

Preferred Quantization Rules: Born–Jordan vs. Weyl: the Pseudo-Differential Point of View

Maurice de Gosson*
Universität Wien, NuHAG
Fakultät für Mathematik
A-1090 Wien

Franz Luef†
University of California
Department of Mathematics
Berkeley CA 94720-3840

November 8, 2010

Abstract

There has recently been evidence for replacing the usual Weyl quantization procedure by the older and much less-known Born–Jordan rule. In this paper we discuss this quantization procedure in detail and relate it to recent results of Boggiato, De Donno, and Oliaro on the Cohen class. We begin with a discussion of some properties of Shubin’s τ -pseudodifferential calculus, which allows us to show that the Born–Jordan quantization of a symbol a is the average for $\tau \in [0, 1]$ of the τ -operators with symbol a .

Introduction

Physical background

Already in the early days of quantum mechanics physicists were confronted with the ordering problem for products of observables (i.e. of symbols, in mathematical language). While it was agreed that the correspondence rule $x_j \longrightarrow x_j$, $p_j \longrightarrow -i\hbar\partial/\partial x_j$ could be successfully be applied to the position and momentum variables, thus turning the Hamiltonian function

$$H = \sum_{j=1}^N \frac{1}{2m_j} p_j^2 + V(x_1, \dots, x_N) \quad (1)$$

*Financed by the Austrian Research Agency FWF (Projektnummer P20442-N13).

†Financed by the Marie Curie Outgoing Fellowship PIOF 220464.

into the partial differential operator

$$\hat{H} = \sum_{j=1}^N \frac{-\hbar^2}{2m_j} \frac{\partial^2}{\partial x_j^2} + V(x_1, \dots, x_N) \quad (2)$$

it quickly became apparent that these rules lead to fundamental ambiguities when applied to more general observables involving products of function of x_j and p_j . For instance, what should the operator corresponding to the magnetic Hamiltonian

$$H = \sum_{j=1}^N \frac{1}{2m_j} (p_j - \mathcal{A}_j(x_1, \dots, x_N))^2 + V(x_1, \dots, x_N) \quad (3)$$

be? Even if the simple case of the product $x_j p_j = p_j x_j$ the correspondence rule led to the a priori equally good answers $-i\hbar x_j \partial / \partial x_j$ and $-i\hbar (\partial / \partial x_j) x_j$ which differ by the quantity $i\hbar$; things became even more complicated when one came (empirically) to the conclusion that the right answer should in fact be the “average rule”

$$x_j p_j \longrightarrow -\frac{1}{2} i\hbar \left(x_j \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j} x_j \right) \quad (4)$$

corresponding to the splitting $x_j p_j = \frac{1}{2}(x_j p_j + p_j x_j)$. In 1926 Born and Jordan [3] proposed to more generally quantize monomials $x_j^m p_j^n$ using the rule

$$\text{(BJ)} \quad x_j^m p_j^n \longrightarrow \frac{1}{n+1} \sum_{k=0}^n \hat{p}_j^{n-k} \hat{x}_j^m \hat{p}_j^k \quad (5)$$

where we have set as usual $\hat{x}_j = \text{multiplication by } x_j$ and $\hat{p}_j = i\hbar \partial / \partial x_j$ (see Fedak and Prentis [6] for a readable analysis cast in a “modern” language of Born and Jordan’s argument; the older papers [4] by Castellani and [5] by Crehan also contain valuable information). Born and Jordan’s rules (5) were actually soon superseded (at least in the mathematical literature) by Weyl’s quantization procedure: in his mathematical study of quantum mechanics, Weyl proposed in [13] a very general quantization rule which leads, for monomials, to the replacement of the Born–Jordan prescription (5) by

$$\text{(Weyl)} \quad x_j^m p_j^n \longrightarrow \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} \hat{p}_j^{n-k} \hat{x}_j^m \hat{p}_j^k. \quad (6)$$

Weyl’s correspondence nowadays plays an important role in mathematical analysis (the theory of pseudo-differential operators), and in physics it has

become the preferred quantization scheme, and this mainly for two reasons: first to real observables (or symbols as they are called in mathematics) correspond (formally) self-adjoint operators; this is a very desirable properties since a thumb rule is that to a real observable should correspond an operator with real eigenvalues (which are, in quantum mechanics, the values that the observable can actually take). Another advantage of the Weyl correspondence is of a more subtle nature: it is its symplectic covariance property; this property which is actually *characteristic* of the Weyl correspondence among all other pseudo-differential calculi, says that if we perform a linear symplectic change of variables in the symbol, then the resulting operator is conjugated to the original by a certain unitary operator obtained by the metaplectic representation. A third property, which is in a sense rather unwelcome (Kauffmann [11]) is that the Weyl correspondence is invertible (see e.g. Wong [14]). Invertibility poses severe epistemological problems, because it is not physically founded. It is actually possible to prove (de Gosson and Hiley [9]) that there is a one-to-one correspondence between Hamiltonian flows and the continuous groups of operators in $L^2(\mathbb{R}^N)$ solving the Schrödinger equation with Hamiltonian obtained by Weyl quantization. This result in a sense “trivializes” quantum mechanics making it appear merely as a “copy” of Hamiltonian mechanics. This issue, which is related to “dequantization”, will be briefly discussed at the end of the present paper.

Aims and structure of the paper

Shubin’s τ -pseudo-differential calculus (which we review and complement) in Section 1) suggests to consider variants of the usual Wigner distribution of the type

$$W_\tau(\psi, \phi)(z) = \left(\frac{1}{2\pi\hbar}\right)^N \int_{\mathbb{R}^N} e^{-\frac{i}{\hbar}py} \psi(x + \tau y) \bar{\phi}(x - (1 - \tau)y) dy$$

where τ is a real parameter (the choice $\tau = \frac{1}{2}$ yields the usual cross-Wigner distribution). Recently Boggiatto et al [1] (also see Boggiatto et al [2]) have shown the advantages of using the average

$$Q(\psi, \phi) = \int_0^1 W_\tau(\psi, \phi) d\tau$$

of these τ -distributions on the interval $[0, 1]$. Besides the fact that it belongs to the Cohen class and has the right marginals (which is an essential feature in quantum mechanics), the distribution $Q(\psi, \phi)$ almost entirely eliminates the interference phenomenon (“ghost frequencies”) presented by the distributions $W_\tau(\psi, \phi)$. This property makes of $Q(\psi, \phi)$ a

tool of choice in time-frequency analysis. Recalling that the Wigner transform $W(\psi, \phi) = W_{1/2}(\psi, \phi)$ is related to the Weyl operator $\widehat{A} = \text{Op}(a)$ by the formula

$$(\widehat{A}\psi|\phi)_{L^2} = \langle a, W(\psi, \phi) \rangle$$

this suggests to define a new type of pseudo-differential operator \widetilde{A} by the formula

$$(\widetilde{A}\psi|\phi)_{L^2} = \langle a, Q(\psi, \phi) \rangle;$$

not very surprisingly that operator \widetilde{A} is also an “average”, namely

$$\widetilde{A} = \int_0^1 \widehat{A}_\tau d\tau$$

where $\widehat{A}_\tau = \text{Op}_\tau(a)$ is the Shubin τ -pseudo-differential operator with symbol a . We will show in this paper that this operator \widetilde{A} (which is also studied in Boggiatto et al [1]) is precisely the *Born–Jordan quantization* \widehat{A}_{BJ} of the symbol a (see Section 2), and which is characterized by the rules (5) above), and show that it allows to recover the quantization formulas ((5)) when the symbol is a monomial. We also prove a harmonic decomposition result for the operator $\widetilde{A} = \widehat{A}_{\text{BJ}}$, namely (Proposition 15)

$$\widehat{A}_{\text{BJ}}\psi = \left(\frac{1}{2\pi\hbar}\right)^N \int_{\mathbb{R}^{2N}} \mathcal{F}_\sigma a(z_0) \Theta(z_0) \widehat{T}(z_0) \psi dz_0$$

where $\widehat{T}(z_0)$ is the usual Heisenberg operator, $\mathcal{F}_\sigma a$ the symplectic Fourier transform of the symbol and Θ is the function defined by

$$\Theta(z_0) = \frac{\sin(p_0 x_0 / \hbar)}{p_0 x_0 / \hbar}$$

which also appears (for $\hbar = 1/2\pi$) in the work of Boggiatto et al [1]; the formula above shows, in particular, that the Weyl symbol of \widehat{A}_{BJ} is given by the formula

$$a_{\text{W}} = \left(\frac{1}{2\pi\hbar}\right)^N a * \mathcal{F}_\sigma \Theta.$$

We also discuss the symplectic covariance properties of the Born–Jordan quantization; we prove that this covariance holds for an interesting subgroup of the metaplectic group, namely the group generated by the metaplectic group and the Fourier transform (full symplectic covariance cannot of course be expected since the latter is characteristic of Weyl quantization as has been shown in detail by Wong [14]).

0.1 Notation

We will write $x = (x_1, \dots, x_N)$, $p = (p_1, \dots, p_N)$, and $z = (x, p)$. Scalar products will be denoted xx' , pp' , etc. For instance $p_1x_1 + \dots + p_Nx_N$. The standard symplectic form on $\mathbb{R}^{2N} \equiv \mathbb{R}^N \oplus \mathbb{R}^N$ is given by $\sigma(z, z') = px' - p'x$. The associated symplectic group is denoted $\text{Sp}(2N, \mathbb{R})$. We denote \mathcal{F} the \hbar -dependent unitary Fourier transform defined by

$$\mathcal{F}\psi(p) = \left(\frac{1}{2\pi\hbar}\right)^{N/2} \int_{\mathbb{R}^N} e^{ipx} \psi(x) dx.$$

The scalar product of two functions ψ, ϕ on \mathbb{R}^N is $(\psi|\phi)_{L^2} = \int_{\mathbb{R}^N} \psi(x) \bar{\phi}(x) dx$ and the associated norm is denoted $\|\psi\|_{L^2}$.

1 Pseudo-Differential Operators

1.1 Definitions and first properties

1.1.1 The operators \widehat{A}_τ

The consideration of quantization rules leads us to study pseudo-differential operators of the type

$$\widehat{A}_\tau \psi(x) = \left(\frac{1}{2\pi\hbar}\right)^N \int_{\mathbb{R}^{2N}} e^{\frac{i}{\hbar} p \cdot (x-y)} a(\tau x + (1-\tau)y, p) \psi(y) dy dp \quad (7)$$

where τ is a real parameter and the integral should be understood in some “reasonable” sense (Shubin [12]). We will often use the notation

$$\widehat{A}_\tau = \text{Op}_\tau(a).$$

For instance, if $\psi \in \mathcal{S}(\mathbb{R}^N)$ and $a \in \mathcal{S}(\mathbb{R}^{2N})$ the integral is absolutely convergent. For more general symbols a (for instance $a \in \mathcal{S}'(\mathbb{R}^{2N})$) one can give a meaning to the expression (7) by declaring that the operator \widehat{A} is defined by the distributional kernel

$$K_\tau(x, y) = \left(\frac{1}{2\pi\hbar}\right)^{N/2} (\mathcal{F}_2^{-1} a)((1-\tau)x + \tau y, p) \quad (8)$$

where \mathcal{F}_2^{-1} is the inverse Fourier transform in the second set of variables; it is however more natural in our context to use the method explained after Proposition 1 below, and which makes use of the τ -Wigner transform. We notice that setting $\tau = \frac{1}{2}$ in formula (7) we recover the expression

$$\widehat{A}\psi(x) = \left(\frac{1}{2\pi\hbar}\right)^N \iint_{\mathbb{R}^{2N}} e^{\frac{i}{\hbar} p \cdot (x-y)} a\left(\frac{1}{2}(x+y), p\right) \psi(y) dy dp. \quad (9)$$

of a Weyl operator which is standard in the theory of partial differential operators. Formula (7) can be rewritten

$$\widehat{A}\psi(x) = \left(\frac{1}{2\pi\hbar}\right)^{N/2} \int_{\mathbb{R}^N} e^{\frac{i}{\hbar}px} a(x,p) \mathcal{F}\psi(p) dp \quad (10)$$

where $\mathcal{F}\psi$ is the Fourier transform of ψ ; this is the conventional definition of a pseudo-differential operator found in most texts dealing with partial differential equations and a is then sometimes called the “Kohn–Nirenberg symbol” of the operator \widehat{A} . The Kohn–Nirenberg calculus is used mainly in the microlocal analysis of partial differential equations, and in time-frequency analysis where it is sometimes more tractable for computational purposes than the Weyl correspondence. One immediately checks that Kohn–Nirenberg operators correspond to the simple ordering rule

$$(KN) \quad x_j^m p_j^n \longrightarrow \widehat{x}_j^m \widehat{p}_j^n \quad (11)$$

in the case of monomials.

A well-known property of the Weyl operators $\widehat{A} = \text{Op}_{1/2}(a)$ is that the (formal) adjoint is given by $\widehat{A}^* = \text{Op}_{1/2}(\bar{a})$; in the τ -dependent case we have the more general relation

$$\text{Op}_\tau(a)^* = \text{Op}_{1-\tau}(\bar{a}) \quad (12)$$

valid for every real τ .

1.1.2 The quasi-distribution W_τ

An associated object is the τ -Wigner distribution; it is defined as follows: for a pair (ψ, ϕ) of functions in $\mathcal{S}(\mathbb{R}^N)$ one sets

$$W_\tau(\psi, \phi)(z) = \left(\frac{1}{2\pi\hbar}\right)^N \int_{\mathbb{R}^N} e^{-\frac{i}{\hbar}py} \psi(x + \tau y) \bar{\phi}(x - (1 - \tau)y) dy \quad (13)$$

and when $\psi = \phi$ one writes $W_\tau(\psi, \psi) = W_\tau\psi$. Of course, when $\tau = \frac{1}{2}$ one recovers the usual cross-Wigner transform

$$W(\psi, \phi)(z) = \left(\frac{1}{2\pi\hbar}\right)^N \int_{\mathbb{R}^N} e^{-\frac{i}{\hbar}py} \psi(x + \frac{1}{2}y) \bar{\phi}(x - \frac{1}{2}y) dy. \quad (14)$$

and, as is the case for W the mapping W_τ is a bilinear and continuous mapping $\mathcal{S}(\mathbb{R}^N) \times \mathcal{S}(\mathbb{R}^N) \longrightarrow \mathcal{S}(\mathbb{R}^{2N})$. If $\tau = 0$ we get

$$W_0(\psi, \phi)(z) = \left(\frac{1}{2\pi\hbar}\right)^{N/2} e^{-\frac{i}{\hbar}py} \psi(x) \overline{\mathcal{F}\phi(p)}$$

hence $W_0(\psi, \phi)$ is the Rihaczek–Kirkwood distribution $R(\psi, \phi)$ well-known from time-frequency analysis (Gröchenig [10], Boggiatto et al. [1]); if $\tau = 1$ one gets the so-called dual Rihaczek–Kirkwood distribution $R^*(\phi, \psi)$. It is easily verified that

$$W_\tau(\phi, \psi) = \overline{W_{1-\tau}(\psi, \phi)}.$$

The distribution $W_\tau\psi = W_\tau(\phi, \psi)$ satisfies the usual marginal properties:

$$\int_{\mathbb{R}^N} W_\tau\psi(z)dp = |\psi(x)|^2, \quad \int_{\mathbb{R}^N} W_\tau\psi(z)dx = |\mathcal{F}\psi(p)|^2 \quad (15)$$

(Boggiatto et al. [1])

There is a fundamental relation between Weyl pseudo-differential operators and the cross-Wigner transform, which is actually often used to define $\widehat{A} = \text{Op}(a)$:

$$(\text{Op}(a)\psi|\phi)_{L^2} = \langle a, W(\psi, \phi) \rangle \quad (16)$$

for $\psi, \phi \in \mathcal{S}(\mathbb{R}^N)$. Not very surprisingly this formula extends to the case of τ -operators:

Proposition 1 *Let $\psi, \phi \in \mathcal{S}(\mathbb{R}^N)$, $a \in \mathcal{S}(\mathbb{R}^{2N})$, and τ a real number. We have*

$$(\text{Op}_\tau(a)\psi|\phi)_{L^2} = \langle a, W_\tau(\psi, \phi) \rangle \quad (17)$$

where $\langle \cdot, \cdot \rangle$ is the distributional bracket on \mathbb{R}^{2N} .

Proof. By definition of W_τ we have

$$\begin{aligned} \langle a, W_\tau(\psi, \phi) \rangle = \\ \left(\frac{1}{2\pi\hbar}\right)^N \int_{\mathbb{R}^{3N}} e^{-\frac{i}{\hbar}p \cdot y} a(z) \psi(x + \tau y) \overline{\phi}(x - (1 - \tau)y) dy dp dx \end{aligned}$$

and setting $x + \tau y = y'$, $x - (1 - \tau)y = y'$ this is

$$\begin{aligned} \langle a, W_\tau(\psi, \phi) \rangle = \\ \left(\frac{1}{2\pi\hbar}\right)^N \int_{\mathbb{R}^{3N}} e^{-\frac{i}{\hbar}p \cdot (x' - y')} a((1 - \tau)x' + \tau y', p) \psi(y') \overline{\phi}(x') dy dp dx \end{aligned}$$

hence the equality (17) in view of definition (7) of the operator $\widehat{A}_\tau = \text{Op}_\tau(a)$.

■

Formula (17) allows us to define $\widehat{A}_\tau\psi = \text{Op}_\tau(a)\psi$ for arbitrary symbols $a \in \mathcal{S}'(\mathbb{R}^{2N})$ and $\psi \in \mathcal{S}(\mathbb{R}^N)$ in the same way as is done for Weyl pseudo-differential operators: choose $\phi \in \mathcal{S}(\mathbb{R}^N)$; then $W_\tau(\psi, \phi) \in \mathcal{S}(\mathbb{R}^{2N})$ and the distributional bracket $\langle a, W_\tau(\psi, \phi) \rangle$ is thus well-defined; by definition $\widehat{A}_\tau\psi$ is given by (17), and \widehat{A}_τ is a continuous operator $\mathcal{S}(\mathbb{R}^N) \rightarrow \mathcal{S}'(\mathbb{R}^N)$.

Remark 2 *It follows from the argument above and using Schwartz's kernel theorem that every continuous operator $\mathcal{S}(\mathbb{R}^N) \longrightarrow \mathcal{S}'(\mathbb{R}^N)$ is an operator of the type \widehat{A}_τ for every value of the parameter τ ; the argument goes exactly as for the standard case of Weyl operators treated in Shubin [12], Gröchenig [10].*

In Weyl calculus the introduction of the Wigner transform $W\psi$ of a square integrable function the following very simple and natural interpretation: it is, up to a constant factor, the Weyl symbol of the projection operator Π_ψ of $L^2(\mathbb{R}^N)$ on the ray $\{\lambda\psi : \lambda \in \mathbb{C}\}$. This interpretation extends to the τ -dependent case:

Proposition 3 *Let $\psi \in L^2(\mathbb{R}^N)$.*

(i) *We have $\Pi_\psi = (2\pi\hbar)^N \text{Op}_\tau(W_\tau\psi)$;*

(ii) *The τ -symbol of the operator with kernel $K = \psi \otimes \bar{\psi}$ is $(2\pi\hbar)^N W_\tau(\psi, \phi)$.*

Proof. (i) Let $\phi \in L^2(\mathbb{R}^N)$; by definition $\Pi_\psi\phi = (\phi|\psi)_{L^2}\psi$ that is

$$\Pi_\psi\phi(x) = \int_{\mathbb{R}^N} \psi(x)\bar{\psi}(y)\phi(y)dy$$

hence the kernel of Π_ψ is $K(x, y) = \psi(x)\bar{\psi}(y)$. Using a partial Fourier inversion, formula (8) expressing the kernel of Π_ψ in terms of its τ -symbol π_ψ can be rewritten

$$\begin{aligned} \pi_\psi(x, p) &= \int_{\mathbb{R}^N} e^{-\frac{i}{\hbar}py} K(x + \tau y, x - (1 - \tau)y) dy \\ &= \int_{\mathbb{R}^N} e^{-\frac{i}{\hbar}py} \psi(x + \tau y)\bar{\psi}(x - (1 - \tau)y) dy \\ &= (2\pi\hbar)^N W_\tau\psi(x, p). \end{aligned}$$

The assertion (ii) is proven similarly replacing $\psi \otimes \bar{\psi}$ with $\psi \otimes \bar{\phi}$ in the two last equalities above. ■

We also have the Moyal identity:

Proposition 4 *Let $(\cdot|\cdot)_{L^2}$ be the scalar product on $L^2(\mathbb{R}^{2N})$ and $\|\cdot\|_{L^2}$ the associated norm. We have*

$$((W_\tau(\psi, \phi)|W_\tau(\psi', \phi'))_{L^2})_{L^2} = \left(\frac{1}{2\pi\hbar}\right)^N (\psi|\psi')_{L^2} \overline{(\phi|\phi')_{L^2}} \quad (18)$$

and hence in particular

$$\|W_\tau(\psi, \phi)\|_{L^2} = \left(\frac{1}{2\pi\hbar}\right)^{N/2} \|\psi\|_{L^2} \|\phi\|_{L^2} \quad (19)$$

for all $\psi, \psi', \phi, \phi' \in L^2(\mathbb{R}^N)$.

Proof. Let us set

$$A = (2\pi\hbar)^{2N} ((W_\tau(\psi, \phi)|W_\tau(\psi', \phi'))_{L^2}).$$

We have, by definition of W_τ ,

$$\begin{aligned} A &= \int_{\mathbb{R}^{4N}} e^{-\frac{i}{\hbar}p(y-y')} \\ &\quad \times \psi(x + \tau y)\psi'(x + \tau y)\overline{\phi}(x - (1 - \tau)y')\overline{\phi'}(x - (1 - \tau)y')dx dp dy dy'. \end{aligned}$$

The integral in p is $(2\pi\hbar)^N \delta(y - y')$ hence

$$\begin{aligned} A &= (2\pi\hbar)^N \\ &\quad \times \int_{\mathbb{R}^{2N}} \psi(x + \tau y)\psi'(x + \tau y)\overline{\phi}(x - (1 - \tau)y)\overline{\phi'}(x - (1 - \tau)y)dx dy. \end{aligned}$$

Setting $u = x + \tau y$ and $v = x - (1 - \tau)y$ we have $dudv = dxdy$ and hence

$$\begin{aligned} A &= (2\pi\hbar)^N \int_{\mathbb{R}^{2N}} \psi(u)\psi'(u)\overline{\phi}(v)\overline{\phi'}(v)dudv \\ &= (2\pi\hbar)^N (\psi|\psi')_{L^2} \overline{(\phi|\phi')}_{L^2} \end{aligned}$$

which proves (18); formula (19) follows. ■

1.1.3 Ordering of monomials

Since we are dealing in this paper with ordering issues let us find the τ -pseudo-differential operator corresponding to the monomial symbols $x_j^m p_j^n$ considered in the Introduction:

Proposition 5 *Let m and n be two non-negative integers. We have*

$$\text{Op}_\tau(x_j^m p_j^n) = \sum_{k=0}^m \binom{m}{k} \tau^k (1 - \tau)^{m-k} \widehat{x}_j^k \widehat{p}_j^n \widehat{x}_j^{m-k} \quad (20)$$

or, equivalently,

$$\text{Op}_\tau(x_j^m p_j^n) = \sum_{k=0}^n \binom{n}{k} (1 - \tau)^k \tau^{n-k} \widehat{p}_j^k \widehat{x}_j^n \widehat{p}_j^{n-k} \quad (21)$$

where $\widehat{x}_j^\ell \psi = x_j^\ell \psi$ and $\widehat{p}_j^\ell \psi = (-i\hbar \partial_{x_j})^\ell \psi$.

Proof. It is sufficient to assume $N = 1$ so we write $x_j^m = x^m$ and $p_j^n = p^n$. Let us set $a_{m,n}(z) = x^m p^n$; we have using the binomial formula

$$a_{m,n}(\tau x + (1 - \tau)y, p) = \sum_{k=0}^m \binom{m}{k} \tau^k (1 - \tau)^{m-k} x^k y^{m-k} p^n. \quad (22)$$

Setting $b_{m,n,k}(z) = x^k y^{m-k} p^n$ we have (in the sense of distributions)

$$\text{Op}_\tau(b_{m,n,k})\psi(x) = \frac{1}{2\pi\hbar} x^k \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{\frac{i}{\hbar} p(x-y)} p^n dp \right] y^{m-k} \psi(y) dy.$$

Using the Fourier inversion formula

$$\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{\frac{i}{\hbar} p(x-y)} p^n dp = (-i\hbar)^n \delta^{(n)}(x - y) \quad (23)$$

we thus have

$$\text{Op}_\tau(b_{m,n,k})\psi = x^k (-i\hbar)^n \partial_x^n (x^{m-k} \psi).$$

Formula (20) follows inserting this expression in (22). To prove that this formula is equivalent to (21) the easiest method consists in remarking that we have the conjugation formula

$$\mathcal{F} \text{Op}_\tau(a) \mathcal{F}^{-1} = \text{Op}_{1-\tau}(a \circ J^{-1})$$

(which will be proven in Proposition 7 below). Since we have $a_{m,n}(J^{-1}z) = (-1)^m x^m p^n$ and, using the standard properties of the Fourier transform,

$$\mathcal{F} \text{Op}_\tau(a_{m,n}) \mathcal{F}^{-1} = (-1)^m \widehat{p}^k \widehat{x}^n \widehat{p}^{m-k}$$

formula (21) follows. ■

Remark 6 Taking $\tau = \frac{1}{2}$ in either formula (20) or (21) we recover Weyl's ordering rule (6). Similarly, taking $\tau = 0$, one gets the Kohn–Nirenberg ordering rule (11).

1.2 Symplectic covariance properties

1.2.1 Conjugation with Fourier transform

As already mentioned in the Introduction a characteristic property of Weyl quantization is symplectic covariance. This property can be described as follows: let $\text{Sp}(2N, \mathbb{R})$ be the standard symplectic group of \mathbb{R}^{2N} : it is the group of linear automorphisms s of \mathbb{R}^{2N} such that $s^T J s = J$ where J is the

matrix $\begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}$; equivalently $s \in \text{Sp}(2N, \mathbb{R})$ if and only if $\sigma(sz, sz') = \sigma(z, z')$ for all $z, z' \in \mathbb{R}^{2N}$ where $\sigma(z, z') = Jz \cdot z'$ is the standard symplectic form. The group $\text{Sp}(2N, \mathbb{R})$ is connected and $\pi_1[\text{Sp}(2N, \mathbb{R})] = (\mathbb{Z}, +)$ so that it has a connected covering group $\text{Sp}_2(2N, \mathbb{R})$ of order 2. That group has a faithful representation by a group of unitary operators on $L^2(\mathbb{R}^N)$, the metaplectic group $\text{Mp}(2N, \mathbb{R})$. Let $\pi^{\text{Mp}} : \text{Mp}(2N, \mathbb{R}) \longrightarrow \text{Sp}(2N, \mathbb{R})$ be the natural projection; to every $s \in \text{Sp}(2N, \mathbb{R})$ thus correspond two elements $\pm S$ of $\text{Mp}(2N, \mathbb{R})$ such that $\pi^{\text{Mp}}(\pm S) = s$. The symplectic covariance of Weyl calculus means that if $\widehat{A} = \text{Op}(a)$ then $S^{-1}\widehat{A}S = \text{Op}(a \circ s)$. This property is equivalent to the following property of the cross-Wigner transform:

$$W(S\psi, S\phi)(z) = W(\psi, \phi)(s^{-1}z) \quad (24)$$

(it is actually an easy exercise to deduce this equivalence from formula (16). This property is characteristic of Weyl calculus: let \widehat{A} be a linear continuous operator $\mathcal{S}(\mathbb{R}^N) \longrightarrow \mathcal{S}'(\mathbb{R}^N)$ and write it as a τ -operator \widehat{A}_τ associated with a symbol a (formula (7); cf. Remark 2). Then if $S^{-1}\widehat{A}S$, again viewed as a τ -operator, has symbol $a \circ s$ we must have $\tau = \frac{1}{2}$. For this reason one cannot expect a general symplectic covariance property for the τ -pseudo-differential calculus unless $\tau = \frac{1}{2}$. For instance Boggiatto et al. prove in [1] that $W_{1-\tau}(\mathcal{F}\psi)(p, -x) = W_\tau\psi(x, p)$ when \mathcal{F} is the Fourier transform; in fact the same argument shows that, more generally,

$$W_{1-\tau}(\mathcal{F}\psi, \mathcal{F}\phi)(p, -x) = W_\tau(\psi, \phi)(x, p).$$

Now, the modified Fourier transform $F = e^{-iN\pi/4}\mathcal{F}$ is in $\text{Mp}(2N, \mathbb{R})$ and we have precisely $\pi^{\text{Mp}}(F) = J$ hence the formula above can be written in a more symplectic fashion as

$$W_{1-\tau}(F\psi, F\phi)(z) = W_\tau(\psi, \phi)(J^{-1}z) \quad (25)$$

which reduces to (24) in the case $s = J$ if and only if $\tau = \frac{1}{2}$. Formula (25) has the following interesting consequence for τ -pseudo-differential operators:

Proposition 7 *Let $\widehat{A}_\tau = \text{Op}_\tau(a)$, $a \in \mathcal{S}'(\mathbb{R}^{2N})$. We have*

$$\mathcal{F} \text{Op}_\tau(a) \mathcal{F}^{-1} = \text{Op}_{1-\tau}(a \circ J^{-1}) \quad (26)$$

Proof. Let $\psi, \phi \in \mathcal{S}(\mathbb{R}^N)$; since \mathcal{F} is unitary we have

$$(\mathcal{F} \text{Op}_\tau(a) \mathcal{F}^{-1} \psi | \phi)_{L^2} = (\text{Op}_\tau(a) \mathcal{F}^{-1} \psi | \mathcal{F}^{-1} \phi)_{L^2}$$

hence, using twice (25),

$$\begin{aligned} (\mathcal{F} \text{Op}_\tau(a) \mathcal{F}^{-1} \psi | \phi)_{L^2} &= \langle a, W_\tau(\psi, \phi) \circ J \rangle \\ &= \langle a \circ J^{-1}, W_{1-\tau}(\psi, \phi) \rangle \\ &= (\text{Op}_{1-\tau}(a \circ J^{-1}) \psi | \phi)_{L^2} \end{aligned}$$

which implies (26) since ψ and ϕ are arbitrary. ■

Remark 8 Formula (26) in Proposition 7 allows us to give a very short proof of the fact that Weyl operators are the only pseudo-differential operators satisfying the property of symplectic covariance (cf. the proof in Wong [14]). Indeed, replacing \mathcal{F} by F in (26) we see that $F \text{Op}_\tau(a) F^{-1} = \text{Op}_\tau(a \circ J^{-1})$ if and only if $\tau = \frac{1}{2}$. One concludes by noting that $F \in \text{Mp}(2N, \mathbb{R})$.

1.2.2 Covariance under the metilinear group

However, symplectic covariance subsists for an important subgroup of the metaplectic group $\text{Mp}(2N, \mathbb{R})$. Let m_L be the automorphism of \mathbb{R}^{2N} defined, for $L \in \text{GL}(N, \mathbb{R})$, by $m_L(x, p) = (L^{-1}x, L^T p)$. One immediately verifies that $m_L \in \text{Sp}(2N, \mathbb{R})$. Moreover, each m_L is the projection onto $\text{Sp}(2N, \mathbb{R})$ of the two operators $M_{L, \mu}$ and $M_{L, \mu+2} = -M_{L, \mu}$ in $\text{Mp}(2N, \mathbb{R})$ defined by

$$M_{L, \mu} \psi(x) = i^\mu \sqrt{|\det L|} \psi(Lx);$$

here μ (the ‘‘Maslov index’’, see de Gosson [7]) is 0 or 2 (modulo 4) if $\det L > 0$ and 1 or 3 (modulo 4) if $\det L < 0$. The operators $M_{L, \mu}$ satisfy the multiplication rule $M_{L, \mu} M_{L', \mu'} = M_{L'L, \mu+\mu'}$ and thus form a group of unitary operators, the *metilinear group* $\text{ML}(2N, \mathbb{R})$.

Proposition 9 Let $M_{L, \mu} \in \text{ML}(2N, \mathbb{R})$. We have

$$W_\tau(M_{L, \mu} \psi, M_{L, \mu} \phi)(z) = W_\tau(\psi, \phi)(m_L^{-1} z) \quad (27)$$

for $\psi, \phi \in \mathcal{S}(\mathbb{R}^N)$ and

$$M_{L, \mu}^{-1} \text{Op}_\tau(a) M_{L, \mu} = \text{Op}_\tau(a \circ m_L) \quad (28)$$

for $a \in \mathcal{S}'(\mathbb{R}^{2N})$.

Proof. We have

$$\begin{aligned} W_\tau(M_{L, \mu} \psi, M_{L, \mu} \phi)(z) &= \left(\frac{1}{2\pi\hbar}\right)^N |\det L| \\ &\quad \times \int_{\mathbb{R}^N} e^{-\frac{i}{\hbar} p \cdot y} \psi(L(x + \tau y)) \overline{\phi}(L(x - (1 - \tau)y)) dy \end{aligned}$$

that is, setting $y' = Ly$,

$$W_\tau(M_{L,\mu}\psi, M_{L,\mu}\phi)(z) = \left(\frac{1}{2\pi\hbar}\right)^N \int_{\mathbb{R}^N} e^{-\frac{i}{\hbar}p \cdot L^{-1}y'} \psi(Lx + \tau y') \bar{\phi}(Lx - (1-\tau)y') dy'$$

hence (27). To prove formula (28) we begin by noting that

$$(M_{L,\mu}^{-1} \text{Op}_\tau(a) M_{L,\mu} \psi | \phi)_{L^2} = (\text{Op}_\tau(a) M_{L,\mu} \psi | M_{L,\mu} \phi)_{L^2}$$

that is, using (17) in Proposition 1, formula (27), and again formula (17):

$$\begin{aligned} (M_{L,\mu}^{-1} \text{Op}_\tau(a) M_{L,\mu} \psi | \phi)_{L^2} &= \int_{\mathbb{R}^N} a(z) W_\tau(M_{L,\mu}\psi, M_{L,\mu}\phi)(z) dz \\ &= \int_{\mathbb{R}^N} a(z) W_\tau(\psi, \phi)(m_L^{-1}z) dz \\ &= \int_{\mathbb{R}^N} a(m_L z) W_\tau(\psi, \phi)(z) dz \\ &= (\text{Op}_\tau(a \circ m_L) \psi | \phi)_{L^2} \end{aligned}$$

hence (28). ■

1.3 Cohen class property

1.3.1 Definition of the Cohen class

Let $Q : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^{2n})$ be a sesquilinear form and set $Q\psi = Q(\psi, \psi)$. Recall that $Q\psi$ belongs to the Cohen class if it is of the type $Q\psi = W\psi * \theta$ for some distribution $\theta \in \mathcal{S}'(\mathbb{R}^{2n})$. Sufficient conditions for a distribution to belong to Cohen's class are

$$Q\psi(z - z_0) = Q(\widehat{T}(z_0)\psi)(z) \quad (29)$$

$$|Q(\psi, \phi)(0, 0)| \leq C \|\psi\|_{L^2} \|\phi\|_{L^2} \quad (30)$$

where C is a constant (see e.g. Gröchenig [10] or de Gosson [8]). Taking $Q(\psi, \phi) = W_\tau(\psi, \phi)$ condition (29) is easily seen to hold, but condition (30) only holds when $\tau \neq 0$ and $\tau \neq 1$; in fact a straightforward calculation using the Cauchy–Schwarz inequality yields the estimate

$$|W_\tau(\psi, \phi)(0)| \leq \left(\frac{1}{2\pi\hbar}\right)^N \frac{1}{\tau^{N/2}(1-\tau)^{N/2}} \|\psi\|_{L^2} \|\phi\|_{L^2}. \quad (31)$$

Boggiatto et al. [1] however show by a direct calculation that when $\hbar = 1/2\pi$ one has

$$W_\tau(\psi, \phi) = W(\psi, \phi) * \alpha_\tau \quad (32)$$

with

$$\alpha_\tau(z) = \left(\frac{2}{|2\tau-1|} \right)^N e^{2\pi i \frac{2}{2\tau-1} px}.$$

when $\tau \neq \frac{1}{2}$. It follows that:

Proposition 10 *For $\psi, \phi \in \mathcal{S}(\mathbb{R}^N)$ we have*

$$W_\tau(\psi, \phi) = W(\psi, \phi) * \theta_\tau \quad (33)$$

where

$$\theta_\tau(z) = \left(\frac{1}{|2\tau-1|\pi\hbar} \right)^N e^{i \frac{2}{2\tau-1} px} \quad (34)$$

when $\tau \neq \frac{1}{2}$. When $\tau = \frac{1}{2}$ we have $\theta_\tau = \delta$.

Proof. Let us denote $W_\tau^{2\pi}(\psi, \phi)$ the transform $W_\tau(\psi, \phi)$ when $\hbar = 1/2\pi$; it is related to the general case by the obvious formula

$$W_\tau^{2\pi}(\psi, \phi)(x, p) = (2\pi\hbar)^N W_\tau(\psi, \phi)(x, 2\pi\hbar); \quad (35)$$

the result immediately follows from (32) using elementary changes of variables. The case $\tau = \frac{1}{2}$ is straightforward since $W_{1/2}(\psi, \phi) = W(\psi, \phi)$. ■

1.3.2 Applications to \widehat{A}_τ

We are going to establish an important representation result for τ -pseudo-differential operators using the Heisenberg operator. Let us first prove the following Lemma which is a straightforward consequence of the Proposition above:

Lemma 11 *Let $\widehat{A}_\tau = \text{Op}_\tau(a)$ with $a \in \mathcal{S}'(\mathbb{R}^{2N})$. The Weyl symbol a_W of \widehat{A}_τ is given by $a_W = a * \theta_\tau$.*

Proof. In view of the equality (16) the Weyl symbol of \widehat{A}_τ is determined by formula (7):

$$(\widehat{A}_\tau \psi | \phi)_{L^2} = \langle a_W, W(\psi, \phi) \rangle$$

for $\psi, \phi \in \mathcal{S}(\mathbb{R}^N)$. In view of (17) we also have

$$(\widehat{A}_\tau \psi | \phi)_{L^2} = \langle a, W_\tau(\psi, \phi) \rangle$$

that is, taking (33) into account,

$$(\widehat{A}_\tau \psi | \phi)_{L^2} = \langle a, W(\psi, \phi) * \theta_\tau \rangle = \langle a * \theta_\tau^\vee, W(\psi, \phi) \rangle$$

where $\theta_\tau^\vee(z) = \theta_\tau(-z)$ hence $a_W = a * \theta_\tau^\vee$; since $\theta_\tau(-z) = \theta_\tau(z)$ we have $a_W = a * \theta_\tau$ as claimed. ■

Proposition 12 *The action of a pseudo-differential operator \widehat{A}_τ on $\psi \in \mathcal{S}(\mathbb{R}^N)$ is given by*

$$\widehat{A}_\tau \psi = \left(\frac{1}{2\pi\hbar}\right)^N \int_{\mathbb{R}^{2N}} \mathcal{F}_\sigma a(z_0) \widehat{T}_\tau(z_0) \psi dz_0 \quad (36)$$

where

$$\mathcal{F}_\sigma a(z) = \left(\frac{1}{2\pi\hbar}\right)^N \int_{\mathbb{R}^{2N}} e^{-\frac{i}{\hbar}\sigma(z,z')} a(z') dz' = \mathcal{F}a(Jz) \quad (37)$$

is the symplectic Fourier transform of $a \in \mathcal{S}'(\mathbb{R}^{2n})$ and $\widehat{T}_\tau(z_0)$ is the modified Heisenberg operator defined by

$$\widehat{T}_\tau(z_0) \psi(x) = e^{\frac{i}{2\hbar}(2\tau-1)p_0 x_0} \widehat{T}(z_0) \psi(x) \quad (38)$$

that is

$$\widehat{T}_\tau(z_0) \psi(x) = e^{\frac{i}{\hbar}(p_0 x - (1-\tau)p_0 x_0)} \psi(x - x_0). \quad (39)$$

Proof. Assume that $\tau = \frac{1}{2}$, then $\widehat{T}_\tau(z_0) = \widehat{T}(z_0)$ and formula (36) becomes

$$\widehat{A} \psi = \left(\frac{1}{2\pi\hbar}\right)^N \int_{\mathbb{R}^{2N}} \mathcal{F}_\sigma a(z_0) \widehat{T}(z_0) \psi dz_0$$

which is the expression of a Weyl operator well-known in harmonic analysis (see e.g. de Gosson [7, 8]). When $\tau \neq \frac{1}{2}$ we argue as follows: in view of Lemma 11 we have

$$\begin{aligned} \widehat{A}_\tau \psi &= \left(\frac{1}{2\pi\hbar}\right)^N \int_{\mathbb{R}^{2N}} \mathcal{F}_\sigma(a * \theta_\tau)(z_0) \widehat{T}(z_0) \psi dz_0 \\ &= \int_{\mathbb{R}^{2N}} \mathcal{F}_\sigma a(z_0) \mathcal{F}_\sigma \theta_\tau(z_0) \widehat{T}(z_0) \psi dz_0 \end{aligned}$$

where we have used the formula $\mathcal{F}_\sigma(a * \theta_\tau) = (2\pi\hbar)^N \mathcal{F}_\sigma a \mathcal{F}_\sigma \theta_\tau$. which follows at once from the usual formula giving the Fourier transform of a convolution product. A straightforward calculation shows that we have

$$\mathcal{F}_\sigma \theta_\tau(z_0) = \left(\frac{1}{2\pi\hbar}\right)^N e^{\frac{i}{2\hbar}(2\tau-1)p_0 x_0}$$

and hence

$$\widehat{A}_\tau \psi = \left(\frac{1}{2\pi\hbar}\right)^N \int_{\mathbb{R}^{2N}} \mathcal{F}_\sigma a(z_0) e^{\frac{i}{2\hbar}(2\tau-1)p_0 x_0} \widehat{T}(z_0) \psi dz_0$$

which is precisely (36). ■

A straightforward calculation shows that the adjoint of the operator $\widehat{T}_\tau(z_0)$ is given by

$$\widehat{T}_\tau(z_0)^* = \widehat{T}_{1-\tau}(-z_0); \quad (40)$$

from this immediately follows the formula for the adjoint of $\widehat{A}_\tau = \text{Op}_\tau(a)$:

$$\text{Op}_\tau(a)^* = \text{Op}_{1-\tau}(\bar{a}). \quad (41)$$

Remark 13 *A straightforward computation using either (38) or (39) shows that the modified Heisenberg operators $\widehat{T}_\tau(z_0)$ satisfy for all values of τ the commutation relations*

$$\widehat{T}_\tau(z_0)\widehat{T}_\tau(z_1) = e^{\frac{i}{\hbar}\sigma(z_0, z_1)}\widehat{T}_\tau(z_1)\widehat{T}_\tau(z_0). \quad (42)$$

These operators thus correspond to (equivalent) representations of the Heisenberg group.

1.3.3 The τ -dependent cross-ambiguity transform

The cross-Wigner transform $W(\psi, \phi)$ has a “dual companion”, the cross-ambiguity transform $A(\psi, \phi)$ which is explicitly given by the integral formula

$$A(\psi, \phi)(z) = \left(\frac{1}{2\pi\hbar}\right)^N \int_{\mathbb{R}^N} e^{-\frac{i}{\hbar}p \cdot x'} \psi\left(x' + \frac{1}{2}x\right) \overline{\phi\left(x' - \frac{1}{2}x\right)} dx'. \quad (43)$$

One toggles between both using the symplectic Fourier transform:

$$A(\psi, \phi) = F_\sigma W(\psi, \phi) \quad , \quad W(\psi, \phi) = F_\sigma A(\psi, \phi). \quad (44)$$

We are going to generalize this formula to the τ -dependent case; let us first recall the following alternative definition of the cross-ambiguity transform (see de Gosson [7, 8]):

$$A(\psi, \phi)(z) = \left(\frac{1}{2\pi\hbar}\right)^n (\psi | \widehat{T}(z) \phi)_{L^2}. \quad (45)$$

This formula suggests that we define

$$A_\tau(\psi, \phi)(z) = \left(\frac{1}{2\pi\hbar}\right)^n (\psi | \widehat{T}_\tau(z) \phi)_{L^2} \quad (46)$$

where $\widehat{T}_\tau(z)$ is the modified Heisenberg operator (38). A straightforward calculation gives the explicit expression

$$\begin{aligned} A_\tau(\psi, \phi)(z_0) &= \left(\frac{1}{2\pi\hbar}\right)^N e^{-\frac{i}{2\hbar}(2\tau-1)p_0 \cdot x_0} \\ &\quad \times \int_{\mathbb{R}^N} e^{-\frac{i}{\hbar}p_0 \cdot x'} \psi(x' + \tau x_0) \overline{\phi(x' - (1-\tau)x_0)} dx'. \end{aligned}$$

2 Born–Jordan Quantization

In what follows the parameter τ is restricted to the closed interval $[0, 1]$.

2.1 The Born–Jordan operators \widehat{A}_{BJ}

2.1.1 Definition of \widehat{A}_{BJ}

Let $\widehat{A}_\tau = \text{Op}_\tau(a)$ be the pseudo-differential operator defined by formula (7). By definition the Born–Jordan operator $\widehat{A}_{\text{BJ}} = \text{Op}_{\text{BJ}}(a)$ is the average of the operators \widehat{A}_τ for $\tau \in [0, 1]$:

$$\widehat{A}_{\text{BJ}}\psi = \left(\frac{1}{2\pi\hbar}\right)^N \int_0^1 \widehat{A}_\tau\psi d\tau. \quad (47)$$

Note that it immediately follows from formula (12) for the adjoint of that we have

$$\text{Op}_{\text{BJ}}(a)^* = \text{Op}_{\text{BJ}}(\bar{a}) \quad (48)$$

hence, in particular, $\widehat{A}_{\text{BJ}} = \text{Op}_{\text{BJ}}(a)$ is (formally) self-adjoint if and only if the symbol a is real. This important property is thus common to Born–Jordan and Weyl calculus, and makes \widehat{A}_{BJ} a good candidate for a physical quantization procedure. But more about that later.

To justify the chosen terminology we have to show that the quantization $a \rightarrow \text{Op}_{\text{BJ}}(a)$ contains as a particular case the original Born–Jordan prescription (5) described in the Introduction. That is we have to prove that

$$\text{Op}_{\text{BJ}}(x_j^m p_j^n) = \frac{1}{n+1} \sum_{k=0}^n \widehat{p}_j^{n-k} \widehat{x}_j^m \widehat{p}_j^k. \quad (49)$$

Recall that we have shown in Proposition 5 (formula (21)) that

$$\text{Op}_\tau(x_j^m p_j^n) = \sum_{k=0}^n \binom{n}{k} (1-\tau)^k \tau^{n-k} \widehat{p}_j^k \widehat{x}_j^n \widehat{p}_j^{n-k}.$$

It follows that

$$\text{Op}_{\text{BJ}}(x_j^m p_j^n) = \sum_{k=0}^n \binom{n}{k} B(n-k+1, k+1) \widehat{p}_j^k \widehat{x}_j^n \widehat{p}_j^{m-k}$$

where B is the beta function. It follows that

$$\begin{aligned} B(k+1, n-k+1) &= \frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+2)} \\ &= \frac{k!(n-k)!}{(n+1)!} \end{aligned}$$

and hence

$$\text{Op}_{\text{BJ}}(x_j^m p_j^n) = \frac{1}{n+1} \sum_{k=0}^m \widehat{p}_j^k \widehat{x}_j^n \widehat{p}_j^{n-k}$$

which is the same thing as (49). Notice that using the formula (20) which is equivalent to formula (21) the same argument yields the alternative equality

$$\text{Op}_{\text{BJ}}(x_j^m p_j^n) = \frac{1}{m+1} \sum_{k=0}^m \widehat{x}_j^k \widehat{p}_j^n \widehat{x}_j^{m-k}. \quad (50)$$

2.1.2 Comparison of Born–Jordan and Weyl quantization

A quadratic Hamiltonian

$$H(z) = \frac{1}{2} M z^2 = (x, p) M (x, p)^T \quad (51)$$

where $M = M^T$ is a real $2N \times 2N$ matrix has identical Weyl and Born–Jordan quantizations; in fact writing the Hamiltonian as

$$H(z) = \sum_j \alpha_j p_j^2 + \beta_j x_j^2 + 2\gamma_j p_j x_j$$

we see that $\text{Op}_{\text{BJ}}(H) = \text{Op}(H)$ when the γ_j are all zero; when there are cross-terms $x_j p_j$ follows using formula (49) (or (50)) with $m = n = 1$; this shows that the Born–Jordan quantization of $x_j p_j$ is $\frac{1}{2}(\widehat{x}_j \widehat{p}_j + \widehat{p}_j \widehat{x}_j)$ which is the same result as that obtained using Weyl quantization (cf. formula (6)). in both case the corresponding operator is given by

$$\widehat{H} = \frac{1}{2}(\widehat{x}_j, \widehat{p}_j) M (\widehat{x}_j, \widehat{p}_j)^T.$$

Born–Jordan and Weyl quantization are also identical for “physical” Hamiltonians of the type “kinetic energy + potential”. Suppose more generally that H is a symbol of the type (1) that is

$$H = \sum_{j=1}^N \frac{1}{2m_j} p_j^2 + V(x) \quad (52)$$

Then $\widehat{H} = \text{Op}_{\text{BJ}}(H) = \text{Op}(H)$ that is

$$\widehat{H} = \sum_{j=1}^N \frac{-\hbar^2}{2m_j} \frac{\partial^2}{\partial x_j^2} + V(x). \quad (53)$$

This can be seen by noting that $\text{Op}_{\text{BJ}}(p_j^2) = -\hbar^2 \partial^2 / \partial x_j^2$ taking $m = 0$ and $n = 2$ in formula (50) and that by (7)

$$\begin{aligned} \text{Op}_\tau(V)\psi(x) &= \left(\frac{1}{2\pi\hbar}\right)^N \int_{\mathbb{R}^{2N}} e^{\frac{i}{\hbar}p \cdot (x-y)} V(\tau x + (1-\tau)y) \psi(y) dy dp \\ &= \int_{\mathbb{R}^N} V(\tau x + (1-\tau)y) \psi(y) \delta(x-y) dy \\ &= V(x)\psi(x); \end{aligned}$$

integrating in τ from 0 to 1 yields $\text{Op}_{\text{BJ}}(V)\psi = V\psi$ and hence (53). We are going to see that the Born–Jordan and Weyl quantizations of the magnetic Hamiltonian (3) also coincide; let us first prove the following elementary Lemma which we will also apply to quadratic Hamiltonians:

Lemma 14 *Let $\mathcal{A} : \mathbb{R}^N \times \mathbb{R}_t \longrightarrow \mathbb{R}$ be a smooth function. Then*

$$\text{Op}_{\text{BJ}}(p_j \mathcal{A})\psi = \text{Op}(p_j \mathcal{A})\psi = -\frac{i\hbar}{2} \left[\frac{\partial}{\partial x} (\mathcal{A}\psi) + \mathcal{A} \frac{\partial}{\partial x} \psi \right]. \quad (54)$$

Proof. It is sufficient to assume $N = 1$. Using definition (7) of $\widehat{A}_\tau = \text{Op}_\tau(a)$ we have

$$\begin{aligned} \text{Op}_\tau(p\mathcal{A})\psi(x) &= \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} e^{\frac{i}{\hbar}p \cdot (x-y)} p \mathcal{A}(\tau x + (1-\tau)y, t) \psi(y) dy dp \\ &= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}p \cdot (x-y)} p dp \right] \mathcal{A}(\tau x + (1-\tau)y, t) \psi(y) dy. \end{aligned}$$

In view of the Fourier formula (23) the expression between the square brackets is $-i\hbar \delta'(x-y)$ hence

$$\begin{aligned} \text{Op}_\tau(p\mathcal{A})\psi(x) &= -i\hbar \int_{-\infty}^{\infty} \delta'(x-y) \mathcal{A}(\tau x + (1-\tau)y, t) \psi(y) dy \\ &= -i\hbar \int_{-\infty}^{\infty} \delta(x-y) \frac{\partial}{\partial y} [\mathcal{A}(\tau x + (1-\tau)y, t) \psi(y)] dy \\ &= -i\hbar \left[(1-\tau) \frac{\partial}{\partial x} (\mathcal{A}\psi) + \tau \mathcal{A} \frac{\partial}{\partial x} \psi \right]. \end{aligned}$$

Formula (54) follows setting in the Weyl case $\tau = \frac{1}{2}$ and integrating from 0 to 1 in the Born–Jordan case. ■

It follows from the Lemma above that: both Weyl and Born–Jordan quantizations of a (time-dependent) magnetic Hamiltonian

$$H(z, t) = \sum_{j=1}^N \frac{1}{2m_j} (p_j - \mathcal{A}_j(x, t))^2 + V(x, t) \quad (55)$$

are the same. In fact, expanding the terms $(p_j - \mathcal{A}_j(x, t))^2$ we get

$$H = \sum_{j=1}^N \frac{1}{2m_j} p_j^2 - \sum_{j=1}^N \frac{1}{m_j} p_j \mathcal{A}_j + \sum_{j=1}^N \mathcal{A}_j^2 + V.$$

We have seen above that the terms p_j^2 and $\mathcal{A}_j^2 + V$ have identical quantizations; in view of formula (54) this also true of the cross-terms $p_j \mathcal{A}_j$, leading in both cases to the formula

$$\widehat{H} = \sum_{j=1}^N \frac{1}{2m_j} \left(-i\hbar \frac{\partial}{\partial x_j} - \mathcal{A}_j(x, t) \right)^2 + V(x, t) \quad (56)$$

well-known from quantum mechanics.

2.1.3 Harmonic representation of \widehat{A}_{BJ}

It is customary in harmonic analysis to write a Weyl operator \widehat{A} with symbol a in the form

$$\widehat{A}\psi = \left(\frac{1}{2\pi\hbar}\right)^N \int_{\mathbb{R}^{2N}} \mathcal{F}a(x_0, p_0) e^{\frac{i}{\hbar}(\widehat{x}x_0 + \widehat{p}p_0)} \psi dp_0 dx_0; \quad (57)$$

this formula goes back to the work of Weyl [13]. It is however preferable for our study of Born–Jordan quantization to use the alternative formulation

$$\widehat{A}\psi = \left(\frac{1}{2\pi\hbar}\right)^N \int_{\mathbb{R}^{2N}} \mathcal{F}_\sigma a(z_0) \widehat{T}(z_0) \psi dz_0 \quad (58)$$

already used in the proof of Proposition 12. This not only because the role of the Heisenberg group in this procedure becomes more apparent, but also because practical calculations are easier and more explicit. The equivalence of both formulas is clear (at least at the formal level): replacing $z_0 = (x_0, p_0)$ in (57) with Jz_0 one gets

$$\widehat{A}\psi = \left(\frac{1}{2\pi\hbar}\right)^N \int_{\mathbb{R}^{2N}} \mathcal{F}a(Jz_0) e^{\frac{i}{\hbar}\sigma(\widehat{z}, z_0)} \psi dp_0 dx_0$$

which is precisely (58).

Proposition 15 *Let $\psi \in \mathcal{S}(\mathbb{R}^N)$. (i) We have*

$$\widehat{A}_{\text{BJ}}\psi = \left(\frac{1}{2\pi\hbar}\right)^N \int_{\mathbb{R}^{2N}} \mathcal{F}_\sigma a(z_0) \Theta(z_0) \widehat{T}(z_0) \psi dz_0 \quad (59)$$

where Θ is the function defined by

$$\Theta(z_0) = \frac{\sin(p_0 x_0 / \hbar)}{p_0 x_0 / \hbar}. \quad (60)$$

(ii) The Weyl symbol a_W of \widehat{A}_{BJ} is given by the convolution product

$$a_W = \left(\frac{1}{2\pi\hbar}\right)^N a * \mathcal{F}_\sigma \Theta. \quad (61)$$

Proof. (i) In view of formulas (36) and (38) in Proposition 12 we have

$$\widehat{A}_{\text{BJ}}\psi = \left(\frac{1}{2\pi\hbar}\right)^N \int_{\mathbb{R}^{2N}} \mathcal{F}_\sigma a(z_0) \left(\int_0^1 e^{\frac{i}{2\hbar}(2\tau-1)p_0 x_0} d\tau \right) \widehat{T}(z_0)\psi dz_0;$$

a straightforward calculation yields

$$\int_0^1 e^{\frac{i}{2\hbar}(2\tau-1)p_0 x_0} d\tau = \frac{2\hbar}{p_0 x_0} \sin \frac{p_0 x_0}{2\hbar}$$

hence formula (59). (ii) Since the symplectic Fourier transform is involutive we have $(\mathcal{F}_\sigma a)\Theta = (\mathcal{F}_\sigma a)\mathcal{F}_\sigma(\mathcal{F}_\sigma \Theta)$ hence (61) since

$$(\mathcal{F}_\sigma a)\mathcal{F}_\sigma(\mathcal{F}_\sigma \Theta) = \left(\frac{1}{2\pi\hbar}\right)^N \mathcal{F}_\sigma(a * \mathcal{F}_\sigma \Theta).$$

■

In Boggiatto et al. [1] the authors consider the average

$$W_{\text{BJ}}(\psi, \phi)(z) = \int_0^1 W_\tau(\psi, \phi)(z) dt$$

(which they denote by $Q(\psi, \phi)$); they show that the bilinear form W_{BJ} belongs to the Cohen class. It immediately follows from (15) that the marginal properties also hold for $W_{\text{BJ}}\psi = W_{\text{BJ}}(\psi, \phi)$:

$$\int_{\mathbb{R}^N} W_{\text{BJ}}\psi(z) dp = |\psi(x)|^2, \quad \int_{\mathbb{R}^N} W_{\text{BJ}}\psi(z) dx = |\mathcal{F}\psi(p)|^2. \quad (62)$$

As expected, Born–Jordan operators can be expressed in terms of their symbol and $W_{\text{BJ}}(\psi, \phi)$:

Proposition 16 *The operator \widehat{A}_{BJ} and the bilinear form W_{BJ} are related by the formula*

$$(\widehat{A}_{\text{BJ}}\psi|\phi)_{L^2} = \langle a, W_{\text{BJ}}(\psi, \phi) \rangle \quad (63)$$

valid for all $\psi, \phi \in \mathcal{S}(\mathbb{R}^N)$.

Proof. In view of formula (17) in Proposition 1 we have

$$(\widehat{A}_\tau\psi|\phi)_{L^2} = \langle a, W_\tau(\psi, \phi) \rangle;$$

integrating this equality from 0 to 1 with respect to the variable τ yields (63). ■

2.1.4 Symplectic covariance of \widehat{A}_{BJ}

Since a Born–Jordan operator is in general distinct from the Weyl operator with same symbol we cannot expect full symplectic covariance to hold for them. However:

Proposition 17 *Let $\widehat{A}_{\text{BJ}} = \text{Op}_{\text{BJ}}(a)$. We have:*

(i) *Let $F \in \text{Mp}(2n, \mathbb{R})$ be the modified Fourier transform $i^{-d/2}\mathcal{F}$; then*

$$F^{-1} \text{Op}_{\text{BJ}}(a)F = \text{Op}_{\text{BJ}}(a \circ J); \quad (64)$$

(ii) *Let $M_{L,\mu} \in \text{ML}(2n, \mathbb{R})$ (the metilinear group) and $m_L = \pi^{\text{Mp}}(M_{L,m}) \in \text{Sp}(2N, \mathbb{R})$; we have*

$$M_{L,\mu}^{-1} \widehat{A}_{\text{BJ}} M_{L,\mu} = \text{Op}_{\text{BJ}}(a \circ m_L). \quad (65)$$

Proof. (i) In view of formula (26) we have

$$F^{-1} \text{Op}_\tau(a)F = \text{Op}_{1-\tau}(a \circ J)$$

hence

$$F^{-1} \left(\int_0^1 \text{Op}_\tau(a) d\tau \right) F = \int_0^1 \text{Op}_{1-\tau}(a \circ J) d\tau = \int_0^1 \text{Op}_\tau(a \circ J) d\tau$$

and formula (64) follows. (ii) In view of formula (28) in Proposition 9 we have

$$M_{L,\mu}^{-1} \left(\int_0^1 \text{Op}_\tau(a) d\tau \right) M_{L,\mu} = \int_0^1 \text{Op}_\tau(a \circ m_L) d\tau$$

hence the covariance formula (65). ■

Remark 18 *It is possible to give a direct proof of (64) and (65) using the explicit formula (59) for $\widehat{A}_{\text{BJ}}\psi$, the symplectic covariance of Weyl operators, and the fact that the function Θ given by (60) is invariant under the transformations $(x, p) \mapsto (p, -x)$ and $(x, p) \mapsto (L^{-1}x, L^T p)$.*

Recalling that

$$M_{L,\mu}\psi(x) = i^\mu \sqrt{|\det L|} \psi(Lx)$$

the operators F and $M_{L,\mu}$ satisfy the intertwining formula

$$FM_{L,\mu} = M_{(L^T)^{-1}, \mu} F,$$

hence the set $\{F, M_{L,\mu} : \det L \neq 0\}$ is a subgroup of the metaplectic group $\text{Mp}(2n, \mathbb{R})$. The result above says that the Born–Jordan operators are covariant under the action of this group.

Discussion

We have seen that for physical Hamiltonians of the type (52) and (55) both Weyl and Born–Jordan quantizations are the same, and so are the quantizations of the generalized harmonic oscillator (51), so one could wonder whether it is really worth to bother. The reason might come from the fact that Weyl quantization is in a sense “too perfect”. It is, as Kauffmann [11] points out the most “austere” quantization, and this austerity enables it to have very good symmetry properties. In particular it has the property of symplectic covariance, and it is the only pseudo-differential calculus having this feature, as follows from the argument in Wong [14]. As we briefly mentioned in the Introduction Weyl calculus is invertible, and allows (see de Gosson and Hiley [9]) to show that fundamentally Schrödinger’s equation is equivalent to Hamilton’s equations of motion. Such a situation is not physically very tenable (unless one introduces supplementary condition), because quantum and Hamiltonian mechanics are certainly not equivalent theories (at least physically)! It turns out that Born–Jordan quantization is not invertible, in the following sense: let $\widehat{A} : \mathcal{S}(\mathbb{R}^N) \longrightarrow \mathcal{S}'(\mathbb{R}^N)$ be an arbitrary continuous operator, and let $a_W \in \mathcal{S}'(\mathbb{R}^{2N})$ be its Weyl symbol: $\widehat{A} = \text{Op}(a_W)$. If there exists $a \in \mathcal{S}'(\mathbb{R}^{2N})$ such that $\widehat{A} = \text{Op}_{\text{BJ}}(a)$ then in view of formula (61) in Proposition 15 a_W and a are related by the convolution equation

$$a_W = \left(\frac{1}{2\pi\hbar}\right)^N a * \mathcal{F}_\sigma \Theta$$

that is, taking (symplectic) Fourier transforms

$$\mathcal{F}_\sigma a_W = (\mathcal{F}_\sigma a) \Theta$$

However, given an arbitrary a_W this relation does not determine $\mathcal{F}_\sigma a$, that is a . This fact, together with the properties of the distribution $Q(\psi, \phi)$ studied by Boggiatto et al. [1] suggest that Born–Jordan quantization could really make a case against more traditional quantization schemes. This possibility should certainly be studied seriously, and perhaps complemented using recent results in Boggiatto et al [2]) where the authors consider weighted averages of the quasi-distributions W_τ . The study of quantization, both from mathematical and physical perspectives, certainly still has a brilliant future!

Acknowledgement 19 *The authors would like to express their gratitude to Professor S. Kauffmann for several exchanges of ideas on dequantization and quantization, and for an interesting discussion of invertibility of quantization.*

References

- [1] P. Boggiatto, G. De Donno, A. Oliaro, Time-Frequency Representations of Wigner Type and Pseudo-Differential Operators, *Transactions of the Amer. Math. Soc.*, 362(9), 4955–4981 (2010)
- [2] P. Boggiatto, Bui Kien Cuong, G. De Donno, A. Oliaro, Weighted integrals of Wigner representations, *Journal of Pseudo-Differential Operators and Applications*, 2010
- [3] M. Born, P. Jordan, Zur Quantenmechanik, *Zeits. Physik* 34 858–888, 1925
- [4] L. Castellani, Quantization Rules and Dirac’s Correspondence, *Il Nuovo Cimento* 48A(3), 1978, 359–368
- [5] P. Crehan, The parametrisation of quantisation rules equivalent to operator orderings, and the effect of different rules on the physical spectrum, *J. Phys. A: Math. Gen.* 1989 811–822
- [6] W. A. Fedak, J. J. Prentis, The 1925 Born and Jordan paper “On quantum mechanics”, *Am. J. Phys.* 77(2) 2009 128–139
- [7] M. de Gosson, *Symplectic Geometry and Quantum Mechanics*, Birkhäuser, Basel, series “Operator Theory: Advances and Applications” (subseries: “Advances in Partial Differential Equations”), Vol. 166 (2006)
- [8] M. de Gosson, *Symplectic Methods in Harmonic Analysis; Applications to Mathematical Physics*, Birkhäuser 2010
- [9] M. de Gosson, B. Hiley, *Imprints of the Quantum World in Classical Mechanics*, Preprint 2010
- [10] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, (2000)
- [11] S. K. Kauffmann, Unambiguous quantization from the maximum classical correspondence that is self-consistent: the slightly stronger canonical commutation rule Dirac missed, arXiv:0908.3024 (2010)
- [12] M. A. Shubin, *Pseudodifferential Operators and Spectral Theory*, Springer-Verlag, (1987) [original Russian edition in Nauka, Moskva (1978)].

- [13] H. Weyl, Quantenmechanik und Gruppentheorie, Zeitschrift für Physik, 46 (1927)
- [14] M. W. Wong, Weyl Transforms, Springer, 1998.