three dynamically-nested epistemic actions within this model: constructing, recognizing, and building-with. Abstraction is described from a sociocultural point of view. They base their definition on five principles, the last of which is crucial to explaining the genesis of abstraction as a learning process:

1. Abstraction is a chain of actions undertaken by an individual or group and driven by a context-specific motive;
2. Context is a personal and social construct;
3. Abstraction requires theoretical and empirical thought;
4. Abstraction leads from initial entities to a novel structure, and;
5. The structure comes into existence through creation of new internal links within the initial entities (Hershkowitz, Schwarz & Dreyfus 2001: 14).

The actor-oriented abstraction approach is aimed at coordinating individual and social levels of abstraction. It emerged through research directed at describing an alternative interpretation of educational transfer, called actor-oriented transfer (AOT; Lobato 2003). The actor-oriented approach seeks to describe how features of instructional environments, including curricular materials, social-cultural norms, tool use, and classroom discourse, interact to affect the conceptual attributes to which students pay attention. Actor-oriented abstraction includes modifications to the Piagetian construct of reflective abstraction to address criticisms. AOT seeks to describe both individual and social levels of abstraction and uses the device of attention focusing to do so. At the individual level, a student generalizes from regularities in records of experience in relationship to the student’s goals and activity, which is a structuring process from the perspective of the learner (Lobato 2012). The social level of abstraction is addressed by identifying records of focus, which are the mathematical ideas on which students seem to have focused their attention while interacting with particular representations in the classroom environment (Lobato, Ellis & Muñoz 2003). Joanne Lobato and colleagues seek to understand why these records of focus exist and account for them by what they refer to as focusing phenomena in the classroom, including the teacher’s actions, classroom discourse, curricular materials, and available tools. Decontextualization is addressed in this approach by considering...

“Context from the point of view of the actor (learner) rather than as something inherent in the situation… contextualizing could be seen as a dynamic process rather than as a static feature of situations to be removed.” (Lobato 2006: 440)

Building from individual to classroom learning and implications for teacher education

While earlier work in reflective abstraction provided useful tools for describing the conceptual growth of individual students, its weakness was that it was difficult to make recommendations concerning how these ideas could be used to develop curriculum or influence classroom instruction. Earlier approaches to reflective abstraction focused on the creation of learning tasks designed to induce reflective abstraction and conceptual learning, and focused mainly on the learning activities of the individual student when faced with these tasks. Currently, the learning “situation” is considered not only to be the specific learning tasks, but also the individual classroom culture. Classroom discourse is credited as being crucial to the concepts that individual students develop, and the limitations of curricular materials alone in spurring cognitive dissonance are recognized. One can summarize the different approaches to reflective abstraction approaches as encompassing the following components:

- Describing an individual student’s current knowledge of a concept
- Describing an individual student’s problem solving actions
- Designing learning tasks likely to spur reflective abstraction in individual or group settings
- Describing the mechanism of reflective abstraction in the context of tools and language
- Describing the mechanism of reflective abstraction in the context of the classroom environment and culture, including the discourse
- Describing reflective abstraction from the perspective of the actor or the observer.

Cifarelli & Sevim’s work falls into the category of providing detailed knowledge of an individual student’s current knowledge of a concept and the new knowledge that is created on the basis of the student’s problem solving activity. Such studies provide important insights into the types of mathematical tasks that can engender conceptual development in students.

Reflecting on a Radical Constructivist Approach to Problem Solving

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> Upshot • Cifarelli & Sevim outline the distinction between “representation” and “re-presentation” in von Glasersfeld’s thinking. After making this distinction, they identify how a student’s problem solving activity initially involved recognition, then re-presentation, and finally reflective abstraction. I use my commentary about the Cifarelli & Sevim article to identify two ways they could extend their current line of research.

Direction 1: What did Marie reflectively abstract?

Victor Cifarelli and Volkan Sevim argue that Marie’s problem-solving activity involved a transition from recognition to re-presentation, and subsequently that she made a reflective abstraction. Therefore, I wondered: What could Marie re-present after finishing the problem-solving sequence and what reflective abstraction did she

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make? To think about a response to these questions, I note that each of the tasks that Cifarelli & Sevim presented had the potential to be solved in (at least) three structurally different ways, and that Marie seemed to use each of these three ways at different points in her problem solving activity. Each way is structurally different in that a person conceives of different quantities in the situation as a basis for establishing an algebraic equivalence. I use an analysis of Task 1 to illustrate the three potential ways that a person could conceive of the quantities.

« 2 » The first way to solve each of the problems is to consider the unknown in the problem, \( x \), to be the depth of Blue Lake (the smaller of the two lakes), then to use the stated multiplicative relationship to find an algebraic expression, \( 2x \), for the depth of Clear Lake (the larger of the two lakes), and finally to establish an algebraic equivalence based on finding two equivalent ways to express the depth of Clear Lake (a quantity not explicitly stated in the problem), \( 2x + 12 = x + 35 \) (Figure 2). See Marie’s solution of Task 2 for a potential example of this kind of solution.

« 3 » A second, and closely related, way to solve the problems is to establish the unknown in the problem, \( x \), to be the depth of Blue Lake (the smaller of the two lakes), then to use the stated multiplicative relationship to find an algebraic expression, \( 2x \), for the depth of Clear Lake (the larger of the two lakes), and finally to establish an algebraic equivalence based on finding two ways to quantify the difference between the bottom of Blue Lake and the surface of Clear Lake (a quantity not explicitly stated in the problem), \( 2(x + 12) = x + 35 \) (Figure 3). See Marie’s solution of Task 1 for an example of this kind of solution. Based on the problem Marie posed in Task 9, my interpretation would be that Marie reflectively abstracted the first way of reasoning. I make this inference because her problem could most readily be solved by measuring the height of the taller hot air balloon using the height of the shorter hot air balloon, and the two differences given in the problem (3 ft. and 2 ft., respectively). When working with future students like Marie, it would be interesting for Cifarelli & Sevim to see if such students have the potential to reflectively abstract the second or third way of reasoning from their problem solving activity. It would be interesting because both ways of reasoning involve positing a quantity that is not stated in the problem, as Cifarelli & Sevim point out. Moreover, the third way of reasoning has the potential to involve distributive reasoning.

\[ x + 12 = x + 35 \]  
\[ 2(x + 12) = x + 35 \]

Figure 1: One way to conceive of Task 1.

Figure 2: A second way to conceive of Task 1.

Figure 3: A third way to conceive of Task 1.
Radical Constructivism

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and reasoning with complex fraction relationships (e.g., the equations for Task 7 are \(9/7(x + 15) = x + 35\) or \(x + 15 = 7/9(x + 35)\)). The additional kinds of reasoning involved in the second and third way of structuring the quantities are important to algebraic problem solving, and so seem worthy of further investigation with students.

**Direction 2: What role does the teacher play in problem solving?**

« 5 » A second direction for further investigation would be to address more explicitly the role that the teacher plays in facilitating problem solving for students. To suggest how Cifarelli & Sevim might do this, I use Ernst von Glasersfeld's (1987, 1991, 1995a) discussion of symbol formation and Patrick Thompson's (2000) model for communication (see also Tillema 2010). I note that this discussion is compatible with Cifarelli & Sevin's distinction between representation and re-presentation.

« 6 » Von Glasersfeld (1995a: 131) conceptualized a symbol as involving bi-directional relationships among a sound or graphic image, a concept, and a person's representations (Figure 4). The bi-directional relationships in Figure 4 denote that when a person is a sophisticated speech user, any of the three call up the other two (e.g., a sound/graphic image call up both a concept and a person's representations). One of these bi-directional relationships, the relationship between a sound or graphic image and a concept, is built up when a person isolates hearing a sound or seeing a graphic item and coordinates it with another aspect of an experience. For example, a young child might first isolate the sound produced when his or her parent says the word “apple,” and coordinate this sound image with the experience of biting into an apple. A person's apple experiences serve as the basis for the construction of concepts, where a concept is a program of operations (Glasersfeld 1991). In the apple example, the child's concept could consist of his or her use of a unitizing operation applied to the sensation of biting the apple.

« 7 » The other bi-directional links in Figure 1 involve a person's re-presentations – the capacity to replay and examine prior chunks of experience, while maintaining awareness that this replay is of a past experience. The bi-directional link between, for example, a sound/graphic image and a re-presentation simply means that a person can use a written word such as “apple” to call up prior apple experiences such as an experience of going to an apple orchard to pick apples as a child, or any other experiences that have been associated with the particular sound/graphic item.

« 8 » Von Glasersfeld (1991, 1995a) identifies that when a person can use a spoken word or graphic item as a symbol, then the spoken word or graphic item functions as a placeholder. It holds the place for actually having to implement activity that the person may initially have used to establish the concept and associated re-presentations. This placeholder function of symbols can be illustrated through a comparison of two students – one working to establish a difference meaning of subtraction and the other a student who can symbolize a difference meaning of subtraction. A student working to establish a difference meaning of subtraction might need to count the number of inches between two people's heights and may have yet to associate this activity with the graphic item “-”; in contrast, a student who has established the graphic item “-” as a symbol may simply use it to denote any time she is comparing two quantities using subtraction (e.g., two heights) without actually having to produce any activity – the student can use a graphic item such as h1 – h2 to mean the potential to make a comparison of two people's heights without actually having to implement activity that would produce the result of this comparison.

« 9 » Given this definition of symbols, communication, then, entails two or more people engaged in reciprocal assimilation of words/graphic items produced by the other person (Thompson 1999). Each person involved in communicating forms a model of the meaning the other person intends. For example, person A may present a problem in written form to person B; person B may assimilate the problem, and in response produce some symbolic activity for the situation (e.g., algebraic symbols, as Marie did in Task 1); person A may then assimilate person B's activity using the model they have developed of person B and respond to person B based on what they think might be a fruitful direction for person B to pursue.

« 10 » This model for communication seems compatible with what Cifarelli & Sevim outline about their orientation to problem solving. Situating their work on problem solving within a model for communication could help them to investigate questions such as the following:

- What types of questions asked by the teacher were effective in supporting a student's problem solving activities?
- What sequencing of tasks was effective in helping students establish particular ways of knowing in the context of their problem solving activities?
- What types of tools were useful for aiding in student's problem solving activities?

Pursuing answers to these types of questions could position Cifarelli & Sevim to make more explicit how von Glasersfeld's theory could help mathematics educators make decisions about individual and classroom instruction.

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