generalization of the idea of an infinite integral, taken as limit of finite integrals).

Lemma 36. *It is enough to verify that for some dense subspace $\boldsymbol{B}$ of $\boldsymbol{L}^2(\mathbb{R}^d)$ within $\boldsymbol{L}^1(\mathbb{R}^d) \cap \boldsymbol{C}_0(\mathbb{R}^d) \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ one can find that the mapping $f \mapsto \hat{f}$ is well defined and isometric, and with dense range, in order to be able to extend the "classical" Fourier transform and its inverse (given by the integral) to an isometry from $\boldsymbol{L}^2(\mathbb{R}^d)$ onto itself.*

Proof. Since we assume that $\boldsymbol{B} \subseteq \boldsymbol{L}^1(\mathbb{R}^d) \cap \boldsymbol{C}_0(\mathbb{R}^d) \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ one can claim that the direct and the inverse Fourier transform given via (absolutely convergent) Riemannian integrals is valid. For the rest one only has to show that for an arbitrary $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ and any sequence f_n, with $f_n \in \boldsymbol{B}$ with $\|f - f_n\|_2 \to 0$ for $n \to \infty$ one finds that $\|\hat{f} - \hat{f}_n\|_2 = \|f - f_n\|_2 \to 0$, so by the completeness of $\boldsymbol{L}^2(\mathbb{R}^d)$ the Cauchy sequence $\hat{f}_n$ must have a limit, which may be denoted (by a so-called abuse of language) as the Fourier transform $\hat{f}$ of f. The extended (still isometric) mapping has dense range according to our assumptions, and therefore the same argument can be applied to the inverse Fourier transform in order to realize that the extended mapping (often called the *Fourier-Planchere* or just *Plancherel* transform defines an isometric automorphism of $\boldsymbol{L}^2(\mathbb{R}^d)$. Due to the polarization identity

$$\langle f, g \rangle = \sum_{k=0}^{3} i^k \|f + i^k g\|^2.$$

such a mapping also preserves scalar products in general. □

For the proof of Plancherel's theorem one may use $\boldsymbol{B} = \boldsymbol{L}^1(\mathbb{R}^d) \cap \boldsymbol{C}_0(\mathbb{R}^d) \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ or the linear span of all the time-frequency shifted and dilated version of a Gauss function (we do not require any norm on $\boldsymbol{B}$). Ideally one can or should use a space of functions which is invariant under the Fourier transform. Details have been given in the FA (= functional analysis) course WS0506 by HGFei.

Of course the extended Fourier transform is still compatible with convolution resp. pointwise multiplication. In other words, convolution on one side of the FT goes into pointwise product on the other side (and vice versa). As a consequence one obtains a characterization of $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$: A function belongs to $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ if and only if it can be written as a convolution product of two functions in $\boldsymbol{L}^2(\mathbb{R}^d)$, i.e. $h \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ if and only if there exist two functions $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$ such that $h = f * g$. The direct direction (i.e. convolution products are in $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$) is easy, because their Fourier transforms give a function which is a pointwise product of two $\boldsymbol{L}^2$-functions, and hence according to the Cauchy-Schwartz inequality $\hat{h} = \hat{f} \cdot \hat{g} \in \boldsymbol{L}^2 \cdot \boldsymbol{L}^2 \subseteq \boldsymbol{L}^1$, or equivalently $h \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$.

The easiest argument for the converse is again on the Fourier transform side. Write $\hat{h} \in \boldsymbol{L}^1(\mathbb{R}^d)$ as a pointwise product of two $\boldsymbol{L}^2$-functions. If $\hat{h}$ was non-negative there is a natural solution to this problem, just take $\sqrt{\hat{h}}$. If $\hat{h}$ is a complex-valued function, one can apply this trick only to $|\hat{h}|$, and can assign the phase factor to one of the two non-negative square roots (details are left to the reader).

7. Wiener's algebra $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)$

Because it can be defined without the existence of a Haar measure the following space plays an important role within Harmonic Analysis. We define $W(G)$ as follows;

Then we define $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)$ as follows:

WCdefphi **Definition 29.** Let φ be any non-zero, non-negative function on $\mathbb{R}^d$.

WCdefphi1
$$\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1) = \{\, f \in \boldsymbol{C}_0(\mathbb{R}^d) \mid \exists (c_k)_{k\in\mathbb{N}} \in \boldsymbol{\ell}^1, (x_k)_{k\in\mathbb{N}} \text{ in } \mathbb{R}^d, |f(x)| \leq \sum_{k\in\mathbb{N}} c_k \varphi(x - x_k) \,\} \tag{33}$$

We define
$$\|f \mid \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)\| := \inf \{ \|\mathbf{c}\|_{\boldsymbol{\ell}^1} = \sum_{k\in\mathbb{N}} |c_k| \,\}$$
where the infimum is taken over all "admissible dominations" of f as in (33).

It is obvious that $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)$ is continuously embedded into $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ [26] since $\|f\|_\infty \leq (\sum_{k\in\mathbb{N}} |c_k|)\|\varphi\|_\infty$. By a similar argument we have a continuous embedding of $\big(\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\big)$ into $\big(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\big)$.

LEFT OVER MATERIAL

antorovich **Theorem 16.** [Kantorowich Lemma] *Let T_α be a strongly convergent and bounded sequence of invertible operators between Banach spaces, with limit T_0, which is assumed to be invertible itself. Then the inverse operators are strongly convergent as well if the inverse operators are uniformly bounded. If we consider only sequences this is a criterion, because then the strong convergence of $T_n^{-1}(y) \to T_0^{-1}(y)$ for every y implies uniform boundedness of the sequence T_n^{-1}.*

[26] by the same argument $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)$ is also contained in many other Banach spaces of functions with the property that translations are isometric and that the space is solid.

QUESTION: For tight nets of bounded measures w^*-convergence implies pointwise and uniform over compact convergence of their FTs. But also the converse is true!!!! (Exercise). [for non-tight families this is not true, just think of the case $\delta_n, n \to \infty$!

An elementary proof showing that the Gauss function $g_0(t) = e^{-\pi|t|^2}$ is mapped itself by the Fourier transform has been given by Georg Zimmermann, see

`http://www.univie.ac.at/NuHAG/FEICOURS/ws0607/efoft.pdf`

8. The Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$ and Banach Gelfand triples

There are different ways of defining $\boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^1)$ should be renamed `WFLili` `instead of WFlili.`

Material for course October 2006

Given a Banach module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ over a Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ with bounded approximate units we define the essential part $\boldsymbol{B_A}$ and the relative completion of A, which is given as $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{A}, \boldsymbol{B})$. Note that this is again in a natural way a Banach module (with respect to the operator norm) over the original Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$, and that $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ can be mapped into this Banach module in a natural way as a closed subspace (at least of $\boldsymbol{B} = \boldsymbol{B_A}$ and if $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ has bd. approx. units).

ARGUMENT: One has to identify each element $b \in \boldsymbol{B}$ with the operator $T_b \in \mathcal{H}_{\boldsymbol{A}}(\boldsymbol{B})$ obtained by something like the "right regular representation", i.e., the operator $T_b : a \mapsto a \bullet b$. It is always true that $\|T_b\|_{op} \leq \|b\|_{\boldsymbol{B}}$, and by applying T_b to the elements of some bounded approximate unit in $\boldsymbol{A}$ one finds the converse estimate, i.e., $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ can be identified with a closed subspace of all bounded linear operators from $\boldsymbol{A}$ to $\boldsymbol{B}$.

Note that one should not forget that one has to impose the "natural" $\boldsymbol{A}$-module structure on $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{B}_1, \boldsymbol{B}_2)$, before making the claim that the $\boldsymbol{A}$-module $\boldsymbol{B}$ can be embedded via an $\boldsymbol{A}$-module homomorphism (embedding) into the larger $\boldsymbol{A}$-module $\boldsymbol{B}^{\boldsymbol{A}}$.

Exercise: For the case of the pointwise algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}}) = \big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ one finds that $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{A}, \boldsymbol{A}) = \big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$ in a "natural way. Note that the "identity operator always belongs to $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{A}, \boldsymbol{A})$ (obviously it commutes with any other operator), and therefore the "enlargement" from $\boldsymbol{A}$ to $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{A})$ also implies the adjunction of a unit element to the Banach algebra $\boldsymbol{A}$, but typically much more than this. So in a way the (in this case isometric) embedding of $\boldsymbol{C}_0(\mathbb{R}^d)$ into $\boldsymbol{C}_b(\mathbb{R}^d)$ (both with the sup-norm) can be seen as an embedding of $\boldsymbol{A} = \boldsymbol{C}_0(\mathbb{R}^d)$ into the maximal algebra "with the same norm" (and a unit).

Plancherel's theorem can be used (we skip those details) to identify $\mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^2, \boldsymbol{L}^2)$ (or equivalently the set of all bounded linear operators from $\boldsymbol{L}^2(\mathbb{R}^d)$ into $\boldsymbol{L}^2(\mathbb{R}^d)$ which commute with translation) with $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{L}^2, \boldsymbol{L}^2)$ (for $\boldsymbol{A} = \mathcal{F}\boldsymbol{L}^1$, i.e., the operators from $\boldsymbol{L}^2(\mathbb{R}^d)$ into $\boldsymbol{L}^2(\mathbb{R}^d)$ which commute with pointwise multiplications with elements from $\boldsymbol{A} = \mathcal{F}\boldsymbol{L}^1$, or equivalenty, just with the multitplication with pure frequencies resp. characters on $\mathbb{R}^d$, which are the functions $x \mapsto e^{2\pi i s \cdot x}$. These are again pointwise multiplication operators, and it is not hard to find out that a pointwise multiplier h of $\boldsymbol{L}^2(\mathbb{R}^d)$ is has to be a measurable function which is essentially bounded, i.e., $h \in \boldsymbol{L}^\infty(\mathbb{R}^d)$.

Let us just sketch the basic idea behind this fact:

Lemma 37. *Assume that $\boldsymbol{B}$ is a Banach module with respect to a pointwise Banach algebar $\boldsymbol{A}$ (and assume that $\boldsymbol{A}_c(\mathbb{R}^d) = \boldsymbol{C}_c(\mathbb{R}^d) \cap \boldsymbol{A}$ is dense in $\boldsymbol{A}$), and assume that $\boldsymbol{A} \cap \boldsymbol{B}$ contains arbitrary large plateau-functions, i.e., with the property, that for each compact set $K \subseteq \mathbb{R}^d$ there exists some $q \in \boldsymbol{B} \cap \boldsymbol{A}$ such that $q(x) \equiv 1$ on K. Then the elements in $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{B}, \boldsymbol{B})$ are pointwise multipliers with suitable functions h which belong locally to $\boldsymbol{A}$.*

Proof. TO BE GIVEN LATER on! □

For us an important Banach algebra is $\big(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\big)$, endowed with convolution as multiplication (which is commutative, due to the commutativity of addition in $\mathbb{R}^d$). It does not have any units, but we have shown earlier (?) that $\big(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\big)$ has bounded approximate units. Typically such a family is obtained by taking any sequence f_n (or net) of functions, e.g. in $\boldsymbol{C}_c(\mathbb{R}^d)$, with $\int_{\mathbb{R}^d} f_n(x)dx = 1$ and "shrinking support". This can be obtained by choosing the shape of of f_n arbitrarly, but assuming that $f_n(x) = 0$ for $|x| \geq \delta_n$ for some null-sequence $\delta_n \to 0$ for $n \to \infty$. Alternatively one compresses a given function $f \in \boldsymbol{C}_c(\mathbb{R}^d)$ or even in $\boldsymbol{L}^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} f_n(x)dx = 1$, and chooses $f_n = St_{\rho_n} f$, for some sequence $\rho_n \to 0$ for $n \to \infty$. The choice $f(x) = e^{-\pi x^2}$ is a popular choice (which also shows that it is not important for f to be compactly supported).

We will see shortly that $\mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^1) = \mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^1, \boldsymbol{L}^1)$ can be identified with the space $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ of all bounded measures (resp. with $\big(\boldsymbol{C}_0{}'(\mathbb{R}^d),\, \|\cdot\|_{\boldsymbol{M}}\big)$). This result is called "Wendel's theorem" ([16, 27]). It has of course two parts: First of all one has to show that convolution operators induced by elements from $\boldsymbol{M}_b(\mathbb{R}^d)$ leave the closed subspace $\boldsymbol{L}^1(\mathbb{R}^d)$ invariant. In the second part one has to verify that any abstract (bounded and linear) operator on $\big(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\big)$ commuting with all the translation must be such a convolution operator.

Proof. For the first part we have to verify that $\boldsymbol{L}^1(\mathbb{R}^d)$ is a closed ideal of $\boldsymbol{M}_b(\mathbb{R}^d)$, i.e., that $\boldsymbol{M}_b(\mathbb{R}^d) * \boldsymbol{L}^1(\mathbb{R}^d) \subseteq \boldsymbol{L}^1(\mathbb{R}^d)$. One way to do that is to check that $\big(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\big)$ is a homogeneous Banach space, i.e., to show that the group $\mathbb{R}^d$ acts in a continuous and isometric way on $\boldsymbol{L}^1(\mathbb{R}^d)$. This is easy to verify since $\|T_x f\|_1 = \|f\|_1$ for all $x \in \mathbb{R}^d$ and all $f \in \boldsymbol{C}_c(\mathbb{R}^d)$ and also $lim_{x\to 0}\|T_x f - f\|_1 = 0$ for $x \to 0$, for any $f \in \boldsymbol{C}_c(\mathbb{R}^d)$, hence (by approximation for any $f \in \boldsymbol{L}^1(\mathbb{R}^d)$).

That $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ (viewed as a Banach algebra with respect to convolution) is acting boundedly on *any homogeneous Banach space* will be discussed separately.

Alternatively one can even describe $\boldsymbol{L}^1(\mathbb{R}^d)$ as the subset of all bounded measures which have continuous translation, in other words, one can show (see a classical paper by Plessner, 1929) that $\|T_x\mu - \mu\|_{\boldsymbol{M}_b} \to 0$ for $x \to 0$ implies that μ is an "absolutely continuous" measure, i.e., belongs to $\boldsymbol{L}^1(\mathbb{R}^d)$.

The argument for this result typically relies on a compactness argument (w^*- compactness of the unit ball in the dual Banach space $\boldsymbol{M}_b(\mathbb{R}^d) = \big(\boldsymbol{C}_0{}'(\mathbb{R}^d),\, \|\cdot\|_{\boldsymbol{M}}\big)$. One applies the given operator $T \in \mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^1)$ to any *Dirac- sequence* f_n which forms a bounded approximate unit in $\boldsymbol{L}^1(\mathbb{R}^d)$. By the boundedness of (f_n) and the operator T the image sequence $T(f_n)$ is also bounded in $\boldsymbol{L}^1(\mathbb{R}^d)$, hence in the larger (dual) space $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b}))$. By the w^*-compactness of bounded balls in this space we obtain a w^*-convergent subnet, with some limit $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$. It remains to show that this limit is inducing the operator, i.e., one has to verify that $T = C_\mu$. □

Material of Nov. 9-th is partially covered by the paper "Banach spaces of distributions having two module structures J. Funct. Anal. (1983)" ([2]). The main result of this paper is a "chemical diagram" that can be attached to each of the spaces in >>> standard situation:

Some comments on the classical Riemann-Stieltjes integral (in German)

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http://de.wikipedia.org/wiki/Stieltjes-Integral
http://de.wikipedia.org/wiki/Beschr%C3%A4nkte_Variation
http://de.wikipedia.org/wiki/Absolut_stetig
http://de.wikipedia.org/wiki/Satz_von_Radon-Nikodym
http://de.wikipedia.org/wiki/Lebesgue-Integral
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Some of the material in this course has been already given in courses in Heidelberg (1980), Maryland (1989/90) or at the university of Vienna in the last 20 years.

The material concerning the Segal algebra $\boldsymbol{S}_0(G)$ is going back to various original publications by the author, see for example [6], where this particular Segal algebra has been introduced and where it is shown that it is the *minimal* TF-isometric homogeneous Banach space (and many other properties). The role of the dual space has been described already in [5] (both papers downloadable from the NuHAG site). The *double module viewpoint* is described in much detail in [2] (Banach spaces of distributions having two module structures, J. Funct. Anal.). A detailed account of notions of *compactness* (and also a clean description of tightness, etc.) is given in [7]. The first atomic characterization of modulation spaces ($\boldsymbol{S}_0(\mathbb{R}^d)$ is among them) has been given at a conference in Edmonton in 1986 (published then in [8]).

There are many places where especially the role of the Segal algebra $\boldsymbol{S}_0(G)$ for the discussion of basic questions in Gabor Analysis has been described. The very first systematic discussion as been probably given in the Chapter by Feichtinger and Zimmermann in the first Gabor book of 1998 ([11]). Another relevant paper is the one by Feichtinger and Kaiblinger ([9]) where it is shown, that (in the $\boldsymbol{S}_0(\mathbb{R}^d)$ context) the dual window is depending continuously on the lattice constants in the case of Gabor frames resp. Gabor Riesz bases.

Preview: In order read about Gabor multipliers the best source is probably the survey article in the second (blue) Gabor book, “Advances in Gabor Analysis”, by Feichtinger and Nowak ([10]).

General references are: Hans Reiter’s book on Harmonic Analysis (including a very fine and compact introduction to Integration Theory over Locally Compact Groups, but without the proof of the existence of the Haar measure) [18]. An updated version (edited by his former PhD student Ian Stegeman is [20]). The book describes (see also [19]) the concepts of Segal algebras (such as $S_0(G)$), and Beurling algebras $\boldsymbol{L}^1_w(G)$ (with respect to multiplicative weights). Both books are available in the NuHAG library.

A nice introduction into ”abstract harmonic analysis” is that of Deitmar ([4]), and of course always Katznelson (starting with classical Fourier series, but also talking about the Gelfand theory for commutative C^*-algebras) is [15]. It also contains the first “propagation” of the concept of *homogeneous Banach spaces*. A similar unifying viewpoint is taken in the book of Butzer and Nessel ([3]) and of course several of the books of Hans Triebel (see BIBTEX collection and book-list). These two mathematicians can also be seen as pioneers of interpolation theory and the so-called theory of function spaces. (Abstract) Homogeneous Banach spaces are also treated in the Lecture Notes by H. S.

Shapiro (sh71 [23]), having approximation theoretic questions in mind (he makes the associativity of the action of bounded measures an axiom, obviously because he could proof it only in concrete cases).

Solid Banach spaces of function (under the name of *Banach function spaces*) appear in the work of Zaanen: za53 [28] (or za67 [29]), both books should be available in the NuHAG library (Alserbachstrasse 23, Room 8)

A very good source to learn about Besov spaces is Jaak Peetre's book entitled "New Thoughts on Besov Spaces" (pe76 [17]).

Banach modules: Rieffel's work ri69-1 [22] [some tex-nical problem with BIBTEX]

Generally interesting references about "mathematics and signal processing": Richard Holmes: ri79 [21]

Protocol for the course: Nov 9th, 2006 HGFei

TOPIC: **Standard Spaces**

What are standard spaces?? Banach spaces of functions or distributions which allow sufficiently many regularization operators, e.g. localization (by pointwise multiplication) and regularization (by convolution).

Definition 30. A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called a *(restricted) standard space* if

(1) $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ (continuous embeddings);
(2) $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{B} \subseteq \boldsymbol{B}$, with $\|h \cdot f\|_{\boldsymbol{B}} \leq \|h\|_{\mathcal{F}\boldsymbol{L}^1} \|f\|_{\boldsymbol{B}}$;
(3) $\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{B} \subseteq \boldsymbol{B}$ with $\|g * f\|_{\boldsymbol{B}} \leq \|g\|_{\boldsymbol{L}^1} \|f\|_{\boldsymbol{B}}; \text{for } g \in \boldsymbol{L}^1(\mathbb{R}^d), f \in \boldsymbol{B}$;

It is clear that almost all the spaces used "normally" in Fourier analysis are such "standard spaces". It is sufficient that a space (of locally integrable functions or Radon measures) is isometrically invariant under the time-frequency shifts $\pi(\lambda) = M_\omega T_t$ for $\lambda = (t, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and that e.g. the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ is contained in $\boldsymbol{B}$ as a dense subspace, to ensure that the above conditions are satisfied. Let us formulate this claim as a lemma:

ndardonRd1 **Lemma 38.** *? DUPLICATE!? Assume that BspN is a Banach space of locally integrable functions on* $\mathbb{R}^d$ *such that* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ *is contained in* $\boldsymbol{B}$ *as a dense subspace and that* $\|M_\omega T_t f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}$ *for all* $\lambda = (t, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. *Then it is a standard space.*

Which kind of objects do we want: Banach spaces of continuous functions? Banach spaces of locally (Lebesgue-) integrable functions? Banach spaces of Radon measures, or (tempered?) distributions? Should we allow even ultra-distributions?

Wishes: We would like to be able to do functional analysis, so with each spaces we would like to have the dual space in the same family (as long as it can be viewed as a Banach space of distributions, hence only if it can be completely characterized by the sum of the local actions).

With each space the Fourier transform of the space should be in the same family, etc. etc.

Formal suggestion in this direction is given by the following definition, which describes at "reasonable" generality a family of Banach spaces which is *not* restricted to Banach spaces of functions, because such a family will typically *not* be closed under duality (an exception is the family of $\boldsymbol{L}^p$-classes, but already the dual space of $\boldsymbol{C}_0(\mathbb{R}^d)$ contains discrete measures which are not represented by (integrals against measurable) functions.

Definition 31. A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called a *(restricted) standard space* if

(1) $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ (continuous embeddings);
(2) $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{B} \subseteq \boldsymbol{B}$, with $\|h \cdot f\|_{\boldsymbol{B}} \leq \|h\|_{\mathcal{F}\boldsymbol{L}^1} \|f\|_{\boldsymbol{B}}$;
(3) $\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{B} \subseteq \boldsymbol{B}$ with $\|g * f\|_{\boldsymbol{B}} \leq \|g\|_{\boldsymbol{L}^1} \|f\|_{\boldsymbol{B}}; \text{for } g \in \boldsymbol{L}^1(\mathbb{R}^d), f \in \boldsymbol{B}$;

Remark 15. The main idea behind this specific definition (also the reason why it is called for a while the "restricted standard situation" is the fact that the fact that $\boldsymbol{S}_0(\mathbb{R}^d)$ and its dual $\boldsymbol{S}_0'(\mathbb{R}^d)$, or that the whole Banach algebra $\boldsymbol{L}^1(\mathbb{R}^d)$ is acting on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ via convolution, can be seen as a matter of convenience. In this way we avoid a number of technical conditions involving weights and still have a fairly large collection of examples available. We will be able to demonstrate the roles of pointwise multiplication and convolutive action in the present context, and it will be easy for the reader to generalize the observations to more general situations.

A typical alternative view in the context of $\mathcal{G} = \mathbb{R}^d$ is the following setting:

Definition 32 (convenient description)**.** A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called a *tempered standard space* on $\mathbb{R}^d$ if

(1) $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ (continuous embeddings);
(2) $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \cdot \boldsymbol{B} \subseteq \boldsymbol{B}$
(3) $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) * \boldsymbol{B} \subseteq \boldsymbol{B}$

Aside from the fact that one needs some functional analytic argument in order to establish the equivalence between this "convenient" and another more technical one (which is however what one needs in order to make those concepts useful). It will be convenient for this purpose to make use of polynomial (submultiplicative) weights w_s, given by $w_x : x \mapsto (1 + |z|^2)^{s/2}$:

Definition 33 (technical definition)**.** A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called a *tempered standard space* on $\mathbb{R}^d$ if

(1) $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ (continuous embeddings);
(2) There $s \geq 0$ such that $\boldsymbol{L}^1_{w_s}(\mathbb{R}^d)$ acts on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ by convolution and $\|g * f\|_{\boldsymbol{B}} \leq C_s \|g\|_{1,w_s} \|f\|_{\boldsymbol{B}}; \text{for } g \in \boldsymbol{L}^1_{w_s}, f \in \boldsymbol{B}$, for some $C_s > 0$;
(3) $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \cdot \boldsymbol{B} \subseteq \boldsymbol{B}$ and there exists some constant $\|h \cdot f\|_{\boldsymbol{B}} \leq \|\widehat{h}\|_{1,w_r} \|f\|_{\boldsymbol{B}}$;

Remark 16. The typical examples of *reduced standard spaces* arise from Banach spaces of say tempered distributions which are isometrically invariant under TF-shifts, i.e., which satisfy

$$\|\pi(t,\omega) f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}} \quad \forall f \in \boldsymbol{B}.$$

and which contain $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ or $\boldsymbol{S}_0(\mathbb{R}^d)$ as a dense subspace. In fact, in such a case on can argue that the isometric invariance of the space implies that the continuity of the mapping

(t,ω) into $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ resp. $\boldsymbol{S}_0(\mathbb{R}^d)$ implies that one can extend the strong continuity to all of $(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}})$, in other words, one obtains a so-called *time-frequency homogeneous Banach space* $(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}})$ in this case, and the mapping (t,ω) to $\pi(t,\omega)(f)$ is continuous for *every* $f \in \boldsymbol{B}$.

One can also discuss from a technical side the need of assuming that the embedding from $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ into $(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}})$ should be a continuous one with respect to the occurring natural tolopogies. In fact, it should be enough, for example, to verify that $(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}})$ itself is continuously embedded into the space of all locally integrable functions and that for each norm convergent sequence (f_n) in $(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}})$ there exists a subsequence (f_{n_k}) which is pointwise convergent almost everywhere. etc. ...

Starting from the observation that for any of the module actions, arising from the pointwise algebra $\boldsymbol{A} = \mathcal{F}\boldsymbol{L}^1$ and the convolutional Banach module structure over $\boldsymbol{L}^1$ one can build two types of completions and also two typos of "essential parts". We will write $\boldsymbol{B}^{\boldsymbol{A}}$ for the $\boldsymbol{A}-$completion of $\boldsymbol{B}$, and $\boldsymbol{B}_{\boldsymbol{A}}$ for the essential part with respect to the pointwise module action. It is easy to verify that an element is in $\boldsymbol{B}_{\boldsymbol{A}}$ if and only if it can be approximated by elements with compact support, or if and only if any bounded sequence of plateau-like functions (forming a bounded approximate unit in $\mathcal{F}\boldsymbol{L}^1$) acts as approximation to the identity operator on the given element.

Analogously we define the completion and the essential part with respect to the Banach algebra $\boldsymbol{L}^1$. Since the action of this algebra usually comes from the group action (by translation), we will use the symbols $\boldsymbol{B}^{\mathcal{G}}$ and $\boldsymbol{B}_{\mathcal{G}}$.

Combining those four operations in a serial way we can come up with a large number of new spaces, derived from any of those spaces. Since the operations of completion and essential part with respect to the *same* algebra action are canceling each other (similar to the operation of taking a closure resp. the interior of a "nice set") we can concentrate in our discussion on "mixed series", such as: ${\boldsymbol{B}_{\mathcal{G}}}^{\boldsymbol{A}}{}_{\mathcal{G}}$, or even longer chains of operations of a similar kind.

The result that has been derived in brfe83 [2] can be summarized in the following way: JUST the LAST OCCURRENCE of each algebra operation counts, i.e., the last occurrence of the symbol $\mathcal{G}$ and the last occurrence if $\boldsymbol{A}$. So we have ${\boldsymbol{B}_{\mathcal{G}}}^{\boldsymbol{A}}{}_{\mathcal{G}} = \boldsymbol{B}^{\boldsymbol{A}}{}_{\mathcal{G}}$ or ${\boldsymbol{B}_{\mathcal{G}}}^{\boldsymbol{A}}{}_{\mathcal{G}\boldsymbol{A}} = \boldsymbol{B}_{\mathcal{G}\boldsymbol{A}}$.

The most important spaces in this family are the minimal space, which turns out to be the "double essential part", where the order of algebra operations does *not* play a role anymore. It coincides with the closure of the test functions in the standard spaces. So we have

oubesspart
$$\boldsymbol{B}_{\boldsymbol{A}\mathcal{G}} = \overline{\boldsymbol{S}_0}^{\boldsymbol{B}} = \boldsymbol{B}_{\mathcal{G}\boldsymbol{A}} \tag{34}$$

On the other hand here is a (single) double completion, which coincides with the w^*-relative completion of $\boldsymbol{B}$ within $\boldsymbol{S}_0'$ (details to be presented at another time).

doublcompl
$$\boldsymbol{B}^{\boldsymbol{A}\mathcal{G}} = \widetilde{\boldsymbol{B}} = \boldsymbol{B}^{\mathcal{G}\boldsymbol{A}} \tag{35}$$

where we define the vague or w^*-relative completion of $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ in $\boldsymbol{S}_0'$ as follows:

el-complet (36)
$$\widetilde{\boldsymbol{B}} = \{\sigma \in \boldsymbol{S}_0' \,|\, \sigma = w^* - lim_\alpha f_\alpha, \sup_\alpha \|f_\alpha\|_{\boldsymbol{B}} < \infty\} \tag{36}$$

The infimum over all the bounds $\sup_\alpha \|f_\alpha\|_{\boldsymbol{B}}$ makes $\widetilde{\boldsymbol{B}}$ into a standard space, which contains $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ as a closed subspace.

Example:
Starting from $\boldsymbol{C}_0(\mathbb{R}^d)$ one can find that is relative completion is just $\big(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\big)$.

A WIKIPEDIA contribution:

http://en.wikipedia.org/wiki/Harmonic_analysis
http://en.wikipedia.org/wiki/Tempered_distribution
http://en.wikipedia.org/wiki/Fourier_analysis
http://en.wikipedia.org/wiki/Discrete-time_Fourier_transform
http://en.wikipedia.org/wiki/Colombeau_algebra

Next we are going to show that the convolution action of bounded discrete measures on a homogenous Banach space can be extended to all of the measures in order to generate an action of MGN on such a Banach space $(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}})$.

eas-homBR0 **Theorem 17.** *Assume that $(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}})$ is a homogeneous Banach-space with respect to some "abstract" group action ρ, i.e. we assume that $x \to \rho(x)f$ is continuous from $\mathcal{G}$ into $(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}})$, isometric in the sense that $\|\rho(x)f\|_{\boldsymbol{B}} = \|f\|_Bsp$ for all $x \in \mathcal{G}$ and $f \in \boldsymbol{B}$, and that $\rho(x_1x_2) = \rho(x_1)\rho(x_2)$. Of course we can define $\mu \bullet_\rho f$ to be the (unique) limit of this Cauchy net in $(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}})$. Then one has*

meas-onB1
$$\|\mu \bullet_\rho f\|_{\boldsymbol{B}} \leq \|\mu\|_{\boldsymbol{M}} \|f\|_{\boldsymbol{B}}, \quad \text{for all } f \in \boldsymbol{B}. \tag{37}$$

In fact, $(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}})$ becomes a Banach module over $(Mbsp(G), |ebbes|_M bsp)$ in this way.

Proof. We are going first to define the action of $\mu \in \boldsymbol{M}(\mathcal{G})$ on an individual element $f \in \boldsymbol{B}$, by verifying that the net

$$D_\Psi(\mu) \bullet_\rho f := \sum_{i \in I} \mu(\psi_i)\rho(x_i)f$$

is convergent, as $|\Psi| \to 0$.

□

Proof. The idea is to consider the action of $D_\Psi \mu$ on f as Riemann-type sums for the integral $\int_{\mathcal{G}} \rho(x) f d\mu(x)$. Therefore it is natural to check that the action of bounded discrete measures is OK (this is an easy consequence of the assumptions) and then to compare two such expressions, namely $D_\Psi(\mu) \bullet_\rho f$ and $D_\Phi(\mu) \bullet_\rho f$ by making use of their joint refinement, constituted by the (double indexed family) $(\psi_i \phi_j)$.

Let us first estimate the norm of $D_\Psi(\mu) \bullet_\rho f$. Using the isometry of the action of ρ on $(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}})$ one has, independently from Ψ:

rmeasconv1
$$\|D_\Psi(\mu) \bullet_\rho f\|_{\boldsymbol{B}} \leq \sum_{i \in I} |\mu(\psi_i)| \|\rho(x_i)f\|_{\boldsymbol{B}} \leq \|f\|_{\boldsymbol{B}} \sum_{i \in I} |\mu(\psi_i)| \leq \|\mu\|_{\boldsymbol{M}} \|f\|_{\boldsymbol{B}}. \tag{38}$$

Assume next that there are two families $\Psi = (\psi_i)_{i \in I}$ and $\Phi = (\phi_j)_{j \in J}$ are given, with central points $(x_i)_{i \in I}$ and $(y_j)_{j \in}$. Then we can define the *joint refinement* $\Psi - \Phi$ as the family $(\psi_i \phi_j)_{(i,j) \in I \diamond J}$, where we can agree to call $I \diamond J$ the family of all index pairs such that $\psi_i \cdot \phi_j \neq 0$ (because all the other products are trivial and should be neglected). In fact, if both Ψ and Φ are sufficiently "fine" BUPUs one has: [27]

ann-estim1
$$\|D_\Psi \mu \bullet_\rho f - D_\Phi(\mu) \bullet_\rho f\|_{\boldsymbol{B}} = \sum_{(i,j) \in I \diamond J} \|\rho(x_i)f - \rho(y_j)f\|_{\boldsymbol{B}} |\mu(\psi_i \phi_j)| \leq \tag{39}$$

$$\sup_{(i,j) \in I \diamond J} \|\rho(x_i)[f - \rho(y_j - x_i)f]\|_{\boldsymbol{B}} \sum_{(i,j) \in I \diamond J} \|(\psi_i \phi_j)\mu\|_{\boldsymbol{M}} \leq \varepsilon \|\mu\|_{\boldsymbol{M}},$$

if only Ψ resp. Φ are fine enough. Due to the completeness of $(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}})$ one finds that there is a uniquely determined limit, which we will call $\mu \bullet_\rho f$. It is then obvious that

stf-estim1
$$\|\mu \bullet_\rho f\|_{\boldsymbol{B}} = \lim_{|\Psi| \to 0} \|D_\Psi \mu \bullet_\rho f\|_{\boldsymbol{B}} \leq \limsup \|D_\Psi \mu\|_{\boldsymbol{M}} \|f\|_{\boldsymbol{B}} = \|\mu\|_{\boldsymbol{M}} \|f\|_{\boldsymbol{B}}. \tag{40}$$

[27] Using that $\psi_i = \sum_{j \in j} \psi_i \phi_j$, hence $\sum_{(i,j) \in I \diamond J} \psi_i \phi_j \equiv 1$ and $\sum_{(i,j) \in I \diamond J} \|(\psi_i \phi_j)\mu\|_{\boldsymbol{M}} = \|\mu\|_{\boldsymbol{M}}$.

Of course it remains to show that the so defined action is associative, i.e. that

assocat
$$(\mu * \mu_2)\bullet_\rho f = (\mu_1)\bullet_\rho(\mu_2 \bullet_\rho f), \quad \mu_1, \mu_2 \in \boldsymbol{M}(\mathcal{G}), f \in \boldsymbol{B}, \tag{41}$$

but his is clear because the associativity is valid for the discrete measures $D_\Psi \mu$ and $D_\Phi(\mu)$. [28]

Remark 17. In the derivation above we have used the isometric property and the fact that $\rho(x_1 x_2) = \rho(x_1) \circ \rho(x_2)$. It would have been no problem if this identity was only true "up to some constant of absolute value one", i.e. if one has a *projective representation* of $\mathcal{G}$ only, such as the mapping $\lambda = (t, \omega) \mapsto \pi(\lambda) = M_\omega T_t$ from $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ into the unitary operators on the Hilbert space $\big(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\big)$.

For the next step we need a simple observation from abstract Hilbert space theory.

Lemma 39. *Assume that a (complex) linear mapping between two Hilbert space over the complex numbers, $\mathscr{H}_1 \to \mathscr{H}_2$ is isometric, i.e. satisfies*

som-embed1
$$\|T(h)\|_{\mathscr{H}_2} = \|h\|_{\mathscr{H}_1} \quad \forall h_1 \in \mathscr{H}_1. \tag{42}$$

Then the adjoint mapping $T' : \mathscr{H}_2 \to \mathscr{H}_1$ is the inverse on the range, i.e. one has $T'(Tf) = f \quad \forall h_1 \in \mathscr{H}_1$.

Proof. The claim follows from the fact that an isometric embedding also preserves scalar products, as a consequence of the polarization identity

arization
$$\langle f, g\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \langle f + {}^k g, f + i^k g\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \|f + i^k g\|^2 \tag{43}$$

Hence

som-adjinv
$$\langle T'(Tf), g\rangle = \langle Tf, Tg\rangle = \langle f, g\rangle \quad \forall f, g \in \mathscr{H}_1. \tag{44}$$

Since this is true for every $g \in \mathscr{H}_1$ the required claim is valid. Usually one says that $T'(h_2)$ is defined in the weak sense for $h_2 \in \mathscr{H}_2$, through the identity

weak-Tadj
$$\langle T' h_2, h_1\rangle_{\mathscr{H}_1} = \langle h_2, T(h_1)\rangle_{\mathscr{H}_2}, \quad h_1 \in \mathscr{H}_1, h_2 \in \mathscr{H}_2. \tag{45}$$

□

Application: $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ is defined via its STFT: $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ belongs to $\boldsymbol{S}_0(\mathbb{R}^d)$ if and only if $V_{g_0} f \in \boldsymbol{L}^1(\mathbb{R}^{2d})$, where g_0 is the Gauss-function (or any other nonzero Schwartz-function). Since $f \mapsto V_g f$ is isometric (assuming that $\|g_0\|_2 = 1$) we have according to the above lemma the (weak) reconstruction formula

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_{g_0} f(\lambda) \pi(\lambda) g_0 \, d\lambda,$$

[28] Note that H.S.Shapiro (cf. [sh71 23]) is making this associativity an extra axiom, apparently because he could not proof it directly for technical reasons, because he defines the action of the bounded measures on an "abstract homogeneous Banach space". H.C. Wang exhibits in [wa77 26] an example of what he calls a *semi-homogeneous Banach space* (without strong continuity of the action of $\mathcal{G}$ on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, which does not allow the extension to all of the bounded measures. Indeed, it is a Banach space of measurable and bounded functions on $\mathbb{R}$ which is non-trivial, but which does not contain any non-zero *continuous* function!

but if $V_{g_0} f \in \boldsymbol{L}^1(\mathbb{R}^{2d}) \subset \boldsymbol{M}(\mathbb{R}^{2d})$ then we have

$$f = V_{g_0} f \bullet_\pi g_0$$

in the spirit of the above abstract statement (for $\rho = \pi$). It follows that one has for every TF-homogeneous Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, i.e. for every Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ such that $\|\pi(\lambda) f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}$ and $\|\pi(\lambda) f - f\|_{\boldsymbol{B}} \to 0$ for $\lambda \to 0$:

o-minimal1

$$\|f\|_{\boldsymbol{B}} = \|V_{g_0} f \bullet_\pi g_0\|_{\boldsymbol{B}} \leq \|V_{g_0} f\|_{\boldsymbol{L}^1(\mathbb{R}^{2d})} \|g_0\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{S}_0} \|g_0\|_{\boldsymbol{B}}. \tag{46}$$

Proof. The proof relies on the fact that for any net of convergent nets Ψ_β (all sufficiently "fine") the overall family $(D_{\Psi_\beta} \mu_\alpha$ is still bounded and uniformly tight! [29] Moreover it is clear that for each fixed β the net $D_{\Psi_\beta} \mu_\alpha$ is w^*-convergent to $D_{\Psi_\beta} \mu_0$. Given the tightness of the family only a finite number of indices of the family $(\psi_i^\beta)_{i \in I}$ is relevant for the convergence, hence $\mu_\alpha(\psi_i^\beta) \to \mu_0(\psi_i^\beta)$ implies that

$$D_{\Psi_\beta} \mu_\alpha \bullet_\rho f \to D_{\Psi_\beta} \mu_0 \bullet_\rho f.$$

FURTHER DETAILS HAVE TO BE CHECKED IN A CLEAN FORM LATER ON! Requiring the argument that one has for any $\mu \in \boldsymbol{M}(\mathcal{G})$: http://www.univie.ac.at/nuhag-php/cm/package.php

$$\|\mu \bullet_\rho f - D_\Psi \mu \bullet_\rho f\|_{\boldsymbol{B}} \leq \varepsilon \|\mu\|_{\boldsymbol{M}}$$

depending only on the element $f \in \boldsymbol{B}$ and the level of "refinement" of Ψ but NOT on the individual choice of μ. $\square$

Next we want to show that there is an important form of continuity from in this action from $\boldsymbol{M}(\mathcal{G}) \times \boldsymbol{B} \to \boldsymbol{B}$, with respect to the w^*-topology on $\boldsymbol{M}(\mathcal{G})$.

st-to-norm

Theorem 18. *Assume that $(\mu_\alpha)_{\alpha \in I}$ is a bounded and tight net of bounded measures, which is w^*-convergent to some limit measure $\mu_0 \in \boldsymbol{M}(\mathbb{R}^d)$. Then one has for every $f \in \boldsymbol{B}$:*

t-to-norm1

$$\|\mu_\alpha \bullet_\rho f - \mu_0 \bullet_\rho f\|_{\boldsymbol{B}} \to 0 \quad \textit{for} \quad \alpha \to \infty. \tag{47}$$

$\square$

[29] It is a good exercise to check the technical details yourself!

Some pointwise estimates

Pointwise estimates: Convolution preserves monotonicity We have to define $|\mu|$ for a given measure, and have to show that $\| \, |\mu| \, \|_{\boldsymbol{M}} = \|\mu\|_{\boldsymbol{M}}$ for each $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$.

point-conv
$$|\mu * f| \leq |\mu| * |f| \tag{48}$$

oint-conv1
$$|f| \leq |g| \Rightarrow |\mu| * |f| \leq |\mu| * |g| \tag{49}$$

ost-estim1
$$[(D_\Psi f - f) * g](x) \;\leq\; [|f| * \mathrm{osc}_\delta(g)](x), \, \forall x \in \mathbb{R}^d. \tag{50}$$

or in short, a pointwise estimate of the form

st-estim1b
$$[(D_\Psi f - f) * g] \;\leq\; [|f| * \mathrm{osc}_\delta(g)]. \tag{51}$$

MAIN ESTIMATE

in-estim11
$$|D_\Psi \mu * f - \mu * f| \leq |\mu| * |\operatorname{Sp}_\Psi f - f| \leq |\mu| * \mathrm{osc}_\delta f \tag{52}$$

if $\mathrm{diam}(\psi) \leq \delta$.

It relies on a couple of "simple" estimates, such as

$$\mathrm{osc}_\delta \check{f} = (\mathrm{osc}_\delta f)\check{}$$

and

$$\mathrm{osc}_\delta(T_x f) = T_x(\mathrm{osc}_\delta f)$$

Obviously

$$|\operatorname{Sp}_\Psi f(x) - f(x)| \;\leq\; \mathrm{osc}_\delta f(x), \, \forall x \in \mathbb{R}^d.$$

Moreover the fact that the discretization operator $D_\Psi : \boldsymbol{M}_b(\mathbb{R}^d) \mapsto \boldsymbol{M}_b(\mathbb{R}^d)$ is the adjoint of the spline operator $\operatorname{Sp}_\Psi : \boldsymbol{C}_0(\mathbb{R}^d) \mapsto \boldsymbol{C}_0(\mathbb{R}^d)$, implies also that we have:

conv-Dpsi
$$D_\Psi \mu * f = \mu * \operatorname{Sp}_\Psi f \tag{53}$$

Lemma 40. *If $f \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)$ then also $\mathrm{osc}_\delta f \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)$.*

Lemma 41. *A function $f \in \boldsymbol{C}_b(\mathbb{R}^d)$ belongs to $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)$ if and only if $f \in \boldsymbol{L}^p(\mathbb{R}^d)$ and $\mathrm{osc}_\delta f \in \boldsymbol{L}^p(\mathbb{R}^d)$.*

9. Discretization and the Fourier transform

TEST: We shall define here $\sqcup\!\sqcup_a := \sum_k \delta_{ak}$ and $\sqcup\!\sqcup^a = \frac{1}{a}\sum_n \delta_{\frac{n}{a}}$. Then $\mathcal{F}\sqcup\!\sqcup_a = \sqcup\!\sqcup^a$. In fact, one has for $a = 1$ according to Poisson's formula $\mathcal{F}\sqcup\!\sqcup_1 = \sqcup\!\sqcup^1$, and the general formula follows from this by a standard dilation argument: Mass preserving compression St_ρ is converted into "value-preserving" dilation D_ρ on the Fourier transform side, and $D_\rho \sqcup\!\sqcup^1 = \sqcup\!\sqcup^{1/\rho}$.

Let us put a few observations of importance at the beginning of this section:

- The periodic and discrete (unbounded) measures are exactly those which arise as periodic repetitions of a fixed finite sequence of the form $\sum_{k=0}^{N-1} a_k\delta_k$.
- The Fourier transform of such a sequence can be calculated directly using the FFT
- for any (sufficiently nice) function f (e.g. $f \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)$) one has for $b = 1/a$:

frepper1
$$\mathcal{F}\big[\sqcup\!\sqcup_{aN} * (\sqcup\!\sqcup^a \cdot f)\big] = \sqcup\!\sqcup^{Nb} \cdot (\sqcup\!\sqcup_b * \hat{f}) = \sqcup\!\sqcup_b * (\sqcup\!\sqcup^{Nb} \cdot \hat{f}) \tag{54}$$

The last step in the proof of formula frepper1 54 is easily verified directly: sampling and periodization commute if (and only if) the periodization constant (bN in our case) is a multiple of the sampling period (in our case b).

The question of approximately obtaining the continuous Fourier transform $\hat{f}$ of a "nice function" f from the FFT of it's sampled version can be derived from this fact.

xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

Given $h > 0$ and some prescribed function ψ on $\mathbb{R}^d$, such as a cubic B-spline, the quasi-interpolation $Q_h f = Q_h^\psi f$ of a continuous function f on $\mathbb{R}^d$ is defined by

eqdefQh
$$Q_h f(x) = \sum_{k\in\mathbb{Z}} f(hk)\psi(x/h - k), \qquad x \in \mathbb{R}^d. \tag{55}$$

For suitable ψ, this formula describes an approximation to f from its samples on the fine grid $h\mathbb{Z}^d \subset \mathbb{R}^d$.

theoS0 **Theorem 19.** *Assume that $\psi \in \boldsymbol{S}_0(\mathbb{R}^d)$ satisfies $\sum_{k\in\mathbb{Z}^d}\psi(x-k) \equiv 1$, i.e., that the family $(T_k\psi)_{k\in\mathbb{Z}^d}$ forms a partition of unity. Then for all $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ we have $\|Q_h f - f\|_{\boldsymbol{S}_0} \to 0$ as $h \to 0$.*

Note, that under the same restrictions on ψ one also has convergence of the quasi-interpolation scheme in the Fourier algebra $\mathcal{F}\boldsymbol{L}^1$ for all $f \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$.

Consequently one has $Q_h^*\sigma \to \sigma$ in the weak*-sense for each $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$. But $Q_h^*\sigma = \sum_k \sigma(\psi_k)\delta_{hk}$. Hence the discrete measures are w^*-dense in $\boldsymbol{S}_0'(\mathbb{R}^d)$.

10. Quasi-Interpolation

The piecewise linear interpolation operator for data available on the lattice of integers $\mathbb{Z}$, say $(c_k)_{k\in\mathbb{Z}}$, can be described as a sum of shifted triangular functions $\Delta(0)=1, \Delta(k)=0$ for $k\notin\mathbb{Z}$. Hence it can be written as a convolution product of the form

$$\big(\sum_{k\in\mathbb{Z}} c_k\delta_k\big)*\Delta.$$

It is easy to show that the resulting sum (the interpolant) belongs to $\boldsymbol{L}^p(\mathbb{R})$ if the sequence $\mathbf{c}$ is from $\boldsymbol{\ell}^p(\mathbb{Z})$. But this is true for much more general functions than then triangular function. It suffices to have $\varphi\in\boldsymbol{W}(\boldsymbol{C}_0,\boldsymbol{\ell}^1)(\mathbb{R})$ in order to find out that $\sum_{k\in\mathbb{Z}} T_k\varphi$ belongs to $\boldsymbol{W}(\boldsymbol{C}_0,\boldsymbol{\ell}^p)(\mathbb{R})$ for $\mathbf{c}\in\boldsymbol{\ell}^p(\mathbb{Z})$. In fact, this assumption implies $\sum_k c_k\delta_k\in\boldsymbol{W}(\boldsymbol{M},\boldsymbol{\ell}^p)$ and hence the convolution relations for Wiener amalgam spaces imply:

$$f=\sum_{k\in\mathbb{Z}} T_k\varphi=\big(\sum_{k\in\mathbb{Z}} c_k\delta_k\big)*\varphi\in\boldsymbol{W}(\boldsymbol{C}_0,\boldsymbol{\ell}^p)(\mathbb{R}).$$

As a consequence f is a continuous function and can be sampled, e.g., over the integers, but in most cases $f(k)$ will be perhaps close to, but different from the original sequence $(c_k)_{k\in\mathbb{Z}}$, hence the name *quasi-interpolation.* [30]

The so-called quasi-interpolation operators make sense for functions from $\boldsymbol{W}(\boldsymbol{C}_0,\boldsymbol{\ell}^p)(\mathbb{R}^d)$, to choose the appropriate generality from now on. For those functions one can guarantee that for some $C>0$ and all $p\in[1,\infty]$ one has:

$$\|(f(k))_{k\in\mathbb{Z}^d}\|_p\le C\|f\,|\,\boldsymbol{W}(\boldsymbol{C}_0,\boldsymbol{\ell}^p)\|\quad\forall f\in\boldsymbol{W}(\boldsymbol{C}_0,\boldsymbol{\ell}^p)(\mathbb{R}^d).$$

The same is true for any other lattice $\Lambda\lhd\mathbb{R}^d$, with

$$\|(f(\lambda))_{\lambda\in\Lambda}\|_p\le C_\Lambda\|f\,|\,\boldsymbol{W}(\boldsymbol{C}_0,\boldsymbol{\ell}^p)\|\quad\forall f\in\boldsymbol{W}(\boldsymbol{C}_0,\boldsymbol{\ell}^p)(\mathbb{R}^d).$$

Hence the operator

$$f\mapsto\sum_{\lambda\in\Lambda} f(\lambda)T_\lambda\varphi$$

is a well defined operator on $\boldsymbol{W}(\boldsymbol{C}_0,\boldsymbol{\ell}^p)(\mathbb{R}^d)$ (even uniformly bounded with respect to the range $p\in[1,\infty]$. We will call such an operator the *quasi-interpolation operator* with respect to the pair (Λ,φ).

Among the quasi-interpolation operators those which arise from BUPUs, i.e., from functions $\varphi\in\boldsymbol{W}(\boldsymbol{C}_0,\boldsymbol{\ell}^1)(\mathbb{R}^d)$ satisfying

$$\sum_{\lambda\in\Lambda} T_\lambda\varphi(x)\equiv 1$$

are the most important ones. We are going to show that quasi-interpolation operators with respect to "fine lattices" Λ are good approximation operators.

The interesting phenomenon is the behaviour of piecewise linear interpolation over lattices of the form $\alpha\mathbb{Z}^d$, for $\alpha\to 0$.

[30]Note that SINC is not covered by this example, although for $p\in(1,\infty)$ it shares more or less all the properties described above.

Let us recall that $(T_k\varphi)_{k\in\Lambda}$ is a BUPU for some $\varphi \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(\mathbb{R}^d)$ if and only if $\hat{\varphi}$ is a Lagrange interpolator over the orthogonal lattice $\Lambda^\perp = \{\chi \,|\, \langle \chi, \lambda\rangle \equiv 1 \quad \forall \lambda \in \Lambda\}$, i.e., that

Lagr-intp1
$$\widehat{\varphi}(\lambda') = \delta_{0,\lambda'} \quad \forall \lambda' \in \Lambda^\perp. \tag{56}$$

Proof. We can reinterpret the BUPU condition as $\sqcup\!\sqcup_H * \varphi \equiv 1$, which turns into

$$\mathcal{F}(\sqcup\!\sqcup_H) \cdot \hat{\varphi} = \mathcal{F}(1) = \delta_0.$$

Since $\mathcal{F}(\sqcup\!\sqcup_H) = C_H \sqcup\!\sqcup_{H^\perp}$ this condition reduces to (using $f \cdot \delta_x = f(x)\delta_x$):

$$C_H \sqcup\!\sqcup_{H^\perp} \cdot \hat{\varphi} = \sum_{h' \in H^\perp} \hat{\varphi}(h')\delta_{h'} = \delta_0,$$

which in turn is true if and only if $\hat{\varphi}(h') = 0$ for $h' \neq 0$ for all $h' \in H^\perp$.

□

Remark 18. The condition described above is invariant with respect to pointwise powers on the Fourier transform side, i.e., $\hat{\varphi}$ satisfies (Lagr-intp1 56) then the same is true for $\hat{\varphi}^2 = \widehat{\varphi * \varphi}$.

The quasi-interpolation operator $Q_{\Lambda,\varphi}$ can thus be described as the mapping

quasint
$$f \mapsto (\sqcup\!\sqcup_H \cdot f) * \varphi) \tag{57}$$

Note that this operators is bounded on $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}^d)$ because $\sqcup\!\sqcup_H \cdot f \in \boldsymbol{W}(\boldsymbol{M}, \boldsymbol{\ell}^\infty)(\mathbb{R}^d) \cdot \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}^d) \subseteq \boldsymbol{W}(\boldsymbol{M}, \boldsymbol{\ell}^p)(\mathbb{R}^d)$, hence

$$(f \cdot \sqcup\!\sqcup_H) * \varphi \subseteq \boldsymbol{W}(\boldsymbol{M}, \boldsymbol{\ell}^p) * \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1) \subseteq \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}^d).$$

It is of interest to check the behaviour of quasi-interpolation for the lattices $h\mathbb{Z}^d$, with $h \to 0$:

Given $h > 0$ and some prescribed function ψ on $\mathbb{R}^d$, such as a B-spline, the quasi-interpolation $Q_h f = Q_h^\psi f$ of a continuous function f on $\mathbb{R}^d$ is defined by

eqdefQh
$$Q_h f(x) = \sum_{k\in\mathbb{Z}} f(hk)\psi(x/h - k), \qquad x \in \mathbb{R}^d. \tag{58}$$

For suitable ψ, this formula describes an approximation to f from its samples on the fine grid $h\mathbb{Z}^d \subset \mathbb{R}^d$.

theoS0 **Theorem 20.** *Assume that $\psi \in \boldsymbol{S}_0(\mathbb{R}^d)$ satisfies $\sum_{k\in\mathbb{Z}^d} \psi(x-k) \equiv 1$, i.e., that the family $(T_k\psi)_{k\in\mathbb{Z}^d}$ forms a partition of unity. Then for all $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ we have $\|Q_h f - f\|_{\boldsymbol{S}_0} \to 0$ as $h \to 0$.*

Note, that under the same restrictions on ψ one also has convergence of the quasi-interpolation scheme in the Fourier algebra $\mathcal{F}\boldsymbol{L}^1$ for all $f \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$.

11. Advantages of a Distributional Fourier Transform

Whereas most books in the field of Fourier analysis describe the Fourier transform at various levels, typically starting from the classical case of the Fourier transform for periodic functions [4,14]. Sometimes the need of a generalized Fourier transform is motivated by the fact, that certain objects (like the "pure frequencies") do not have a Fourier transform in the usual sense, because first of all the classical Fourier transform is bound to diverge, while on the other hand the Fourier transform (which is in the generalized calculus a Dirac measure) is not an ordinary function, but has to be a kind of generalized function (in fact a bounded measure in that case), cf. [1].

In this section we want to emphasize (by demonstrating the situation, valid even for locally compact groups in full generality, through the example $\mathcal{G} = \mathbb{R}^d$).

Not only does the distributional Fourier transform (and it suffices to know the $\boldsymbol{S}_0'$-theory for this purpose) allow to define the Fourier transform for decaying objects (like functions in any of the $\boldsymbol{L}^p$-spaces), but also for periodic objects (such as periodic functions belonging locally to $\boldsymbol{L}^p$), even with different periods (which brings us already close to the discussion of almost periodic functions).

As a central topic let us therefore discuss the Fourier transform of periodic functions (or measures, or distributions) as the "infinite limit" of it's periodic repetitions. First of let us recall that it is easy to find for each lattice $\Lambda = \boldsymbol{A} * \mathbb{Z}^d$, for some non-singular $d \times d$-matrix $\boldsymbol{A}$ a fundamental domain (equal to $Q = \boldsymbol{A} * [0,1)^d$) and also bounded partitions of unity of the form $(\varphi_\lambda)_{\lambda\in\Lambda} = (T_\lambda \varphi)_{\lambda\in\Lambda}$, with $\varphi \in \boldsymbol{C}_c(\mathbb{R}^d)$ or even in $\mathcal{S}(\mathbb{R}^d)$. For the next lemma we need $\varphi \in \boldsymbol{S}_0(\mathbb{R}^d)$, or even better, in $\mathcal{S}(\mathbb{R}^d)$.

Lemma 42. *A function f (or distribution in $\boldsymbol{S}_0'(\mathbb{R}^d)$)* [31] *is periodic with respect to $\Lambda \lhd \mathbb{R}^d$ if and only if it is of the form*

$$f = \sum_{\lambda\in\Lambda} T_\lambda f^\circ_\lambda, \tag{59}$$

for some compactly supported pseudo-measure $f^\circ \in \mathcal{F}\boldsymbol{L}^\infty(\mathbb{R}^d)$.

Proof. If f is Λ-periodic, i.e., if $T_\lambda f = f$ for all $\lambda\in\Lambda$, then we can choose $f^\circ = f\varphi$, for a function φ with compact support, generating a Λ-BUPU as described above, because

$$f = \sum_{\lambda\in\Lambda} T_\lambda(T_{-\lambda}(f\varphi_\lambda)) = \sum_{\lambda\in\Lambda} T_\lambda f^\lambda = \sum_{\lambda\in\Lambda} T_\lambda f^\circ.$$

because $f^\lambda = T_{-\lambda}(f \cdot \varphi_\lambda) = T_{-\lambda} f \cdot T_{-\lambda}\varphi_\lambda = f \cdot \varphi =: f^\circ$ by the periodicity of f.

Conversely, let f° a compactly supported pseudo-measure or even in $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \boldsymbol{\ell}^1)$. Since $\boldsymbol{W}(\boldsymbol{M}, \boldsymbol{\ell}^\infty) * \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \boldsymbol{\ell}^1) \subset \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^\infty) = \boldsymbol{S}_0'$ the partial sums of the periodization are both uniformly in $\boldsymbol{S}_0'$ as well as w^*-convergent, as can been seen from the interpretation

$$\lim_{F\to\Lambda} \sum_{\lambda\in F} T_\lambda f^\circ = \lim_{F\to\Lambda} \left((\sum_{\lambda\in F} \delta_\lambda) * f^\circ \right).$$

[31] A distribution from $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ which is $\Lambda-$periodic for any co-compact lattice Λ in $\mathbb{R}^d$ does in fact belong automatically to $\boldsymbol{S}_0'(\mathbb{R}^d)$!

In fact, one can reduce the general case of w^*-convergence to the case where (first of all) f° as compact support, but in testing that the action on arbitrary test functions $h \in \boldsymbol{S}_0(\mathbb{R}^d)$ those in $(\boldsymbol{S}_0)_c = \boldsymbol{A}_c(\mathbb{R}^d)$ are sufficient.
PROBABLY SOME MORE DETAILS TO BE GIVEN □

We remark that a compactly supported pseudo-measure has the property that its Fourier transform is indeed a bounded and continuous function. Indeed, we can find some $\varphi \in \boldsymbol{S}_0(\mathbb{R}^d)$ such that $f^\circ \cdot \varphi$, hence $\mathcal{F}(f^\circ \cdot \varphi) = \widehat{f^\circ} * \hat{\varphi} \in \boldsymbol{L}^\infty * \boldsymbol{S}_0 \subseteq \boldsymbol{C}_b(\mathbb{R}^d)$. One can also show that the Fourier coefficients of the periodic version are just the values $\widehat{f^\circ}$ over the orthogonal lattice $\Lambda^\perp$. Let us describe this in detail:

Using now the fact that the distributional FT of $⊔⊔_\Lambda$ coincides with a multiple of $⊔⊔_{\Lambda^\perp}$, i.e., $\mathcal{F}⊔⊔_\Lambda = C_\Lambda ⊔⊔_{\Lambda^\perp}$ in the $\boldsymbol{S}_0'$-sense, we find that for any periodic function or distribution f we have

FT (60) $$\mathcal{F}f = \mathcal{F}(⊔⊔_\Lambda * f^\circ) = \mathcal{F}(⊔⊔_\Lambda) \cdot \hat{f}^\circ = C_\Lambda ⊔⊔_{\Lambda^\perp} \cdot \hat{f}^\circ.$$

Consequently we have $\mathrm{supp}(\hat{f}) \subseteq \mathrm{supp}(⊔⊔_{\Lambda^\perp}) = \Lambda^\perp$. In fact, we can give a more explicit description of $\hat{f}$: by carrying out the pointwise multiplication $⊔⊔_{\Lambda^\perp} \cdot \hat{f}^\circ$ we find that

$$\hat{f} = \sum_{\lambda^\perp \in \Lambda^\perp} \hat{f}^\circ(\lambda^\perp)\, \delta_{\lambda^\perp}.$$

But for $f^\circ \in \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \boldsymbol{\ell}^1)$ (by the Hausdorff-Young principle for generalized amalgams) we know that $\hat{f}^\circ \in \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^\infty) \subset \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^\infty) \subset \boldsymbol{C}_b(\mathbb{R}^d)$. Hence we can even claim that $\hat{f}$ is a sum of Dirac measures located at the points of $\Lambda^\perp$, with a bounded sequence of coefficients in $\boldsymbol{\ell}^\infty(\Lambda)$.

In fact (this has to be shown separately), one can be shown that this is a complete *characterization* all the tempered distributions $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ with $\mathrm{supp}(\sigma) \subseteq \Lambda^\perp$.

Lemma 43. *(Characterization of distributions supported on discrete subgroups)*
A distribution $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ satisfies $\mathrm{supp}(\sigma) \subseteq \Lambda$ *if and only if it is of the form*

$$\sigma = \sum_{\lambda \in \Lambda} c_\lambda \delta_\lambda$$

for some sequence $\mathbf{c} = (c_\lambda)_{\lambda \in \Lambda} \in \boldsymbol{\ell}^\infty(\Lambda)$.

Summarizing we have a mapping from the periodic elements (in $\boldsymbol{S}_0'(\mathbb{R}^d)$) into $\boldsymbol{\ell}^\infty(\Lambda)$, of the form $f \to (\hat{f}^\circ(\lambda))_{\lambda\in\Lambda}$ (which is of course independent of the choice of f°). Assume we have a regular distribution, "coming from" some $f \in \boldsymbol{L}^1(\mathbb{T})$. Let us discuss (for simplicity) the situation for $d = 1$ and $\Lambda = \mathbb{Z}$. Then we have $Q = [0, 1)$ and we can choose $f^\circ = f \cdot \mathbf{1}_Q \in \boldsymbol{L}^1(\mathbb{R})$. Since $\mathbb{Z}^\perp = \mathbb{Z}$ the Fourier transform of a periodic (local) $\boldsymbol{L}^1$-function are a sequence on $\mathbb{Z}$ again: $\hat{f}^\circ(n) = \int_0^1 f(t)e^{-2\pi i t}dt$ which coincides with the usual ("classical") definition of Fourier coefficients for locally integrable, $\mathbb{Z}$-periodic functions.

Note that in our context local square (or generally p-) integrability corresponds to additional properties on f° which in such a case will belong locally to $\boldsymbol{L}^2(\mathbb{R})$ (resp. $\boldsymbol{L}^p(\mathbb{R})$), resp. to $\boldsymbol{L}^2(\mathbb{T})$ or $\boldsymbol{L}^p(\mathbb{T})$. In our terminology $f^\circ \in \boldsymbol{W}(\boldsymbol{L}^2, \boldsymbol{\ell}^1)(\mathbb{R})$ resp. $W(\boldsymbol{L}^p, \ell^1)(\mathbb{R})$, and again by the Hausdorff-Young principle we obtain that $\hat{f}^{\circ} \in \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^\infty)(\mathbb{R})$, or the Fourier coefficients of f resp. the values of $(\hat{f}^{\circ}(n))_{n\in\mathbb{Z}}$ belong to $\boldsymbol{\ell}^{p'}(\mathbb{Z})$.

A topic of interest in connection with standard spaces is the following one, which is based on the fact that for any (restricted) standard space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ we have the following chain of continuous inclusion:

WB-incl0
$$\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^1) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^\infty) \tag{61}$$

and the fact that we have

WB-incl2
$$\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^p) \hookrightarrow \boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^q) \quad \text{if} \quad p \leq q. \tag{62}$$

w-uppindex **Definition 34.** Given a (restricted) standard space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ we define the *lower* resp. *upper index* as follows:

low-index
$$\operatorname{lowind}(\boldsymbol{B}) := \sup\{\, p \,|\, \boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^p) \hookrightarrow \boldsymbol{B}\} \tag{63}$$

and

upp-index
$$\operatorname{uppind}(\boldsymbol{B}) := \inf\{\, q \,|\, \boldsymbol{B} \hookrightarrow \boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^q)\} \tag{64}$$

In most cases the supremum resp. infimum will *not be attained*. However, we will have in any case

lowlequpp
$$\operatorname{lowind}_{\boldsymbol{B}} \leq \operatorname{uppind}_{\boldsymbol{B}} \tag{65}$$

The following condition is in general slightly stronger than the case of equality of indices:

Definition 35. A Banach space is called to be of *global type* p if one has

deftypep
$$\boldsymbol{B} = \boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^p). \tag{66}$$

Aside from the trivial facts that $\boldsymbol{L}^p(\mathbb{R}^d)$ is of course of type p for any p, one can check that $\boldsymbol{M}_b(\mathbb{R}^d)$ is of type 1, while the usual $\boldsymbol{L}^2$-Sobolev spaces are of type 2. They have therefore been called $\boldsymbol{\ell}^2$-puzzles by P. Tchamitchian in [24, 25] (?) tc84,tc87

One of the interesting and non-trivial facts (no proof is given here) is that one has (the most interesting perhaps being the case $p = 1$), see [12]: gr92-2

Lemma 44. *For* $1 \leq p \leq 2$ $\operatorname{lowind}(\mathcal{F}\boldsymbol{L}^p) = p = \operatorname{lowind}(\mathcal{F}\boldsymbol{L}^{p'})$ *while* $\operatorname{uppind}(\mathcal{F}\boldsymbol{L}^p) = p' = \operatorname{uppind}(\mathcal{F}\boldsymbol{L}^{p'})$. *Hence, except for the case* $p = 2$ *the space* $\mathcal{F}\boldsymbol{L}^p$ *is never of a particular type.*

Another interesting family of spaces is the set of all "multipliers" on $\boldsymbol{L}^p$, for say $1 \leq p < \infty$, which we denote by $H_{\boldsymbol{L}^1}(\boldsymbol{L}^p, \boldsymbol{L}^p)$, which is indeed another (restricted) standard space. It is well known that it coincides (by duality) with $H_{\boldsymbol{L}^1}(\boldsymbol{L}^{p'}, \boldsymbol{L}^{p'})$, that $\mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^1) = \boldsymbol{M}_b(\mathbb{R}^d)$ and $\mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^2, \boldsymbol{L}^2) = \mathcal{F}\boldsymbol{L}^\infty(\mathbb{R}^d)$. Hence one may *conjecture* that lowind and uppind of these spaces equals p and p', but to my knowledge nothing is known about it for $1 < p < 2$.

Another interesting question could be the upper and lower index for modulation spaces (which can be shown to be of local type $\mathcal{F}\boldsymbol{L}^q$). Since the space $\boldsymbol{M}^p(\mathbb{R}^d) = \boldsymbol{M}^s_{p,q}(\mathbb{R}^d)$ for $p = q$ are Fourier invariant an coincide with $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \boldsymbol{\ell}^p)$ they are clearly of *type* p.

12. Ideas on BUPUS, Wiener Amalgam and Spline-Type Spaces

BUPUs are a universal tool for many questions in the theory of function spaces, they are quite useful in order to develop concepts for (conceptual) harmonic analysis, and they are crucial for the definition of Wiener amalgam spaces.

Here is a short list of properties of these families that make them so important:

- They can be defined over arbitrary locally compact groups $\mathcal{G}$. In fact, there use is implicit in the construction of the Haar measure following Cartan resp. A. Weil;
- By means of BUPUs it is easy to show that the discrete measures are w^*-dense in the space of bounded measures $(\boldsymbol{M}_b(\mathcal{G}), \|\cdot\|_{\boldsymbol{M}_b})$.
- Obviously they are extremely useful in defining Wiener amalgam spaces (the discrete description is much more general, at least in order to introduce the spaces, than the "continuous" description);
- Spline-Type are quite important as well; they are obtained as "closed linear span" of a set of function and their translates, within some larger function space, say $\big(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\big)$. If we find a *Riesz projection basis* for such a space than typically the discrete $\boldsymbol{\ell}^p$-norm on the coefficients, and the $\boldsymbol{L}^p(\mathbb{R}^d)$ or also the $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}^d)$-norm are equivalent on the corresponding spline-type space.

For most purposes any result that is valid for regular BUPUs on $\mathbb{R}^d$ can be easily transferred to general statements about arbitrary BUPUs over locally compact, or at least locally compact Abelian groups.

13. Riesz Bases and Banach Frames

In the terminology introduced in XXX this means that $\mathbf{R}$ is injective, but not surjective, and $\mathbf{C}$ is a left inverse of $\mathbf{R}$. Thus we have the following commutative diagram.

$$\begin{tikzcd}
\boldsymbol{X} \arrow[d, "\mathbf{P}"'] \arrow[dr, "\mathbf{C}"] & \\
\boldsymbol{X}_0 \arrow[r, shift left, "\mathbf{C}"] & \boldsymbol{Y} \arrow[l, shift left, "\mathbf{R}"]
\end{tikzcd}$$

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