Approximate Foveated Images and Reconstruction of their Uniform Pre-Images

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Abstract

Approximate foveated images can be obtained from uniform images via the approximation of some integral operators. In this paper it is shown that these operators belong to a well studied operator algebra, and the problem of restoration of the approximate uniform pre-images is considered. Under common assumptions on smoothness of the integral operator kernels, necessary and sufficient conditions are established for such procedure to be feasible.

Key words: Foveated images, spline Galerkin approximation, singular integral, Toeplitz algebra
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1 Introduction

Conventional images have a uniform resolution: every part of the image is resolved with the same accuracy. In contrast to this, biological visual systems are typically space-dependent structured: high resolution is available only at a relatively small area of interest, and the resolution decreases when moving away from this area. The high resolution visual center is called fovea. Foveated distributions of resolution are used in signal and image processing in order to reduce the (global) information of the visual field while preserving the (local) resolution at the fovea. The corresponding non-uniform images are called foveated images.

Technically, foveated images are obtained from uniform images in different ways, cf. [2,5,10,11,13]. In the present paper, we consider an approach which is based on an integral transform

\[
(T\varphi)(x) = \int_{\mathbb{R}} k(t, x)\varphi(t) \, dt, \quad x \in \mathbb{R}^s ,
\]

with an accordingly specified kernel function \( k \). The function \( \varphi \) in (1) is considered as the initial, or uniform image, and \( T\varphi \) represents its foveated image. The function \( \varphi \) is also called the pre-image of the foveated image \( T\varphi \). In case \( s = 1 \), one often speaks about ‘signals’ instead of ‘images’.

We will consider kernel functions \( k \) of the operator (1) of the following form. Let \( g \) be a function on the real line \( \mathbb{R} \). For given parameters \( \gamma \in \mathbb{R} \) and \( \beta \in (0, \infty) \), let \( \omega = \omega_{\gamma, \beta} \) denote the function

\[
\omega_{\gamma, \beta}(x) := \beta|x - \gamma|, \quad x \in \mathbb{R}.
\]

Consider the integral operator

\[
(T\varphi)(x) = \int_{\mathbb{R}} \frac{1}{\omega_{\gamma, \beta}(x)} g \left( \frac{t-x}{\omega_{\gamma, \beta}(x)} \right) \varphi(t) \, dt.
\]

In the present setting, the parameter \( \gamma \) represents the point of the highest resolution - the fovea. The parameter \( \beta \) determines the speed at which the resolution falls off when the distance from the fovea grows, cf. [5].

To ensure the boundedness of the operator \( T \) on the space \( L^2(\mathbb{R}) \), it is usually assumed (see [5] again) that \( g \) is in \( L^1(\mathbb{R}) \) and bounded. We are going to impose different conditions on the kernel of \( T \). These condition will not only provide
the boundedness of $T$ on an appropriate function space; they will moreover guarantee that the operator $T$ belongs to a well-studied operator algebra.

It is sufficient to examine the operator (2) with fovea 0. Indeed, for $\gamma \in \mathbb{R}$, let $U_{\gamma}$ denote the shift operator 

$$(U_{\gamma}f)(x) = f(x - \gamma), \quad x \in \mathbb{R}.\$$

Then $T$ can be rewritten as $T = U_{\gamma}T_{g,\beta}U_{-\gamma}$, where $T_{g,\beta}$ is the integral operator (2) with fovea $\gamma = 0$, i.e.,

$$(T_{g,\beta}f)(x) = \int_{-\infty}^{+\infty} f(t) \frac{1}{\beta|x|} g\left(\frac{t-x}{\beta|x|}\right) dt. \tag{3}$$

Thus, without loss of generality, we can restrict our attention to the operator $T_{g,\beta}$. As we will see later, this operator belongs to a class of integral operators which is quite well understood. This observation will enable us to examine the properties of the foveated image $T\varphi$ in more detail.

To create foveated images, Chang, Mallat and Yap [5] proposed to consider an approximation $\varphi_n$ to the signal $\varphi$ with subsequent approximation of $T\varphi_n$. More precisely, for each positive number $m$, let $Q_m$ be the linear operator on $L^2(\mathbb{R})$ defined by

$$(Q_m f)(x) := \begin{cases} f(x) & \text{if } x \in [-m, m] \\ 0 & \text{otherwise.} \end{cases}$$

Further, let $P_n$ stand for the orthogonal projection of $L^2(\mathbb{R})$ onto a finite dimensional subspace. Then Chang, Mallat and Yap consider the following approximate foveated image $\psi_n^m$ of the signal $\varphi$:

$$\psi_n^m(x) := (P_n Q_m T Q_m P_n \varphi)(x), \quad x \in \mathbb{R}, \tag{4}$$

with $T$ given by (2). Note that the operators $P_n Q_m T Q_m P_n$ with $n \in \mathbb{N}$ can be viewed as Galerkin approximations for the operator $Q_m T Q_m$. Sequences of Galerkin approximations have been studied in connection with different problems of analysis and mathematical physics.

The present paper is addressed to the following inverse problem. Assume that an approximate foveated image $\psi_n^m$ is known. Is it possible to reconstruct its approximate uniform pre-image $\varphi_n^m = Q_m P_n \varphi$? It is obvious that, in general, the answer to this question is negative. However, one can try to find
conditions which would make such a reconstruction feasible. Another relevant problem is that of the quality of the approximate uniform images one obtains. More precisely, if $\varphi^m_n, n \in \mathbb{N}$, is an approximation for $\varphi$, what can be said about the errors $\varphi - \varphi^m_n$, at least for large $n$? In the present paper these problems are studied for the Galerkin approximations. For the sake of simplicity, piecewise constant splines are used to approximate the uniform and foveated images (but other approximation spaces could be used as well; see the final section where approximations based on splines of higher order are briefly discussed). It is also worth mentioning that the Galerkin scheme can be replaced by other approximation procedures, for example by collocation, quadrature or qualocation, which are often more convenient and less expensive from the computational point of view.

**Outline of the paper.** Section 2 establishes conditions for the boundedness of the foveation operator $T_{g, \beta}$ in the spaces under consideration. We will see that the same conditions ensure that $T_{g, \beta}$ belongs to a well-known operator algebra. In Section 3, we are going to introduce Galerkin projections, and we will recall notions and results relating to the stability of approximation methods. Section 4 is devoted to the study of the stability of an operator sequence associated with the operator $T_{g, \beta}$ and with approximate foveated images. The stability of this sequence provides a theoretical base for approximate restoration of uniform pre-images of foveated images. In Section 5, we consider a concrete operator of the form (2) for which all stability conditions can be effectively verified. In the concluding Section 6, we will generalize the obtained approximate restoration results to splines of higher order.

**Some notations.** Throughout this paper, let $p$ and $\alpha$ be real parameters with $1 < p < \infty$ and $0 < \alpha + 1/p < 1$. Given an interval $I \subseteq \mathbb{R}$, we write $L^p(I, \alpha)$ for the Banach space of all Lebesgue measurable functions $f : I \to \mathbb{C}$ such that

$$
\|f\|_p = \|f\|_{p, \alpha, I} := \int_I |f(t)|^p |t|^{\alpha p} dt < \infty. \quad (5)
$$

The Banach dual $L^p(I, \alpha)^*$ of $L^p(I, \alpha)$ will be identified with $L^q(I, -\alpha)$ where $1/p + 1/q = 1$ with respect to the sesqui-linear form

$$
\langle f, g \rangle := \int_I f(t) \overline{g(t)} dt.
$$

Given a linear space $X$ and a positive integer $r$, we denote by $X_r$ the linear space of column vectors of length $r$ with components from $X$, and we let $X^{r \times r}$ refer to the linear space of $r \times r$ matrices with entries from $X$. Further, $\mathfrak{B}(X)$ denotes the Banach algebra of all bounded linear operators on the Banach space $X$, and $\text{im } A$ stands for the range of the operator $A \in \mathfrak{B}(X)$. 

4
2 Integral Operators of Mellin type

The goal of this section is to represent the operator \( T_{g, \beta} \) defined by (3) in a special form which will prove to be crucial for the further analysis of that operator.

We will allow the operator \( T_{g, \beta} \) to act on the weighted space \( L^p(\mathbb{R}, \alpha) \) whereas Chang, Mallat and Yap [5] studied this operator on the Hilbert space \( L^2(\mathbb{R}) \). The use of the weighted \( L^p \)-spaces in the present paper is needed to force the invertibility of certain auxiliary integral operators which will be considered below. The invertibility of these operators plays a dominant role in our considerations, and it turns out that this invertibility indeed depends on the choice of the underlying space. An archetypal example of an operator where this phenomenon can be observed is the singular integral operator of Cauchy type on the semi axis (which will be examined in detail in Section 5). This operator is invertible on the weighted space \( L^p(\mathbb{R}^+, \alpha) \) if and only if \( \alpha + 1/p \neq 1/2 \). In particular, this operator fails to be invertible on \( L^2(\mathbb{R}^+) \), but it is invertible on \( L^p(\mathbb{R}^+) \) for every \( p \in (1, \infty) \setminus \{2\} \).

In what follows, we identify the space \( L^p(\mathbb{R}, \alpha) \) with the space \( L^p_2(\mathbb{R}^+, \alpha) \) which consists of all pairs \( (f_1, f_2)^T \) with \( f_1, f_2 \in L^p(\mathbb{R}^+, \alpha) \). We provide the space \( L^p_2(\mathbb{R}^+, \alpha) \) with the norm

\[
\| (f_1, f_2)^T \|^p := \| f_1 \|^p_{p, \alpha, \mathbb{R}^+} + \| f_2 \|^p_{p, \alpha, \mathbb{R}^+}.
\]

Then the mapping

\[
\eta : L^p(\mathbb{R}, \alpha) \to L^p_2(\mathbb{R}^+, \alpha), \quad \eta(f) : s \mapsto (f(s), f(-s))^T
\]

becomes an isometric bijection the inverse of which acts via

\[
(\eta^{-1}[(f_1, f_2)^T])(s) = \begin{cases} f_1(s) & \text{if } s \in \mathbb{R}^+ \\ f_2(-s) & \text{if } s \in \mathbb{R}^- \end{cases}
\]

Thus, the mapping

\[
\psi_\eta : \mathfrak{B}(L^p(\mathbb{R}, \alpha)) \to \mathfrak{B}(L^p_2(\mathbb{R}^+, \alpha)), \quad A \mapsto \eta A \eta^{-1},
\]

is an isometric algebra isomorphism. Therefore, the properties of the operator \( \psi_\eta(A) \) reflect completely the corresponding properties of \( A \), and vice versa.
However, there are some instances where the operator $\psi_\eta(A)$ has a nicer structure than the operator $A$ itself. In particular, it will turn out in a few moments that the entries of the operator $\psi_\eta(A_{g, \beta})$ are Mellin convolution operators, a class of operators which we are going to introduce now.

Let $M$ and $M^{-1}$ denote the direct and inverse Mellin transform, respectively, i.e.,

$$(Mf)(z) = \int_0^{+\infty} x^{1/p+\alpha+1-iz} f(x) \, dx, \quad z \in \mathbb{R},$$

and

$$(M^{-1}f)(x) = \frac{1}{2\pi} \int_0^{+\infty} x^{-1/p-\alpha+iz} f(z) \, dz, \quad x \in \mathbb{R}^+.$$

It is well known (see, e.g., [8], pp. 47-48) that if $b$ is an $L^p(\mathbb{R})$-Fourier multiplier, then

$$\mathcal{M}(b) := M^{-1}bM$$

defines a bounded linear operator $\mathcal{M}(b)$ on $L^p(\mathbb{R}^+, \alpha)$, the so-called Mellin operator with symbol $b$. For example, every bounded function of finite total variation on $\mathbb{R}$ is a Fourier multiplier.

In case the kernel function $k := M^{-1}b$ belongs to $L^1(\mathbb{R}^+)$ with respect to the measure $ds/s$, the Mellin operator (8) can be represented as the integral operator

$$(\mathcal{M}(b)f)(s) = \int_0^{+\infty} k(s/\sigma)f(\sigma) \frac{d\sigma}{\sigma}, \quad s \in \mathbb{R}^+.$$

(Note that the Cauchy singular integral operator considered in Section 5 can also be represented in this form. However, this operator has to be understood in the Cauchy principal value sense.)

**Proposition 1** Let the operator $T_{g, \beta}$ be defined by (3), and let $G_{g, \beta}$ be the matrix function defined by

$$G_{g, \beta}(t) := \frac{1}{\beta t} \begin{pmatrix} g \left( \frac{1}{3} \left( \frac{t}{\beta} - 1 \right) \right) & g \left( \frac{1}{3} \left( -\frac{1}{\beta} - 1 \right) \right) \\ g \left( \frac{1}{3} \left( \frac{t}{\beta} + 1 \right) \right) & g \left( \frac{1}{3} \left( -\frac{1}{\beta} + 1 \right) \right) \end{pmatrix}, \quad t \in \mathbb{R}^+.$$
Assume that the entries of the matrix function $B_{g, \beta} := MG_{g, \beta}$ are continuous functions of finite total variation which have finite limits at $\pm \infty$. Then

$$\psi_\eta(T_{g, \beta}) = \mathcal{M}(B_{g, \beta}),$$

i.e. $\psi_\eta(T_{g, \beta})$ is the block Mellin operator with symbol $B_{g, \beta}$.

**Proof.** The operator $T_{g, \beta}\eta^{-1}$ acts as follows:

$$(T_{g, \beta}\eta^{-1}[(f_1, f_2)^T])(s)$$

$$= \frac{1}{\beta} \int_0^{+\infty} f_1(t) \frac{t - s}{|s|} g \left( \frac{t - s}{\beta |s|} \right) \frac{dt}{t} + \frac{1}{\beta} \int_{-\infty}^0 f_2(-t) \frac{t - s}{|s|} g \left( \frac{t - s}{\beta |s|} \right) \frac{dt}{t}. \quad (11)$$

The second term on the right-hand side of this equality can be rewritten as

$$\frac{1}{\beta} \int_0^{+\infty} f_2(u) \frac{u - s}{|s|} g \left( \frac{u - s}{\beta |s|} \right) \frac{du}{u}.$$ 

Hence, by (6),

$$\eta \left( \frac{1}{\beta} \int_{-\infty}^0 f_2(-t) \left( \frac{t}{|t|} \right) g \left( \frac{t - s}{\beta |t|} \right) \frac{dt}{t} \right)(s)$$

$$= \left( \begin{array}{c} \frac{1}{\beta} \int_0^{+\infty} f_2(u) \left( \frac{u}{s} \right) g \left( \frac{1}{\beta} \left( \frac{-u}{s} - 1 \right) \right) \frac{du}{u} \\ \frac{1}{\beta} \int_0^{+\infty} f_2(u) \left( \frac{u}{s} \right) g \left( \frac{1}{\beta} \left( \frac{-u}{s} + 1 \right) \right) \frac{du}{u} \end{array} \right), \quad s \in \mathbb{R}^+ \quad \text{(12)}$$

Performing analogous transformations for the first term on the right-hand side of (11), one gets

$$\eta T_{g, \beta}\eta^{-1} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \quad \text{(13)}$$

where the operators $T_{rl}$ are of the form (9) with kernels defined by the corresponding entries of the matrix (10). Moreover, the matrix $B_{g, \beta}(z)$ can be
represented in the form
\[
B_{g, \beta}(z) = B^{(1)} + B^{(2)} \coth \pi(z + i(1/p + \alpha)) + B_{g, \beta}^{\circ}(z), \quad z \in \mathbb{R}, \quad (14)
\]
where \(B^{(1)}\) and \(B^{(2)}\) are constant matrices, and where the entries of \(B_{g, \beta}^{\circ}\) are continuous functions of finite total variation having limit zero at \(\pm \infty\). Indeed, this representation follows easily from the fact that the function \(\coth\) has limits \(\pm 1\) at \(\pm \infty\). Note that similar representations can be obtained by choosing any other function \(f\) with \(f(\pm \infty) \neq f(\mp \infty)\) instead of the \(\coth\). The advantage of our choice is that the \(\coth\) function is the Mellin symbol of the Cauchy singular integral operator. This fact will simplify some of our further arguments essentially. \(\square\)

Thus Proposition 1 ensures that the operator \(\psi_\eta(T_{g, \beta})\) belongs to the algebra of Mellin operators described in [6], [8, pp. 47 – 51].

3 Galerkin approximations of the foveated images

Let \(\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm \infty\}\) be the compactification of the real axis by the two points \(\pm \infty\), and let \(C(\overline{\mathbb{R}})\) refer to the algebra of all complex-valued functions \(f\) which are continuous on \(\mathbb{R}\) and possess finite limits \(f(\pm \infty)\) at \(\pm \infty\).

From now on we will always assume that the matrix function \(B_{g, \beta}\) satisfies condition (14). Thus, \(T_{g, \beta}\) is a bounded block Mellin operator on \(L^p_2(\mathbb{R}^+, \alpha)\) whenever \(1 < p < \infty\) and \(0 < \alpha + 1/p < 1\). If, in addition, \(\det B_{g, \beta}(t) \neq 0\) for all \(t \in \overline{\mathbb{R}}\), then the operator \(T_{g, \beta}\) is invertible, and its inverse is the Mellin operator \(M(B_{g, \beta}^{-1})\). Note that the invertibility of the operator \(T_{g, \beta}\) is not involved if one only considers the approximate foveated image \(\psi_{\eta}^m\). However, the condition of invertibility cannot be avoided if one wants to restore the approximate uniform pre-image \(Q_mP_n\phi\).

Let \(\chi : \mathbb{R} \to \{0, 1\}\) be the characteristic function of the interval \([0, 1]\), and let \(I\) denote one of the sets \(\mathbb{R}\) or \(\mathbb{R}^+\). For each positive integer \(n\), we consider the functions
\[
\varphi_{nj}(t) := \chi(nt - j), \quad j \in \mathbb{Z}.
\]

Note that \(\varphi_{nj}\) is the characteristic function of the interval \([j/n, (j + 1)/n]\). By \(S_n(\mathbb{R})\) we denote the smallest closed subspace of \(L^p(\mathbb{R}, \alpha)\) which contains all functions \(\varphi_{nj}\) with \(j \in \mathbb{Z}\), and we write \(S_n(\mathbb{R}^+)\) for the smallest closed subspace of \(L^p(\mathbb{R}^+, \alpha)\) which contains all functions \(\varphi_{nj}\) with \(j \geq 0\). Further,
we introduce the Galerkin projections $P_n^I : L^p(I, \alpha) \to S_n(I)$ by

$$P_n^I f := n \sum_{k \in \mathbb{Z} \cap I} \langle f, \varphi_{nk} \rangle \varphi_{nk}. \quad (9)$$

To simplify notations, we abbreviate $P_n^R$ to $P_n$ and $P_n^R+$ to $P_n^+$. What we are interested in are approximations $\psi_n^m$ of the foveated image of the signal $\varphi \in L^p(\mathbb{R}, \alpha)$ of the form

$$\psi_n^m := P_n Q_m T_{g, \beta} Q_m P_n \varphi. \quad (15)$$

As already mentioned, these are the approximate foveated images considered in [5] (where they were based on a projection onto a space of wavelets in place of the spline projection $P_n$).

Assume that $\psi_n^m$ is known. Are there any conditions which allow us to restore the approximate uniform image $Q_m P_n \varphi$ (provided the parameter $\beta$ and the smoothing function $g$ are known)? To put these questions into an appropriate context, we have to recall some notions from numerical analysis. Let $X$ be a Banach space, and let $(L_n)_{n \in \mathbb{N}}$ be a sequence of projections on $X$ which converges strongly to the identity operator. As usual, strong convergence of a sequence $(A_n)_{n \in \mathbb{N}}$ to an operator $A \in \mathcal{B}(X)$ means that $\lim_{n \to \infty} A_n x = Ax$ for every $x \in X$. Consider the operator equation

$$Ax = y, \quad x, y \in X, \quad A \in \mathcal{B}(X) \quad (16)$$

and the sequence of its approximations

$$A_n L_n x_n = L_n y, \quad x_n \in \text{im } L_n, \quad A_n \in \mathcal{B}(\text{im } L_n). \quad (17)$$

Regarding the approximation operators $A_n$, one usually assumes that the equations (17) are consistent with equation (16) in the sense that the sequence $(A_n L_n)$ converges strongly to the operator $A$.

**Definition 1** The approximation method (17) is stable if there is a number $n_0$ such that the operators $A_n : \text{im } L_n \to \text{im } L_n$ are invertible for all $n \geq n_0$ and if

$$C := \sup_{n \geq n_0} \|A_n^{-1}\| < \infty. \quad (18)$$

The notion of stability plays a fundamental role in the analysis of approximation methods. For example, let the operator $A$ be invertible and the approxi-
mation method (17) be stable. If \(x^*\) and \(x_n^*\) are the solutions of the equations (16) and (17), respectively, then the error estimate

\[
\|x^* - x_n^*\| \leq C\|Ax^* - A_nL_nx^*\| + \|x^* - L_nx^*\|
\]  

holds.

4 Stability of the sequence \((P_nQ_nT_{g,\beta}Q_nP_n)\)

The object of this section is the stability of the sequence \((P_nQ_nT_{g,\beta}Q_nP_n)\). We start with examining the stability of a related approximation method, namely

\[
P_nT_{g,\beta}P_n\varphi_n = P_n\psi, \quad n \in \mathbb{N}, \quad \varphi_n \in \text{im} P_n
\]  

where \(P_n\) are the above defined spline projections and \(\psi\) is the foveated image of the initial signal \(\varphi\). Here we assume that the user knows \(P_n\psi\) and wants to restore \(\varphi_n\).

**Step 1: Stability of the Sequence** \((P_nT_{g,\beta}P_n)\)

**Proposition 2** Let \(B_{g,\beta} \in C^{2\times2}(\mathbb{R})\), and let the entries of \(B_{g,\beta}\) have finite total variation on \(\mathbb{R}\). Then the approximation method (20) is stable if and only if the operator \(P_1T_{g,\beta}P_1 : S_1(\mathbb{R}) \to S_1(\mathbb{R})\) is invertible.

**Proof.** Let \(l^p(\mathbb{Z}, \alpha)\) refer to the set of all sequences \((\xi_j)_{j \in \mathbb{Z}}\) of complex numbers such that

\[
\|(\xi_j)\|^p := \sum_{j \in \mathbb{Z}} |\xi_j|^p(1 + |j|)^{\alpha p} < \infty.
\]

For every natural number \(n\), we consider the operators

\[
E_n : l^p(\mathbb{Z}, \alpha) \to S_n(\mathbb{R}), \quad (\xi_j)_{j \in \mathbb{Z}} \to \sum_{j \in \mathbb{Z}} \xi_j\varphi_{nj}.
\]

It is well-known and easy to see (c.p. [1]) that these operators possess continuous inverses \(E_{-n} := E_n^{-1} : S_n(\mathbb{R}) \to l^p(\mathbb{Z}, \alpha)\) and that there is a constant \(C\) such that

\[
\|E_n\| \leq C n^{-(1/p+\alpha)} \quad \text{and} \quad \|E_{-n}\| \leq C n^{1/p+\alpha}.
\]
Hence, the operators

\[ P_n T_{g, \beta} P_n : \quad S_n(\mathbb{R}) \to S_n(\mathbb{R}) \]

are invertible if and only if the corresponding operators

\[ E_{-n} P_n T_{g, \beta} P_n E_n : \quad l^p(\mathbb{Z}, \alpha) \to l^p(\mathbb{Z}, \alpha) \]

are so. Consider the matrix representation \( A_n = (A_{jk}^{(n)})_{j, k \in \mathbb{Z}} \) of the operator \( E_{-n} P_n T_{g, \beta} P_n E_n \) with respect to the standard basis of \( l^p(\mathbb{Z}, \alpha) \). Straightforward calculations yield

\[
A_{jk}^{(n)} = \frac{n}{\beta} \int_{\mathbb{R}} \chi(nx - j) \left( \frac{1}{|x|} g \left( \frac{1}{\beta} \left( \frac{t-x}{|x|} \right) \right) \right) \chi(nt - k) \, dt \, dx
\]

\[ = \frac{1}{\beta} \int_{j}^{j+1+k+1} \int_{k}^{1} \frac{1}{|x|} g \left( \frac{1}{\beta} \left( \frac{t-x}{|x|} \right) \right) \, dt \, dx. \tag{22} \]

One observes that the entries of the matrices \( A_n \) are independent of \( n \). Therefore, the approximation method (20) is stable if and only if the operator \( E_{-1} P_1 T_{g, \beta} P_1 E_1 \) is continuously invertible. Taking into account the estimates (21), we get the claim. \( \square \)

**Step 2: Fredholm Properties of the Operator** \( E_{-1} P_1 T_{g, \beta} P_1 E_1 \)

Now we are going to consider the operator \( E_{-1} P_1 T_{g, \beta} P_1 E_1 \) in more detail. As was already mentioned, \( \eta T_{g, \beta} \eta^{-1} = \mathcal{M}(B_{g, \beta}) = (T_{rl})_{r, l=1}^2 \) where every \( T_{rl} \) is a Mellin convolution operator on \( L^p(\mathbb{R}^+, \alpha) \) with symbol from \( C(\mathbb{R}) \). It is easy to check that, for every \( n \),

\[ \eta P_n \eta^{-1} = \text{diag}(P_n^+, P_n^+) \]

Hence, \( E_{-1} P_1 T_{g, \beta} P_1 E_1 \) can be identified with an operator \( D_{g, \beta} = (D_{rl})_{r, l=1}^2 \) where the operators \( D_{rl} \) act on \( l^p_2(\mathbb{N}, \alpha) \).

The further study of these operators is based on the fact that they belong to an algebra generated by Toeplitz operators. Let us recall briefly what a Toeplitz operator is. For a more detailed information concerning such operators the reader should consult [3,4]. Write \( T \) for the complex unit circle. For \( a \in L^\infty(\mathbb{T}) \),
let \( a_k \) denote the \( k \)th Fourier coefficient of \( a \),

\[
a_k := \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta})e^{-ik\theta} \, d\theta, \quad k \in \mathbb{Z}.
\]

If the function \( a \) is piecewise continuous on \( T \) (which means that \( a \) has finite one-sided limits at each point of \( T \)) and has a finite total variation, then the operator which acts on the finitely supported sequences in \( l^p(\mathbb{N}, \alpha) \) via

\[
(x_n)_{n \in \mathbb{N}} \mapsto (y_n)_{n \in \mathbb{N}} \quad \text{with} \quad y_n := \sum_{k \in \mathbb{N}} a_{n-k}x_k
\]

extends by continuity to a bounded linear operator \( T(a) \) acting on all of \( l^p(\mathbb{N}, \alpha) \). Thus, the matrix representation of \( T(a) \) with respect to the standard basis of \( l^p(\mathbb{N}, \alpha) \) is given by

\[
T(a) = \begin{pmatrix}
a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\
a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\
a_2 & a_1 & a_0 & a_{-1} & \cdots \\
a_3 & a_2 & a_1 & a_0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

We let \( T^p(\alpha) \) stand for the smallest closed subalgebra of \( \mathfrak{B}(l^p(\mathbb{N}, \alpha)) \) which contains all Toeplitz operators \( T(a) \) with a generating function \( a \) having finite total variation on \( T \), being continuous on \( T \setminus \{1\} \), and possessing finite one-sided limits at \( 1 \in T \). Thus, the precise formulation of the above vague statement on \( D_{g,\beta} \) is that this operator belongs to \( T^p(\alpha)^{2 \times 2} \). A proof of this fact is in [8], Sections 2.2.3 and 2.4.3.

If the symbol \( B_{g,\beta} \) of the operator \( \psi_{\eta}(T_{g,\beta}) \) does not satisfy the conditions of Proposition 1 then it cannot be guaranteed that the operator \( D_{g,\beta} \) belongs to the Toeplitz algebra \( T^p(\alpha)^{2 \times 2} \). In this case it is not clear how to describe the algebra generated by \( D_{g,\beta} \), and all considerations will become much more involved.

It remains a serious problem to decide whether an operator in \( T^p(\alpha) \) is invertible. But there is a very comfortable criterion for the Fredholmness of operators in \( T^p(\alpha) \) which we will recall next. To each Toeplitz operator \( A = T(a) \) as above (i.e., \( a \) has finite total variation, is continuous on \( T \setminus \{1\} \), and possesses finite one-sided limits \( a(1 \pm 0) \) taken with respect to the counter-clockwise orientation of \( T \)) we associate the function \( A^{\sharp} : T \times \mathbb{R} \to \mathbb{C} \) which maps \((t, z)\) into \( a(t)\) if \( t \neq 1 \) and into

\[
\frac{a(1 + 0) + a(1 - 0)}{2} - \frac{a(1 + 0) - a(1 - 0)}{2} \coth \pi(z + i(1/p + \alpha))
\]

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if $t = 1$. Thus, one makes the range of $a$ to a closed curve in $\mathbb{C}$ by joining the points $a(1 \pm 0)$ by a certain circular arc depending on the parameter $1/p + \alpha$.

If now $A \in T^p(\alpha)$ is a finite sum of products of Toeplitz operators $A_{ij}$, then we define

$$A^\sharp = (\sum \prod A_{ij})^\sharp := \sum \prod A_{ij}^\sharp.$$ 

The mapping $A \mapsto A^\sharp$ is correctly defined, and it extends by continuity onto all of $T^p(\alpha)$. The function $A^\sharp$ is also called the symbol of the operator $A$. The relevance of the symbol $A^\sharp$ for the purpose of Fredholmness is as follows: The operator $A \in T^p(\alpha)$ is Fredholm if and only if the point 0 does not belong to the range of $A^\sharp$. Moreover, if one provides the curve $A^\sharp(T \times \mathbb{R})$ with the orientation inherited by the counter-clockwise orientation of $T$, then the Fredholm index of $A \in T^p(\alpha)$ is equal to the negative winding number of the curve $A^\sharp(T \times \mathbb{R})$ with respect to the origin [3, Theorem 6.38].

In order to apply these results to the discretized operator $D_{g, \beta}$ we still have to recall another property of the algebra $T^p(\alpha)$, namely it is commutative modulo compact operators. From this commutativity we conclude that the operator $D_{g, \beta} = (D_{rl}) \in T^p(\alpha)^{2\times2}$ is Fredholm if and only if the operator

$$\det(D_{rl}) := D_{11}D_{22} - D_{12}D_{21} \in T^p(\alpha)$$

is Fredholm, and that their indices coincide. Thus, $D_{g, \beta}$ is a Fredholm operator if and only if 0 does not lie on the curve $(D_{11}^\sharp D_{22}^\sharp - D_{12}^\sharp D_{21}^\sharp)(T \times \mathbb{R})$, and in this case the index of $D_{g, \beta}$ is minus the winding number of that curve.

It remains to compute the symbols of the operators $D_{rl} = E_1 P_1 M(b_{rl}) P_1 E_{-1}$ where $b_{rl}$ is the $rl$th component of the function $B_{g, \beta}$. Write

$$b_{rl}(z) = \mu_{rl} + \nu_{rl} \coth \pi(z + i(1/p + \alpha)) + n_{rl}(z)$$

where $\mu_{rl}, \nu_{rl} \in \mathbb{C}$ are chosen such that $n_{rl}(\pm \infty) = 0$. (Recall that the limits of the coth-function at infinity are $\pm 1$.) Then the symbol of $D_{rl}$ is equal to

$$(t, z) \mapsto \begin{cases} 
\mu_{rl} + \nu_{rl} \sigma(t) & \text{if } t \neq 1 \\
\mu_{rl} - \nu_{rl} \coth \pi(z + i(1/p + \alpha)) + n_{rl}(z) & \text{if } t = 1 
\end{cases} \quad (23)$$

where

$$\sigma(e^{2\pi i y}) := -\frac{\sin^2 \frac{\pi y}{\pi^2}}{\sum_{m \in \mathbb{Z}} \text{sgn}(y + m)(y + m)^2} \quad (24)$$

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for \( y \in (0, 1) \). A detailed computation of these functions can be found in [8], Sections 2.2.3, 2.4 and 2.5.2. Geometrically, this condition is quite simple again since the range of the restriction of this mapping onto \((T \setminus \{1\}) \times \mathbb{R}\) is just the interval \((\mu_{rl} + \nu_{rl}, \mu_{rl} - \nu_{rl})\).

**Step 3: Stability of the sequence** \((P_n Q_m T_{g, \beta} Q_m P_n)\)

We will now apply a similar approach to examine the stability of the approximation method

\[ P_n Q_m T_{g, \beta} Q_m P_n \varphi_n^m = P_n Q_m \psi, \quad m, n \in \mathbb{N}, \quad \varphi_n^m \in \text{im} P_n \quad (25) \]

where again \(P_n\) and \(Q_m\) are the above defined projections and where \(\psi := T_{g, \beta} \varphi\) is the foveated image of the initial signal. Thus, we assume \(P_n Q_m \psi\) to be known to the user who wants to restore \(\varphi_n^m\). Observe that the right hand sides of the equations (25) can be replaced by the approximate foveated images \(\psi_n^m\) defined by (4) without changing the asymptotic solvability properties of these equations. This follows simply from the fact that the sequences \((P_n)\) and \((Q_m)\) of projections converge strongly to the identity operator. Thus, the norms \(\|P_n Q_m \psi - \psi_n^m\|\) become as small as desired if \(m\) and \(n\) are chosen large enough.

It turns out that the problem of stability of the approximation method (25) can be reduced to the stability of a finite section method for the operator \(D_{g, \beta}\) which belongs to the Toeplitz algebra \(T^p(\alpha)^{2 \times 2}\). Towards this end we provide the space \(l^p_2(\mathbb{N}, \alpha)\) with the norm \(\| \langle f, g \rangle \|^p := \|f\|^p + \|g\|^p\). Then the mapping \((f, g) \mapsto h\) with

\[ h(n) := \begin{cases} f(n), & n \geq 0 \\ g(-1 - n), & n < 0 \end{cases} \]

is an isometry from \(l^p_2(\mathbb{N}, \alpha)\) onto \(l^p(\mathbb{Z}, \alpha)\). Analogously, the space \(L^p_2([0, 1], \alpha)\) is identified with \(L^p([-1, 1], \alpha)\).

For \(l \in \mathbb{N}\), define projections \(R_l\) on \(l^p(\mathbb{N}, \alpha)\) by

\[ R_l : (x_k)_{k \in \mathbb{N}} \mapsto (y_k)_{k \in \mathbb{N}} \quad \text{with} \quad y_k := \begin{cases} x_k & \text{if } k < l \\ 0 & \text{if } k \geq l. \end{cases} \]

Since \(P_n Q_m = Q_m P_n\), the calculations from the previous section immediately yield that the operator \(P_n Q_m T_{g, \beta} Q_m P_n\) is invertible if and only if the operator
and, thus, the operator

$$
\begin{pmatrix}
R_{m \cdot n} & 0 \\
0 & R_{m \cdot n}
\end{pmatrix}
\begin{pmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{pmatrix}
\begin{pmatrix}
R_{m \cdot n} & 0 \\
0 & R_{m \cdot n}
\end{pmatrix}
= \begin{pmatrix}
R_{m \cdot n}D_{11}R_{m \cdot n} & R_{m \cdot n}D_{12}R_{m \cdot n} \\
R_{m \cdot n}D_{21}R_{m \cdot n} & R_{m \cdot n}D_{22}R_{m \cdot n}
\end{pmatrix}
$$

(26)

is invertible. Consequently, the stability of the approximation method (25) is equivalent to the stability of the finite section method for the operator $D_{g, \beta} \in T^p(\alpha)^{2 \times 2}$, which has

$$
\begin{pmatrix}
R_{l} & 0 \\
0 & R_{l}
\end{pmatrix}
\begin{pmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{pmatrix}
\begin{pmatrix}
R_{l} & 0 \\
0 & R_{l}
\end{pmatrix}
$$

(27)

as its system matrix. The finite section method for a large class of operators including the operators $(D_{rl})$ from (27) has been studied in [8], Sections 4.1.1 – 4.1.3. A characterization of the stability of the approximation method (27) can be deduced from these general results. Note that, formally, we only get a subsequence of the sequence of the finite section method. But, as has been shown in [9], the sequence formed by the matrices in (27) and its subsequence formed by the matrices in (26) are simultaneously stable or not. Moreover, the finite section method for operators in the Toeplitz algebra is an example of a fractal approximation method; roughly speaking fractality means that every infinite subsequence of the sequence of the approximation matrices allows one to restore the complete sequence up to a sequence which tends to zero in the norm. Thus, every subsequence contains the same “asymptotic information” as the whole sequence. For more facets of this fascinating topic see [9].

Summarizing these results we arrive at the following theorem.

**Theorem 1** Let $B_{g, \beta} \in C^{2 \times 2}(\mathbb{R})$, and let the entries of $B_{g, \beta}$ have finite total variation on $\mathbb{R}$. The approximation method (25) is stable if and only if

(a) the operator $D_{g, \beta}$ is invertible on $l_2^p(\mathbb{N}, \alpha)$,

(b) the operator $(T(\mu_{r_{l}} - \nu_{r_{l}}\sigma))_{r_{l}=1}^{2}$ is invertible on $l_2^p(\mathbb{N}, 0)$, and

(c) the operator $(\chi[0, 1]T_{r_{l}}\chi[0, 1])_{r_{l}=1}^{2}$ is invertible on $L_2^p([0, 1], \alpha)$. 


5 An example

In this section we will illustrate the obtained results by a concrete example. The point is that for the integral operator considered below all conditions concerning the stability of relevant operator sequences and the invertibility of associated operators (as quoted in the previous theorem) can be effectively verified. Thus, this operator seems to be a good candidate to illustrate our approach to foveation operators.

Let the function $g$ be given by

$$g(t) = \frac{1}{\pi it}, \quad t \in \mathbb{R}. \quad (28)$$

In contrast to [5], this function does not belong to the space $L^1(\mathbb{R})$, but it nevertheless satisfies the conditions of Proposition 1. We are also aware of the fact that this is not a smoothing function as considered in [5]. But it is a function, for which the stability conditions mentioned previously take a simple and effective form.

If $g$ is specified as above, then the operator $T_{g, \beta}$ does no longer depend on $\beta$, and we denote it by $T_g$. It turns out that $T_g : L^p(\mathbb{R}, \alpha) \to L^p(\mathbb{R}, \alpha)$ is just the singular integral operator $S_\mathbb{R}$ acting by

$$(S_\mathbb{R} f)(x) = \frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{f(t)}{t - x} dt.$$  

For good (say, Hölder continuous) functions, this integral exists in the Cauchy principal value sense, and it can be extended by continuity to all of $L^p(\mathbb{R}, \alpha)$. The corresponding matrix function $G_g$ has the form

$$G_g(t) = \frac{1}{\pi i} \begin{pmatrix} 1 & -1 \\ 1 - t & 1 + t \\ 1 & 1 \\ 1 + t & t - 1 \end{pmatrix}, \quad t \in \mathbb{R}^+.$$  

We determine the entries of the matrix function $B_g(z) := MG_g(z)$. Using formulae 3.238.1 and 3.238.2 of [7], one obtains

$$b_{11}(z) = \frac{1}{\pi i} \int_0^{+\infty} \frac{x^{1/p + \alpha - 1 - iz}}{1 - x} dx = \coth \pi(z + i(1/p + \alpha)).$$
for \( z \in \mathbb{R} \). Analogously, by 3.194.4 of [7],

\[
b_{21}(z) = \frac{1}{\pi i} \int_0^{+\infty} \frac{x^{1/p + \alpha - 1 - iz}}{1 + x} \, dx = \frac{1}{\sinh \pi(z + i(1/p + \alpha))}.
\]

Hence,

\[
B_g(z) = \begin{pmatrix}
\coth \pi(z + i(1/p + \alpha)) & -1/\sinh \pi(z + i(1/p + \alpha)) \\
1/\sinh \pi(z + i(1/p + \alpha)) & -\coth \pi(z + i(1/p + \alpha))
\end{pmatrix}.
\] (29)

From this representation, some basic properties of the singular integral operator follow almost at once. For example, the entries of the matrix \( B_g \) are continuous and have finite total variation on \( \mathbb{R} \), and their limits at \( \pm \infty \) are

\[
B_g(+\infty) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_g(-\infty) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Therefore, \( S_\mathbb{R} \) is a bounded operator on \( L^p(\mathbb{R}, \alpha) \). Notice in this connection that the continuity conditions for the operator \( T_g \) given in [5], Theorem 1, are too restrictive. The function \( g \) used above is neither bounded nor belongs to \( L^1(\mathbb{R}) \) as required in [5], but nevertheless the operator \( S_\mathbb{R} = T_g \) is bounded on \( L^p(\mathbb{R}, \alpha) \).

As another application of (29), we examine the invertibility of \( S_\mathbb{R} \). As already mentioned, this operator is invertible if and only if the determinant of \( B_g \) does not vanish on \( \mathbb{R} \). Since

\[
\det B_g(z) = \frac{1 - \cosh^2(\pi(z + i(1/p + \alpha)))}{\sinh^2(\pi(z + i(1/p + \alpha)))} = -1,
\]

the operator \( S_\mathbb{R} \) is invertible on each of the spaces \( L^p(\mathbb{R}, \alpha) \) for \( 1 < p < \infty \) and \( 0 < 1/p + \alpha < 1 \). The inverse operator for the operator \( S_\mathbb{R} \) can be written as a block Mellin operator, the Mellin symbol of which is the matrix

\[
\frac{1}{\det B_g(z)} \begin{pmatrix}
-\coth \pi(z + i(1/p + \alpha)) & 1/\sinh \pi(z + i(1/p + \alpha)) \\
-1/\sinh \pi(z + i(1/p + \alpha)) & \coth \pi(z + i(1/p + \alpha))
\end{pmatrix}
\]
Comparing (29) and (30), we obtain

$$S^{-1}_R = S_R.$$  

Of course, all these results are well known and can be proved without having recourse to Mellin techniques. But they illustrate these techniques quite well.

Due to the simple structure of the operator $T_g = S_R$, it is more convenient to study the invertibility of the Galerkin approximations $P_nS_RP_n$ and $P_nQ_mS_RQ_mP_n$ directly and without doubling the dimension. The point is that the operator $E_{-n}P_nS_RP_nE_n$ is again independent of $n$ and that this operator coincides with the Laurent operator $L(\sigma)$ on $l^p(\mathbb{Z}, \alpha)$. By definition, the Laurent operator $L(a)$ with generating function $a \in L^\infty(\mathbb{T})$ is given via its matrix representation

$$L(a) = \begin{pmatrix} \ldots & \ldots & \ldots & \ldots & a_0 & a_{-1} & a_{-2} & a_{-3} & \ldots \\ \ldots & \ldots & a_0 & a_{-1} & a_{-2} & \ldots \\ \ldots & a_1 & a_0 & a_{-1} & a_{-2} & \ldots \\ \ldots & a_2 & a_1 & a_0 & a_{-1} & \ldots \\ \ldots & a_3 & a_2 & a_1 & a_0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

with respect to the standard basis of $l^p(\mathbb{Z}, \alpha)$. As in the Toeplitz operator case, a sufficient condition for the boundedness of this operator is that the function $a$ is piecewise continuous and has a finite total variation. A basic difference between Toeplitz and Laurent operators with piecewise continuous generating functions is that $L(a)L(b) = L(ab)$ (whereas the corresponding result for Toeplitz operators is definitely wrong in general). Consequently, the Laurent operators generate a commutative algebra, whereas the Toeplitz operator algebra $T^p(a)$ is merely commutative modulo compact operators. This implies that the Laurent operator $L(a)$ is invertible if and only if the function $a$ is invertible in $L^\infty(\mathbb{T})$.

Since the essential range of $\sigma$ is the interval $[-1, 1]$, the operators $P_nS_RP_n = E_{-n}L(\sigma)E_n$ cannot be invertible. For that reason, we replace the kernel function (28) by the more general function

$$g(t) = a + b \frac{1}{\pi it}, \quad t \in \mathbb{R}$$  

(32)
with complex constants $a$ and $b$. Then $T_g = aI + bS_R$ and $E_n P_n S_R P_n E_n = L(a + b \sigma)$, and the latter operator becomes invertible if $a$ and $b$ are suitably chosen. At this point, it is sufficient to require that $0$ does not lie on the segment joining $a - b$ to $a + b$.

**Proposition 3** Let the kernel function $g$ be given by (32). Then the sequence $(P_n T_g P_n)$ is stable if and only if $0 \not\in [a - b, a + b]$.

What happens with the operators $P_n Q_m T_g Q_m P_n$? Their invertibility corresponds to the stability of the finite sections sequence $R_m' L(a) R_m'$ where the projections $R_m'$ on $l^p(\mathbb{Z}, \alpha)$ are defined by

$$R_m' : (x_n)_{n \in \mathbb{Z}} \mapsto (y_n)_{n \in \mathbb{Z}} \quad \text{with} \quad y_n := \begin{cases} x_n & \text{if } -m \leq n \leq m - 1 \\ 0 & \text{else.} \end{cases}$$

Evidently, the matrix $R_m' L(a) R_m'$ can be identified with $R_{2n} T(a) R_{2n}$. Thus, the finite section method for the Laurent operator $L(a)$ corresponds to a subsequence of the finite section method for the Toeplitz operator $T(a)$. The stability of the finite section method for that Toeplitz operator is well understood. The following result is a corollary of a more general theorem of [8]. A direct proof which works in case the generating function of the Toeplitz operator has exactly one point of discontinuity can be found in [12].

**Proposition 4** Let $c$ be a piecewise continuous function with finite total variation. Then the finite section method applies to the Toeplitz operator $T(c)$ on $l^p(\mathbb{N}, \alpha)$ if and only if the operator $T(c)$ is invertible on $l^p(\mathbb{N}, \alpha)$ and if the operator $T(\tilde{c})$ with $\tilde{c}(y) := c(1/y)$ is invertible on $l^p(\mathbb{N}, 0)$.

For the kernel function $g$ as in (32), the applicability of the finite section method to $T(a + b \sigma)$ is equivalent to the invertibility of $T(a + b \sigma)$ on $l^p(\mathbb{N}, \alpha)$ and of $T(a - b \sigma)$ on $l^p(\mathbb{N}, 0)$. The invertibility of the first mentioned operator is equivalent to the fact that the point $0$ does not lie in the region which is bounded by

$$[a + b, a - b] \cup \{ a + b \coth \pi(z + i(1/p + \alpha)) : z \in \mathbb{R} \},$$

whereas the invertibility of the second operator is equivalent to the fact that $0$ is not contained in the region bounded by

$$[a - b, a + b] \cup \{ a - b \coth \pi(z + i/p) : z \in \mathbb{R} \}.$$ 

Both regions are bounded by a union of a straight line with a circular arc.
Thus for the function $g$ defined by (32), the conditions for the invertibility of the corresponding operators have a very simple geometrical nature and can be effectively verified. Let us mention once more that the invertibility of these operators depend on the parameters $p$ and $\alpha$ determined by the space under consideration.

6 Splines of Higher Order

The Galerkin approximations considered so far have been based on first order splines, i.e., on piecewise constant splines, generated by the characteristic function $\chi_{[0,1)}$. One expects that the use of higher order splines might give a better approximation for both uniform and foveated signals. Of course, the replacement of the basis functions $\chi_{[0,1)}$ by piecewise polynomials leads to different approximation operators. Thus, the stability problem has to be studied once again. Thereby one observes that the involved operators, used to approximate the foveated images, again belong to the Toeplitz algebra $T^p(\alpha)^{2 \times 2}$ we have already met. Therefore, the study of the stability problem for such operator sequences can again be based upon the approach developed in Section 4. Let us briefly comment on the amendments which have to be made in this situation.

For a given $d \geq 2$, the $d$-th order cardinal $B$-spline $N^{(d)}$ is defined recursively by

$$N^{(d)}(t) := \int_0^1 N^{(d-1)}(t - s) \, ds,$$

where $N^{(1)} = \chi_{[0,1)}$ is the characteristic function of the interval $[0,1)$. The reader can consult [1, Chapter 1] for definitions and basic properties of $B$-splines. Here we only mention some facts we need.

1. The support of $N^{(d)}$ is the interval $[0, d]$.
2. $N^{(d)}(t) > 0$ for all $t \in (0, d)$.
3. $N^{(d)}(-t + d) = N^{(d)}(t)$ for every $t \in \mathbb{R}$.

Fix a positive integer $n$. For $j \in \mathbb{Z}$, introduce functions $\varphi_{nj}$ by

$$\varphi_{nj}(t) := \begin{cases} 
N^{(d)}(nt - j) & \text{if } j \geq 0 \\
N^{(d)}(nt - j - d + 1) & \text{if } j < 0.
\end{cases}$$

20
Let $S^d_n(\mathbb{R}^+)$ be the smallest closed subspace of $L^p(\mathbb{R}^+, \alpha)$ which contains all functions $\varphi_{nj}$ with $j \in \mathbb{Z}$ non-negative, and let $S^d_n(\mathbb{R})$ be the smallest closed subspace of $L^p(\mathbb{R}, \alpha)$ which contains all functions $\varphi_{nj}$ with $j \in \mathbb{Z}$. By analogy with the case $d = 1$, introduce the Galerkin projections $\tilde{P}_n^I : L^p(I, \alpha) \to S^d_n(I)$ by

$$\tilde{P}_n^I f := n \sum_{k \in \mathbb{Z}} \langle f, \varphi_{nk} \rangle \varphi_{nk},$$

and rewrite equation (15) with the projections onto the spaces $S_n(I) = S^{(1)}_n(I)$ replaced by projections $\tilde{P}_n^I$. Again, the operators $\tilde{E}_n : l^p(\mathbb{Z}, \alpha) \to S^d_n(\mathbb{R})$, $(\xi_j)_{j \in \mathbb{Z}} \mapsto \sum_{j \in \mathbb{Z}} \xi_j \varphi_{nj}$

are continuously invertible with inverses $\tilde{E}_{-n}$, and

$$\|\tilde{E}_n\| \leq C n^{-(1/p+\alpha)}, \quad \|\tilde{E}_{-n}\| \leq C n^{1/p+\alpha}$$

with a certain constant $C$.

Repeating and modifying the arguments of Section 4 one obtains that the analog of the approximation method (15), which is now based on the spline $N^{(d)}$ with $d \geq 2$, is stable if and only if the operator

$$\tilde{E}_{-1} \tilde{P}_1 T_{g, \beta} \tilde{P}_1 \tilde{E}_1 : l^p(\mathbb{Z}, \alpha) \to l^p(\mathbb{Z}, \alpha)$$

is invertible. The entries $\tilde{A}_{jk}^{(n)}$ of the matrix representation of that operator with respect to the standard basis of $l^p(\mathbb{Z}, \alpha)$ can be calculated by formulas similar to (22). For example, if $j \geq 0$ and $k \geq 0$, then

$$\tilde{A}_{jk}^{(n)} = \frac{1}{\beta} \int_{j}^{j+d} N^{(d)}(x - j) \int_{k}^{k+d} \frac{1}{|x|} \left( \frac{1}{\beta} \left( \frac{t - x}{|x|} \right) \right) N^{(d)}(t - k) dt dx.$$

A further analysis shows that the operator $\tilde{E}_{-1} \tilde{P}_1 T_{g, \beta} \tilde{P}_1 \tilde{E}_1$ can be identified with an operator $\tilde{D}_{g, \beta} := (\tilde{D}_{rt})_{r, t=1}^{2}$ in the matrix Toeplitz algebra $T^p(\alpha)^{2 \times 2}$ and that the symbol of the operator $\tilde{D}_{rt}$ is given by

$$(t, z) \mapsto \begin{cases} 
\mu_{rt} + \nu_{rt} \bar{\sigma}(t) & \text{if } t \neq 1 \\
\mu_{rt} - \nu_{rt} \coth \pi(z + i(1/p + \alpha)) + c(d)n_{rt}(z) & \text{if } t = 1
\end{cases}$$
with $\mu_{rl}$ and $\nu_{rl}$ as before,

$$\tilde{\sigma}(e^{2\pi i y}) := -\frac{\sin^2 \pi y}{\pi^2 d} \sum_{m \in \mathbb{Z}} \frac{\text{sgn}(y + m)}{(y + m)^2}$$

for $y \in (0, 1)$,

and

$$c(d) = \left( \int_0^d N^{(d)}(t) \, dt \right)^2.$$

Finally, this leads to results similar to Section 4. Thus, Theorem 1 is also valid for $d \geq 2$. The operator $D_{g, \beta}$, however, must be replaced by the corresponding operator $\tilde{D}_{g, \beta}$.

**Concluding Remarks**

Our analysis shows that the use of the operator (2) allows to create and transmit foveated images. It also offers a theoretical possibility to restore their approximate uniform pre-images. Moreover, if the kernel $k$ of the integral operator (2) gives rise to a stable approximation method, then the quality of such restored uniform pre-images can be quite satisfactory as estimate (19) shows.

As was also pointed out, one might use other numerical procedures to approximate foveated images. For example, one can apply procedures based on meshes having higher density around the points of interest. Another possibility is to use wavelets instead of splines. The Galerkin procedures considered in the present paper can also be replaced by other numerical methods, for example by quadrature or by collocation methods. Of course, each new approach will require an additional analysis of the stability for the employed numerical procedures. However, in many cases, these approximation procedures for the operator (2) can be associated with well understood operator algebras, which allows one to find criteria of their stability.

From the practical point of view, a restoration of two-dimensional signals from their foveated images can be more attractive. It is worth noting that the analysis of the stability for the corresponding approximation methods can be done analogously to our previous considerations. However, the operators arising in connection with those methods might have more complicated structure.
References


