

Finite family symmetries in the lepton sector

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Vienna Theory Lunch Seminar, October 11th, 2011



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Outline of the talk

- Theoretical basics of fermion mixing
- Lepton mixing - Experimental results
- Finite family symmetries
- Examples
- Summary and conclusions

Theoretical basics of fermion mixing

Fermion mixing part I: The charged current interaction

Charged current interaction

$$\mathcal{L}_{CC} = -\frac{g}{\sqrt{2}} J_{CC}^\mu W_\mu^- + \text{H.c.}$$

where

$$J_{CC}^\mu = \bar{d}'_L \gamma^\mu u'_L + \bar{s}'_L \gamma^\mu c'_L + \bar{b}'_L \gamma^\mu t'_L \\ + \bar{e}'_L \gamma^\mu \nu'_{eL} + \bar{\mu}'_L \gamma^\mu \nu'_{\mu L} + \bar{\tau}'_L \gamma^\mu \nu'_{\tau L}.$$

Important question at this step:

What is the physical meaning of \bar{e}'_L, \dots ?

Is e'_L our usual electron field?

Only observable which allows to distinguish the types of charged leptons: **mass**.

Charged lepton flavours

The flavours of the charged leptons e, μ, τ are defined via their masses.

Fermion mixing part II: Fermion masses

In the standard model fermion masses are generated via the **Higgs mechanism**.

→ Yukawa coupling:

$$\mathcal{L}_Y^\ell = - \sum_{j=1}^{n_H} \sum_{\alpha\beta=e,\mu,\tau} \bar{D}'_{\alpha L} (\Gamma_j)_{\alpha\beta} \phi_j \beta'_R + \text{H.c.}$$

(most general form with SM Higgs doublets)

Γ_j : complex 3×3 -matrices = Yukawa coupling constants.

$\phi_j = \begin{pmatrix} \varphi_j^+ \\ \varphi_j^0 \end{pmatrix}$: SM Higgs doublets. (In SM: $n_H = 1$.)

$D'_{\alpha L} = \begin{pmatrix} \nu'_{\alpha L} \\ \alpha'_L \end{pmatrix}$: left-handed doublets.

Mass generation by spontaneous symmetry breaking:

$$\phi_j = \begin{pmatrix} \varphi_j^+ \\ h_j + v_j \end{pmatrix}$$

with $\langle 0 | \varphi_j^+ | 0 \rangle = 0$ and $\langle 0 | h_j | 0 \rangle = 0$. $v_j \in \mathbb{C}$ constant = vacuum expectation value (VEV).

$$\Rightarrow \mathcal{L}_Y^\ell = \left(- \sum_{\alpha=e,\mu,\tau} \bar{\alpha}'_L (\mathcal{M}_\ell)_{\alpha\beta} \beta'_L + \text{H.c.} \right) + \dots$$

\mathcal{M}_ℓ : complex 3×3 -matrix = charged lepton mass matrix;
depends on Yukawa-couplings Γ_j and VEVs v_j .

- \Rightarrow In general \mathcal{M}_ℓ not diagonal!
- \Rightarrow Fields e'_L, μ'_L, \dots are not the physical fields.

Fermion mixing part II: Fermion masses

\mathcal{M}_ℓ : arbitrary complex square matrix \Rightarrow can be diagonalized by a **bi-unitary** transformation:

$$\mathcal{M}_\ell = U_{\ell L} \text{diag}(m_e, m_\mu, m_\tau) U_{\ell R}^\dagger.$$

\Rightarrow Defining

$$\begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix} = U_{\ell L}^\dagger \begin{pmatrix} e'_L \\ \mu'_L \\ \tau'_L \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix} = U_{\ell R}^\dagger \begin{pmatrix} e'_R \\ \mu'_R \\ \tau'_R \end{pmatrix}$$

we find

$$\Rightarrow \mathcal{L}_Y^\ell = (-m_e \bar{e}_L e_R - m_\mu \bar{\mu}_L \mu_R - m_\tau \bar{\tau}_L \tau_R + \text{H.c.}) + \dots$$

Fermion mixing part II: Fermion masses

Similar situation in the neutrino sector: Mass eigenfields ν_1, ν_2, ν_3 :

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = U_{\nu L}^\dagger \begin{pmatrix} \nu'_{eL} \\ \nu'_{\mu L} \\ \nu'_{\tau L} \end{pmatrix} \quad (\text{in general})$$

Explicit form of neutrino mass term depends on whether neutrinos are Dirac or Majorana particles.

Dirac-neutrinos: mass term analogous to charged lepton and quark mass terms, i.e.

$$\mathcal{L} = -\bar{\nu}'_{\alpha L} (\mathcal{M}_\nu)_{\alpha\beta} \nu'_{\beta R} + \text{H.c.}$$

Majorana neutrinos:

$$\mathcal{L} = \frac{1}{2} \nu'_{\alpha L}{}^T C^{-1} (\mathcal{M}_\nu)_{\alpha\beta} \nu'_{\beta L} + \text{H.c.}$$

Fermion mixing part III: The lepton mixing matrix

We reformulate the leptonic part of the charged current interaction in terms of the mass eigenfields.

$$\begin{aligned}\mathcal{L}_{CC,\text{leptonic}} &= -\frac{g}{\sqrt{2}} W_\lambda^- (\bar{e}'_L \quad \bar{\mu}'_L \quad \bar{\tau}'_L) \gamma^\lambda \begin{pmatrix} \nu'_{eL} \\ \nu'_{\mu L} \\ \nu'_{\tau L} \end{pmatrix} + \text{H.c.} = \\ &= -\frac{g}{\sqrt{2}} W_\lambda^- (\bar{e}_L \quad \bar{\mu}_L \quad \bar{\tau}_L) \gamma^\lambda \underbrace{U_{\ell L}^\dagger U_{\nu L}}_U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} + \text{H.c.}\end{aligned}$$

$U = U_{PMNS}$: Pontecorvo-Maki-Nakagawa-Sakata matrix =
Lepton mixing matrix

Definition of neutrino flavour

Defined by associated charged lepton in the production/detection process

$$\Rightarrow \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} .$$

\Rightarrow Neutrino flavour eigenstate produced in charged current interaction is a **superposition of mass eigenstates**.

$$|\nu_\alpha\rangle = \sum_{j=1}^3 U_{\alpha j}^* |\nu_j\rangle$$

Neutrino propagation

Neutrinos are produced and detected as flavour eigenstates but they propagate as a **coherent** superposition of mass eigenstates.

$$|\nu_\alpha, x\rangle = \sum_{j=1}^3 U_{\alpha j}^* e^{-i(Et - p_j x)} |\nu_j\rangle.$$

$$p_j = \sqrt{E^2 - m_j^2} \simeq E - \frac{m_j^2}{2E} \quad \text{for } E \gg m_j.$$

Transition probability

$$P_{\nu_\alpha \rightarrow \nu_\beta} = |\langle \nu_\beta | \nu_\alpha, (t, L) \rangle|^2 = \left| \sum_{j=1}^3 U_{\beta j} U_{\alpha j}^* e^{-im_j^2 L/2E} \right|^2.$$

Fermion mixing part IV: Neutrino oscillations

$P_{\nu_\alpha \rightarrow \nu_\beta}$ is

- time independent
- oscillatory in $L/E \Rightarrow$ Neutrino oscillations,
- a function of the mass squared differences $\Delta m_{ij}^2 = m_i^2 - m_j^2$,
- a function of the elements of the mixing matrix U .

Fermion mixing part IV: Neutrino oscillations

Parameterization of U : three mixing angles $\in [0, \pi/2]$

- solar mixing angle θ_{12} ,
- reactor mixing angle θ_{13} ,
- atmospheric mixing angle θ_{23}

the CP phase $\delta \in [0, 2\pi)$ and six additional phases irrelevant for neutrino oscillations.

$$U = D_1 U' D_2,$$

$$U' = U_{23} U_{13} U_{12}$$

$$U_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \quad U_{13} = \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta} & 0 & c_{13} \end{pmatrix},$$

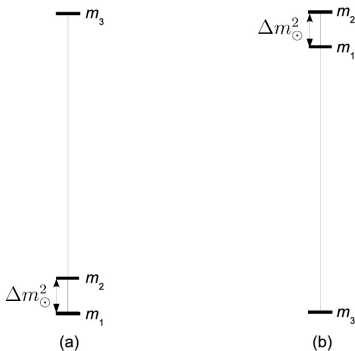
$$U_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad c_{ij} = \cos \theta_{ij}, \quad s_{ij} = \sin \theta_{ij}.$$

Fermion mixing part IV: Neutrino oscillations

Neutrino mass spectrum: By convention

$$m_2 > m_1 \Rightarrow \Delta m_{\odot}^2 := \Delta m_{21}^2 = m_2^2 - m_1^2 > 0.$$

\Rightarrow Two possible spectra: **normal** spectrum (a) and **inverted** spectrum (b).



Experimentally by now not decidable.

Lepton mixing - Experimental results

Fermion mixing part V: Experimental results

Global fit by Schwetz, Tortola, Valle; arXiv:1108.137

parameter	best fit $\pm 1\sigma$	2σ	3σ
Δm_{21}^2 [10^{-5}eV^2]	$7.59^{+0.20}_{-0.18}$	7.24–7.99	7.09–8.19
Δm_{31}^2 [10^{-3}eV^2]	$2.50^{+0.09}_{-0.16}$ $-(2.40^{+0.08}_{-0.09})$	2.25 – 2.68 $-(2.23 - 2.58)$	2.14 – 2.76 $-(2.13 - 2.67)$
$\sin^2 \theta_{12}$	$0.312^{+0.017}_{-0.015}$	0.28–0.35	0.27–0.36
$\sin^2 \theta_{23}$	$0.52^{+0.06}_{-0.07}$ 0.52 ± 0.06	0.41–0.61 0.42–0.61	0.39–0.64
$\sin^2 \theta_{13}$	$0.013^{+0.007}_{-0.005}$ $0.016^{+0.008}_{-0.006}$	0.004–0.028 0.005–0.031	0.001–0.035 0.001–0.039
δ	$\left(\begin{array}{l} -0.61^{+0.75}_{-0.65} \\ -0.41^{+0.65}_{-0.70} \end{array} \right) \pi$	0 – 2π	0 – 2π

Cosmological bounds: $\sum_{j=1}^3 m_j < 1 \text{ eV}$.

Fermion mixing part V: Experimental results

Most important aspects of the experimental results:

- Neutrino oscillations prove that at least two neutrinos must be massive.
- Neutrino masses are about 6 orders of magnitude smaller than the electron mass.
- $\sin^2\theta_{12} \approx \frac{1}{3} \gg 0$.
- $\sin^2\theta_{23} \approx \frac{1}{2} \gg 0$.
- $\sin^2\theta_{13} \approx 0$.

⇒ Suggestion by P.F. Harrison, D.H. Perkins, W.G. Scott (2002):

$$U \approx U_{HPS} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Till about June 2011: U_{HPS} within experimental 2σ -range, but then ...

Surprise: $\sin^2\theta_{13} > 0$ at 3σ !

(T2K-Experiment, arXiv:1106.2822)

Since then: hot topic in flavour physics

Nevertheless: U_{HPS} good “leading order” approximation.

Finite family symmetries

Finite family symmetries

Form of U_{HPS} :

$$U \approx U_{HPS} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

→ Idea: **Symmetry in the lepton sector.**

What types of symmetries can we have?

- Continuous symmetries: either gauged or not
→ **Problem:** Symmetry must be broken \Rightarrow additional **massive gauge bosons**, or (even worse) **massless Goldstone bosons**.
- Discrete groups: No problem with additional bosons if there are no accidental continuous symmetries. However, other problem: **domain walls**.

→ Popular ansatz: Lagrangians that are invariant under the action of **finite groups**.

The heart of such models is the **G -invariant Yukawa coupling**.

G -invariant Yukawa couplings

The Yukawa coupling

$$Y = -\bar{a}\Phi_j\Gamma_j b + \text{H.c.}$$

is called invariant under a finite group G , if there are matrix representations D_a, D_b, D_ϕ of the group G , such that Y is invariant under

$$a \mapsto D_a a, \quad b \mapsto D_b b, \quad \Phi \mapsto D_\phi \Phi.$$

$$Y = -\bar{a}\Phi_j\Gamma_j b + \text{H.c.}$$

Γ_j ... complex 3×3 -matrices

Φ_j ... e.g. standard model Higgs doublets, in general scalars (Higgs-triplets,...)

The scalar sector of the standard model is extended!

SSB: $\Phi_j \mapsto \langle \Phi_j \rangle_0 + H_j$

$$\Rightarrow \mathcal{M} = \langle \Phi_j \rangle_0 \Gamma_j.$$

A chosen finite group G restricts

- the number of scalars,
- the structure of the mass matrices,
- the structure of the mixing matrix.

G is called a **finite family symmetry group**.

How does G restrict Γ_j and Φ ?

Theorem

$Y = -\bar{a}\Phi_j\Gamma_j b + \text{H.c.}$ is a G -invariant Yukawa coupling if and only if Γ_{jkl} are **Clebsch-Gordan coefficients (CGCs)** for

$$(D_a^{-1})^\dagger \otimes (D_b^{-1})^T = D_\phi \oplus \dots$$

$$\iff D_a^\dagger \Gamma_j (D_\phi)_{jk} D_b = \Gamma_k.$$

This equation can be interpreted as an eigenvalue problem to the eigenvalue 1 \Rightarrow solvable using computer algebra systems.

Important question: **Which finite groups are appropriate as family groups?**

Idea: 3 generations of fermions → specialize to finite groups that have **faithful 3-dimensional representations**.

→ Finite subgroups of $U(3)$

Unfortunately not yet classified.

⇒ Finite subgroups of $SU(3)$ (already classified at the beginning of the 20th century by Miller, Dickson and Blichfeldt.¹)

¹*Theory And Applications of Finite Groups*; John Wiley & Sons, New York, 1916

Finite family symmetries

Today: Classification of Blichfeldt refined: Structure of all finite $SU(3)$ -subgroups known:

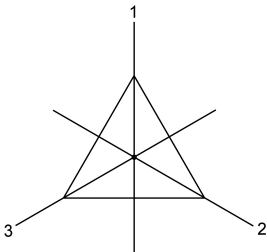
(A) Groups of diagonal matrices: Abelian groups $\mathbb{Z}_m \times \mathbb{Z}_n$.

Examples: Rotations about one axis: \mathbb{Z}_m

(B) Finite subgroups of $U(2)$.

$$\begin{pmatrix} \det A^* & 0 \\ 0 & A \end{pmatrix}, \quad A \in U(2)$$

Examples: Three-dimensional rotation symmetries of regular polygons.

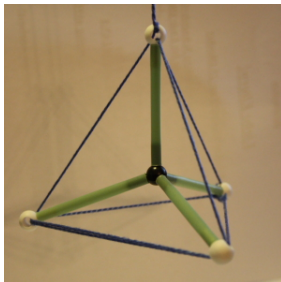


Finite family symmetries

(C) Groups of the form $(\mathbb{Z}_m \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3$.

Examples: $\Delta(3n^2) \cong (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3$,

$\Delta(12) \cong A_4$: symmetry group of the tetrahedron.

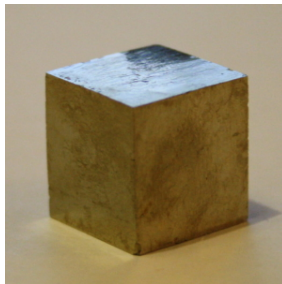
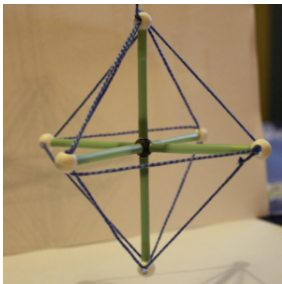


Finite family symmetries

(D) Groups of the form $(\mathbb{Z}_m \times \mathbb{Z}_n) \rtimes S_3$.

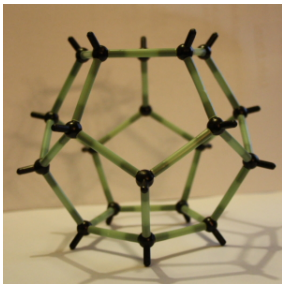
Examples: $\Delta(6n^2) \cong (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes S_3$,

$\Delta(24) \cong S_4$: symmetry group of the octahedron and the cube.



Finite family symmetries

- (E) Six exceptional groups: $\Sigma(m \times 3)$, $m = 36, 72, 216, 360$,
and the simple groups $PSL(2, 7)$ and $A_5 =$ symmetry group of
the dodecahedron/icosahedron.



Examples

Example 1: Abelian groups

Abelian groups have only one-dimensional irreducible representations \Rightarrow **Texture zeros** in mass matrices.

Example: Model based on \mathbb{Z}_6 taken from: W. Grimus, L. Lavoura [arXiv:hep-ph/0412283]

One dimensional representations of \mathbb{Z}_6 : $a \mapsto \pm\omega^m$,
 $\omega = \exp(2\pi i/3)$, $m = 0, 1, 2$.

Particle content and transformation properties:

- n standard model Higgs doublets: $\phi_j \mapsto \phi_j$ (trivial transf.)
- standard model leptons:

$$\begin{pmatrix} D_e \\ D_\mu \\ D_\tau \end{pmatrix} \mapsto \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} D_e \\ D_\mu \\ D_\tau \end{pmatrix}, \quad \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix} \mapsto \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix}$$

- three Higgs triplets Δ_k :

$$\begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\omega^2 \end{pmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix}$$

Example 1: Abelian groups

⇒ Form of the mass matrices:

$$\mathcal{M}_\ell \sim \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix}, \quad \mathcal{M}_\nu \sim \begin{pmatrix} 0 & 0 & \times \\ 0 & \times & \times \\ \times & \times & \times \end{pmatrix}$$

texture zeros

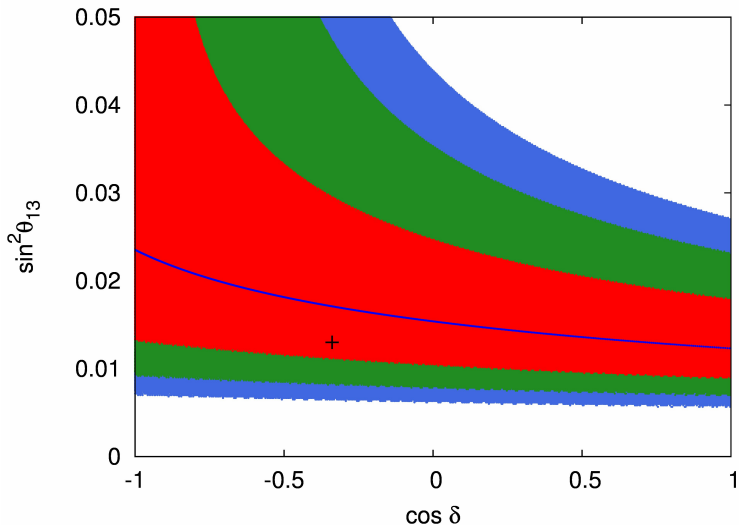
× ... nonzero entries dependent on vacuum expectation values.

Interesting texture zeros in the light of the recent T2K data:

P.O.L., S. Morisi, E. Peinado [arXiv:1109.3393 [hep-ph]]:

The above texture leads to $\theta_{13} > 0$ at 3σ !

Example 1: Abelian groups



Example 2: Non-Abelian groups

Example: Model based on the non-Abelian group S_3 : (W. Grimus, L. Lavoura and A. Singraber [arXiv:0911.5120 [hep-ph]].)

Particle content:

- Standard model fermions $D_{\alpha L}, \alpha_R$ ($\alpha = e, \mu, \tau$).
- Five right-handed neutrinos: $\nu_{eR}, \nu_{\mu R}, \nu_{\tau R}, \nu_{1R}, \nu_{2R}$.
- Four standard model Higgs doublets: $\phi_0, \phi_1, \phi_2, \phi_3$.
- A complex scalar singlet χ .

Symmetries of the model:

- Family lepton number (softly broken).
- Three \mathbb{Z}_2 -symmetries: $\alpha_R \rightarrow -\alpha_R, \phi_\alpha \rightarrow -\phi_\alpha$
- S_3 : two generators

$$\mathbf{1} : a \mapsto 1, \quad b \mapsto 1$$

$$\mathbf{1}' : a \mapsto -1, \quad b \mapsto 1$$

$$\mathbf{2} : a \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad \omega = \exp(2\pi i/3)$$

Example 2: Non-Abelian groups

$$D_{\alpha L} \sim (\mathbf{1} \oplus \mathbf{2}), \quad \alpha_R \sim (\mathbf{1} \oplus \mathbf{2}), \dots$$

$$\begin{pmatrix} \nu_{1R} \\ \nu_{2R} \end{pmatrix} \sim \mathbf{2}, \quad \begin{pmatrix} \chi \\ \chi^* \end{pmatrix} \sim \mathbf{2}.$$

Soft and spontaneous breaking + type I seesaw mechanism \Rightarrow

$$U = \begin{pmatrix} 2c/\sqrt{6} & 1/\sqrt{3} & 2s/\sqrt{6} \\ -c/\sqrt{6} + s/\sqrt{2} & 1/\sqrt{3} & -s/\sqrt{6} - c/\sqrt{2} \\ -c/\sqrt{6} - s/\sqrt{2} & 1/\sqrt{3} & -s/\sqrt{6} + c/\sqrt{2} \end{pmatrix},$$

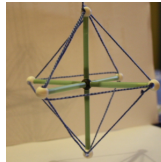
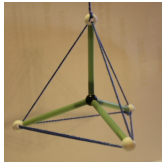
where $c = \cos(\zeta/2)$, $s = \sin(\zeta/2)$, $\zeta = \arg v_\chi$.

For $\zeta = 0 \Rightarrow$ Harrison-Perkins-Scott (tribimaximal) mixing.

Summary and conclusions

- Introduction of neutrino mass terms \Rightarrow **Lepton mixing**.
- Lepton mixing experimentally verified through observation of **neutrino oscillations**. \Rightarrow At least two neutrinos massive.
- **Lepton mixing angles**: Two large angles and one small (but nonzero) one. Completely different to quark sector.
- Nice form of lepton mixing matrix \rightarrow Idea: **Symmetries in the lepton sector?**
- Interesting candidates: **Discrete groups** \Rightarrow No Goldstone bosons.
- **Abelian groups**: **Texture zeros**.
- More predictive models by use of **non-Abelian groups**.
- Main goal for the future: More systematic way of analyzing models with finite family symmetries.

Thank you for your attention!



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