

AN EXOTIC MINIMAL BANACH SPACE OF FUNCTIONS

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ABSTRACT. This note describes a new Banach space \mathcal{B}_0 of square integrable functions on \mathbb{R}^d having many interesting invariance properties. In fact, the Fourier transform, time-frequency shifts, and L^2 -normalized dilations act isometrically on it. For its definition, we make use of a general construction principle for minimal invariant spaces. We demonstrate a variety of properties following immediately from this principle. Furthermore, we give a number of different characterizations, including various atomic decompositions, as well as natural necessary and sufficient conditions for an L^2 -function to belong to this new space. A surprising result is the fact that despite the richness of atoms, the resulting space is a proper subspace of L^2 .

1. INTRODUCTION

Within a given class of Banach function spaces with certain invariance properties, it is of interest to ask for the smallest of its elements (if it exists), since it is the intersection of all the spaces. Often it is possible to describe this minimal space by an *atomic decomposition*, i.e., as the space of absolutely convergent series over a family of atoms. This method is described in detail in Section 2, where we also list several examples of such spaces, among them the real Hardy space $H^1(\mathbb{R})$. Furthermore, we show some general properties of spaces constructed in this manner.

The space \mathcal{B}_0 we want to describe is minimal among those Banach spaces of complex-valued functions on \mathbb{R} that are isometrically invariant under translation, modulation, and L^2 -normalized dilation. \mathcal{B}_0 has a number of properties, presented in Sections 4 and 5, which exhibit a somewhat exotic character of this space. In the proofs of these properties, we often make use of the space \mathcal{S}_0 which can be characterized as the minimal space isometrically invariant under translation and modulation, and thus is a (proper) subspace of \mathcal{B}_0 .

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Notation and conventions. In addition to standard convention, we will use the following notation throughout the paper.

For a Banach space \mathbf{B} , we denote by \mathbf{B}' its dual Banach space of bounded linear functionals, and by $\mathcal{L}(\mathbf{B})$ the space of bounded linear transformations on \mathbf{B} .

For a space $\mathbf{X}(G)$ of functions or distributions on a locally compact group G , we denote by $\mathbf{X}_c(G)$ the subspace of $\mathbf{X}(G)$ of elements with compact support. For $f \in \mathbf{X}(G)$, (*left translation*) by an element $x \in G$ is denoted T_x , i.e., $T_x f(\cdot) = f(x^{-1}\cdot)$. If G is Abelian, one writes $T_x f(\cdot) = f(\cdot - x)$. In the Abelian case, we also define *Modulation* M_ξ by an element ξ of the dual group \widehat{G} via $M_\xi f = \xi \cdot f$. Furthermore, on \mathbb{R}^d , *dilation* $D_\rho = D_\rho^{(1/2)}$ (with \mathbf{L}^2 -normalization) by a positive number $\rho \in \mathbb{R}^+$ is given by

$$D_\rho f(\cdot) = \rho^{d/2} f(\rho \cdot).$$

Occasionally we will choose different normalizations, then we write

$$D_\rho^{(\alpha)} f(\cdot) = \rho^{\alpha d} f(\rho \cdot).$$

So we obtain that $D_\rho^{(1/p)}$ is an isometry of $\mathbf{L}^p(\mathbb{R}^d)$.

For the *Fourier transformation* $\mathcal{F} : f \mapsto \widehat{f}$ on \mathbb{R}^d , we use the normalization

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx$$

which implies that the *Gauss function*

$$g_0(x) = e^{-\pi \|x\|^2} = e^{-\pi(x_1^2 + \dots + x_d^2)}$$

satisfies $\widehat{g}_0 = g_0$.

The image of \mathbf{L}^1 under the Fourier transformation is denoted \mathbf{A} , i.e., we have

$$\mathbf{A}(\widehat{G}) = \mathcal{F}(\mathbf{L}^1(G)) \quad \text{with} \quad \|\widehat{f}\|_{\mathbf{A}} = \|f\|_{\mathbf{L}^1}.$$

For a linear operator T on a space \mathbf{B} of functions or distributions on a locally compact Abelian group G , we define its Fourier transform to be the linear operator on $\widehat{\mathbf{B}}(\widehat{G}) := \widehat{\mathbf{B}(G)}$ given by

$$\widehat{T} = \mathcal{F} \circ T \circ \mathcal{F}^{-1} : \widehat{f} \mapsto \widehat{Tf}.$$

In this sense, one has $\widehat{T}_x = M_{-x}$, $\widehat{M}_y = T_y$, and $\widehat{D}_\rho^{(\alpha)} = D_{1/\rho}^{(1-\alpha)}$.

A *representation* π of a locally compact group G on a Banach space \mathbf{B} is a group homomorphism from G into $\mathcal{L}(\mathbf{B})$. We call π *bounded*, if $\sup_{g \in G} \|\pi(g)\|_{\mathcal{L}(\mathbf{B})} < \infty$. It

is called *strongly continuous*, if for each $x \in \mathbf{B}$, the mapping $G \rightarrow \mathbf{B}$, $g \mapsto \pi(g)x$, is continuous.

A *projective representation* of G on \mathbf{B} is a mapping π from G into $\mathcal{L}(\mathbf{B})$ with the properties that there is a measurable function $\sigma : G \times G \rightarrow \{z \in \mathbb{C} : |z| = 1\}$ satisfying

$$\pi(xy) = \sigma(x, y) \pi(x) \pi(y)$$

and $\pi(e) = \text{Id}$. From a projective representation π of G on \mathbf{B} , it is possible to construct a representation of the group $\tilde{G} = G \rtimes \mathbb{T}$, where $(x, z)(y, w) = (xy, \overline{\sigma(x, y)}zw)$, on \mathbf{B} via $\tilde{\pi}(x, z) = z\pi(x)$. As an example, consider a locally compact Abelian group H , let $\mathbf{B} = \mathbf{L}^2(H)$, and $G = H \times \hat{H}$. Then $\pi : G \rightarrow \mathcal{L}(\mathbf{B})$, $(x, \xi) \mapsto M_\xi T_x$, is a (unitary) projective representation with $\sigma((x, \xi), (y, \eta)) = \overline{\eta(x)}$ ($= e^{-2\pi i \eta \cdot x}$ for $H = \mathbb{R}^d$). For \tilde{G} , we obtain the Heisenberg group, and $\tilde{\pi}$ is the Schrödinger representation. For details, e.g., see [20] or [21].

We will need to consider another class of spaces of functions or distributions, namely, the Wiener amalgam spaces ([9], [12], [13]). First, we choose a suitable *bounded uniform partition of unity* (BUPU). This can, e.g., be a collection $(T_n \psi)_{n \in \mathbb{Z}^d}$ on \mathbb{R}^d , where $\psi \in \mathcal{D} = \mathcal{C}_c^\infty$ (the space of test functions) satisfies $\sum_{n \in \mathbb{Z}^d} T_n \psi \equiv 1$. Let $\mathbf{X}(\mathbb{R}^d)$ be a strongly translation invariant Banach space of functions or distributions on \mathbb{R}^d (i.e., we have that $\|T_x f\|_{\mathbf{X}} = \|f\|_{\mathbf{X}}$ for all $x \in \mathbb{R}^d$ and $f \in \mathbf{X}$) with the property that $\mathcal{D} \cdot \mathbf{X} \subseteq \mathbf{X}$. The *local version* $\mathbf{X}_{loc}(\mathbb{R}^d)$ of $\mathbf{X}(\mathbb{R}^d)$ is the space of all functions or distributions f on \mathbb{R}^d such that for any $\psi \in \mathcal{D}$, we have $(T_x \psi) f \in \mathbf{X}(\mathbb{R}^d)$ for all $x \in \mathbb{R}^d$. Furthermore, let $\mathbf{Y}(\mathbb{Z}^d)$ be a solid shift invariant Banach space of sequences on \mathbb{Z}^d (i.e., we have $T_n \mathbf{Y} = \mathbf{Y}$ for all $n \in \mathbb{Z}^d$, and given an $a \in \mathbf{Y}$, any sequence b on \mathbb{Z}^d with $|b[n]| \leq |a[n]|$ for all $n \in \mathbb{Z}^d$ satisfies $b \in \mathbf{Y}$ with $\|b\|_{\mathbf{Y}} \leq \|a\|_{\mathbf{Y}}$). The *Wiener amalgam space* with local component \mathbf{X} and global component \mathbf{Y} can be defined as

$$\mathbf{W}(\mathbf{X}, \mathbf{Y}) = \left\{ f \in \mathbf{X}_{loc}(\mathbb{R}^d) : \|f\|_{\mathbf{W}(\mathbf{X}, \mathbf{Y})} = \left\| (\|f T_n \psi\|_{\mathbf{X}})_{n \in \mathbb{Z}^d} \right\|_{\mathbf{Y}} < \infty \right\}.$$

The condition $\psi \in \mathcal{D}$ can be weakened depending on \mathbf{X} . Under the assumptions on \mathbf{X} made above, it is sufficient to have $\psi \in \mathcal{L}_c^\infty(\mathbb{R}^d)$ with $\psi \cdot \mathbf{X} \subseteq \mathbf{X}$. Under these conditions, different choices of ψ will generate the same space and equivalent norms.

2. ATOMIC DECOMPOSITIONS AND MINIMAL ATOMIC SPACES

The space we want to describe belongs to a class of spaces obtained by a general construction principle which has been used before (see Section 3). Several interesting properties hold for all such spaces. Therefore we will prove them here for the general case.

Theorem 2.1. *Let \mathbf{B} be a Banach space, and $\Phi = (\varphi_j)_{j \in J}$ a (not necessarily countable) bounded family in \mathbf{B} , i.e., $\|\varphi_j\|_{\mathbf{B}} \leq C$ for all $j \in J$. Define*

$$\mathbf{B} = \mathbf{B}_{\Phi} = \left\{ f = \sum_{j \in J} a_j \varphi_j : \sum_{j \in J} |a_j| < \infty \right\},$$

and let the norm be the infimum of the costs of all admissible representations, i.e.,

$$\|f\|_{\mathbf{B}} = \inf \left\{ \sum_{j \in J} |a_j| : f = \sum_{j \in J} a_j \varphi_j \right\}.$$

Then \mathbf{B} is a Banach space, continuously embedded in \mathbf{B} .

We refer to the φ_j as the atoms generating \mathbf{B} .

Proof. Note that $\ell^1(J)$ is a Banach space, and consider the map

$$S : \ell^1(J) \rightarrow \mathbf{B}, \quad a = (a_j)_{j \in J} \mapsto \sum_{j \in J} a_j \varphi_j.$$

S obviously satisfies $\|S(a)\|_{\mathbf{B}} \leq \sum_j |a_j| \|\varphi_j\|_{\mathbf{B}} \leq C \|a\|_{\ell^1(J)}$, so it is a bounded linear map with closed kernel. By definition, \mathbf{B} is isometric to the quotient space $\ell^1(J)/\ker S$, which implies the claim. \square

Lemma 2.2. *Consider a fixed Banach space \mathbf{B} and two families of atoms Φ_1 and Φ_2 . If $\Phi_1 \subseteq \Phi_2$, then \mathbf{B}_{Φ_1} is a continuously embedded subspace of \mathbf{B}_{Φ_2} .*

Proof. We trivially have $\mathbf{B}_{\Phi_1} \subseteq \mathbf{B}_{\Phi_2}$. Furthermore, for $f \in \mathbf{B}_{\Phi_1}$, the atoms in Φ_2 allow at least the representations of f from the atoms in Φ_1 , we see that $\|f\|_{\mathbf{B}_{\Phi_1}} \geq \|f\|_{\mathbf{B}_{\Phi_2}}$. \square

A natural question to ask is under what conditions we obtain $\mathbf{B} = \mathbf{B}$, with equivalent norms. This is trivially obtained by choosing $\Phi = \{\varphi \in \mathbf{B} : \|\varphi\|_{\mathbf{B}} = 1\}$, since then $f = \|f\|_{\mathbf{B}} \frac{f}{\|f\|_{\mathbf{B}}}$ with $\|f\|_{\mathbf{B}} = \|f\|_{\mathbf{B}}$ for all $f \in \mathbf{B}$. In [2], F.F. Bonsall has presented a much weaker sufficient criterion.

Proposition 2.3. ([2], Thm. 1) *If under the assumptions of Theorem 2.1, there is a constant C_1 such that*

$$\sup_{j \in J} |\psi(\varphi_j)| \geq C_1 \|\psi\|_{\mathbf{B}'} \quad \forall \psi \in \mathbf{B}',$$

then $\mathcal{B} = \mathbf{B}$ with norm equivalence, i.e.,

$$C_1 \|f\|_{\mathcal{B}} \leq \|f\|_{\mathbf{B}} \leq C_2 \|f\|_{\mathcal{B}} \quad \forall f \in \mathbf{B}.$$

Atomic spaces from bounded representations. When studying a class of function spaces invariant under a group of operators, we can determine the smallest such space by its atomic decomposition as follows. Consider a bounded (projective) representation π of a group G on a Banach space \mathbf{B} . Choosing $\varphi \in \mathbf{B}$ (or, if so desired, several φ_j) as *basic atom*, we can use the coherent family $\Phi = (\pi(g)\varphi)_{g \in G}$ as atoms and obtain the Banach space

$$\mathcal{B}_\varphi := \mathcal{B}_{\pi(G)\varphi} = \left\{ f = \sum_{g \in G} a[g] \pi(g)\varphi : a \in \ell^1(G) \right\}$$

with the norm defined as above. It is immediate that \mathcal{B}_φ is the *minimal $\pi(G)$ -invariant Banach space* containing φ . We shall see first how convolution in $\ell^1(G)$ comes into play (although with a “twist”, if π is only a projective representation).

Definition. For \mathbf{B} , G , and π as above, define the (*semidiscrete*) *convolution product* of $a \in \ell^1(G)$ with $\varphi \in \mathbf{B}$ induced by π as

$$a *'_\pi \varphi = \sum_{g \in G} a[g] \pi(g)\varphi.$$

For $a, b \in \ell^1(G)$, define the *twisted convolution product*

$$(a *_\sigma b)[g] = \sum_{h \in G} a[h] b[h^{-1}g] \overline{\sigma(h, h^{-1}g)} = \sum_{hk=g} a[h] \overline{\sigma(h, k)} b[k].$$

If π is a representation, i.e., $\sigma \equiv 1$, we have $a *_\sigma b = a * b$ (standard convolution).

Using this notation, we obtain the convenient description

$$\mathcal{B}_{\pi(G)\varphi} = \ell^1(G) *'_\pi \varphi.$$

Furthermore, it can be seen why it is often reasonable to consider projective representations, since for the representation $\tilde{\pi}$ of $\tilde{G} = G \rtimes \mathbb{T}$, we obviously have

$$\ell^1(\tilde{G}) *'_\pi \varphi = \ell^1(G) *'_\pi \varphi,$$

and using the projective representation, we have a smaller group to deal with.

Lemma 2.4. *For \mathbf{B} , G , and π as above, the following hold.*

(i) *For $a, b \in \ell^1(G)$, we have $a *_{\sigma} b \in \ell^1(G)$ with*

$$\|a *_{\sigma} b\|_{\ell^1(G)} \leq \|a\|_{\ell^1(G)} \|b\|_{\ell^1(G)}.$$

(ii) *Semidiscrete convolution and twisted convolution are related via*

$$a *'_{\pi} (b *'_{\pi} \varphi) = (a *_{\sigma} b) *'_{\pi} \varphi.$$

Proof. (i) Follows immediately from $|\sigma| = 1$.

(ii) Since the change in the order of summation below is well justified, we have

$$\begin{aligned} a *'_{\pi} (b *'_{\pi} \varphi) &= \sum_{g \in G} a[g] \pi(g) \sum_{h \in G} b[h] \pi(h) \varphi = \sum_{g \in G} a[g] \sum_{h \in G} b[h] \overline{\sigma(g, h)} \pi(gh) \varphi \\ &= \sum_{g \in G} a[g] \sum_{k \in G} b[g^{-1}k] \overline{\sigma(g, g^{-1}k)} \pi(k) \varphi \\ &= \sum_{k \in G} \left(\sum_{g \in G} a[g] b[g^{-1}k] \overline{\sigma(g, g^{-1}k)} \right) \pi(k) \varphi = (a *_{\sigma} b) *'_{\pi} \varphi. \end{aligned}$$

□

Theorem 2.5. *Assume that \mathbf{B} , G , and π are as above, let $\varphi, \psi \in \mathbf{B}$, and define $\mathbf{B}_{\varphi}, \mathbf{B}_{\psi}$ accordingly. Then we have the following.*

(i) $\{\pi(g) : g \in G\}$ *is a family of isometries of \mathbf{B}_{φ} .*

(ii) *If $\varphi \in \mathbf{B}_{\psi}$, then \mathbf{B}_{φ} is continuously embedded in \mathbf{B}_{ψ} .*

(iii) *If $\varphi \in \mathbf{B}_{\psi}$ and $\psi \in \mathbf{B}_{\varphi}$, then $\mathbf{B}_{\varphi} = \mathbf{B}_{\psi}$ with equivalent norms.*

Proof. (i) For $g \in G$, we have that T_g is an isometry of $\ell^1(G)$ satisfying

$$\pi(g)(b *'_{\pi} \varphi) = (T_g(b[\cdot] \overline{\sigma(g, \cdot)})) *'_{\pi} \varphi.$$

(This is a special case of Lemma 2.4, if we use $a = \delta_g$.) Thus we have for arbitrary $f = a *'_{\pi} \varphi \in \mathbf{B}_{\varphi}$ that

$$\begin{aligned} \|\pi(g) f\|_{\mathbf{B}_{\varphi}} &= \inf \{ \|b\|_{\ell^1(G)} : b *'_{\pi} \varphi = \pi(g) (a *'_{\pi} \varphi) \} \\ &= \inf \{ \|c\|_{\ell^1(G)} : (T_g(c \overline{\sigma(g, \cdot)})) *'_{\pi} \varphi = \pi(g) (a *'_{\pi} \varphi) \} \\ &= \inf \{ \|c\|_{\ell^1(G)} : \pi(g) (c *'_{\pi} \varphi) = \pi(g) (a *'_{\pi} \varphi) \} \\ &= \inf \{ \|c\|_{\ell^1(G)} : c *'_{\pi} \varphi = f \} \\ &= \|f\|_{\mathbf{B}_{\varphi}}. \end{aligned}$$

(ii) $\varphi \in \mathcal{B}_\psi$ means that $\varphi = b *'_\pi \psi$ for some $b \in \mathcal{L}^1(G)$, and thus we have by Lemma 2.4 for arbitrary $f \in \mathcal{B}_\varphi$ that

$$f = a *'_\pi \varphi = a *'_\pi (b *'_\pi \psi) = (a *_\sigma b) *'_\pi \psi \in \mathcal{B}_\psi.$$

The continuity of the embedding follows from

$$\begin{aligned} \|f\|_{\mathcal{B}_\varphi} &= \inf \{ \|a\|_{\mathcal{L}^1(G)} : a *'_\pi \varphi = f \} \\ &= \inf \{ \|a\|_{\mathcal{L}^1(G)} : (a *_\sigma b) *'_\pi \psi = f \} \\ &\geq \inf \{ \|a *_\sigma b\|_{\mathcal{L}^1(G)} / \|b\|_{\mathcal{L}^1(G)} : (a *_\sigma b) *'_\pi \psi = f \} \\ &\geq \|b\|_{\mathcal{L}^1(G)}^{-1} \inf \{ \|c\|_{\mathcal{L}^1(G)} : c *'_\pi \psi = f \} \\ &= \|b\|_{\mathcal{L}^1(G)}^{-1} \|f\|_{\mathcal{B}_\psi}. \end{aligned}$$

Since this holds for any $b \in \mathcal{L}^1(G)$ satisfying $b *'_\pi \psi = \varphi$, we have that

$$\|f\|_{\mathcal{B}_\psi} \leq \|f\|_{\mathcal{B}_\varphi} \|\varphi\|_{\mathcal{B}_\psi}.$$

(iii) Clear by (ii). □

The above results raise the question under which conditions we have $\mathcal{B}_\varphi = \mathcal{B}_\psi$ for all $\psi \in \mathcal{B}_\varphi$. Theorem 3.4 below shows a particular case where this is true.

Next, we state conditions ensuring \mathcal{B}_φ to be invariant under the Fourier transformation. We will make use of this result in Theorems 3.1 and 4.1.

Theorem 2.6. *Let \mathbf{B} be a Banach space of functions or distributions on a self-dual locally compact Abelian group. Assume that $\widehat{\mathbf{B}} = \mathbf{B}$, and that $\widehat{\pi(G)} \cong \pi(G)$ as families of operators. (I.e., $T \in \pi(G)$ if and only if $\alpha \widehat{T} \in \pi(G)$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$.) Then $\widehat{\mathcal{B}_\varphi} = \mathcal{B}_{\widehat{\varphi}}$. Consequently, if in addition $\widehat{\varphi} = \varphi$ or at least $\varphi \in \mathcal{B}_{\widehat{\varphi}}$ and $\widehat{\varphi} \in \mathcal{B}_\varphi$, then $\widehat{\mathcal{B}_\varphi} = \mathcal{B}_\varphi$.*

Proof. By the assumptions on \mathbf{B} and $\pi(G)$, we can rewrite the Fourier transform of an element of \mathcal{B}_φ as $\mathcal{F}(\sum_{g \in G} a[g] \pi(g) \varphi) = \sum_{g \in G} a[g] \widehat{\pi(g)} \widehat{\varphi} = \sum_{g \in G} b[g] \alpha_g \pi(g) \widehat{\varphi}$, where b is obtained from a by an appropriate permutation. Since this works both ways, we have $\widehat{\mathcal{B}_\varphi} = \mathcal{B}_{\widehat{\varphi}}$. The second claim follows by Theorem 2.5.(iii). □

Examples. As a first example, consider $\mathbf{B} = \mathbf{L}^2(\mathbb{R}^d)$ and $\varphi = g_0$, the Gauss function. Letting $G = \mathbb{R}^d \times \widehat{\mathbb{R}^d}$ with $\pi(x, \xi) = M_\xi T_x$, we obtain for \mathcal{B}_φ the Segal algebra $\mathcal{S}_0(\mathbb{R}^d)$ which we want to describe more closely in the next section, since we can transfer several

ideas and methods from this example to the space we want to examine. Note that we have $\widehat{\mathbf{B}} = \mathbf{B}$, $\widehat{\varphi} = \varphi$, and $\widehat{\pi(G)} \cong \pi(G)$, so we can conclude that $\mathbf{S}_0(\mathbb{R}^d)$ is invariant under the Fourier transformation.

Several other examples for spaces constructed by the above methods can be found in the literature. In [23], vol. I, Y. Meyer uses $\mathbf{B} = \mathbf{C}_0(\mathbb{R}^d)$, $G = \mathbb{R}^d \times \mathbb{R}^+$ with $\pi(x, \rho) = D_\rho^{(0)} T_x$, i.e., \mathbf{L}^∞ -normalization), and $\varphi = g_0$ (the Gauss-function). This yields a space the author denotes “l’algèbre des bosses” (“hump-algebra”). It allows for an easy description in terms of wavelet bases. In the same book, the author shows that $\mathbf{B} = \mathbf{L}^1(\mathbb{R}^d)$, $G = \mathbb{R}^d \times \mathbb{R}^+$ with $\pi(x, \rho) = D_\rho^{(1)} T_x$, and $\varphi = g_0$ yields $\mathbf{B}_\varphi = \mathbf{L}^1(\mathbb{R}^d)$ with equivalent norms. This is an example where an L^p -space ($p = 1$) has an atomic decomposition using a two-parameter group only. Furthermore, it is worth noting that this immediately implies (by an argument similar to the proof of Theorem 2.6) for the choice $\mathbf{B} = \mathbf{A}(\mathbb{R}^d)$, $G = \widehat{\mathbb{R}}^d \times \mathbb{R}^+$ with $\pi(\xi, \rho) = D_\rho^{(0)} M_\xi$, and $\varphi = g_0$ that $\mathbf{B}_\varphi = \mathbf{A}(\mathbb{R}^d)$ with equivalent norms.

Also in [23], vol. I, it is shown that $\mathbf{B} = \mathbf{L}^1(\mathbb{R})$, $G = \mathbb{R}^d \times \mathbb{R}^+$ with $\pi(x, \rho) = D_\rho^{(1)} T_x$, and $\varphi = \chi_{[0,1/2]} - \chi_{[1/2,1]}$ (the so-called Haar-wavelet) yields for \mathbf{B}_φ the homogeneous Besov space $\dot{\mathbf{B}}_{1,1}^0$.

One of the first spaces to become famous for its ℓ^1 -atomic representation is the real Hardy space $\mathbf{H}^1(\mathbb{R})$. It can be characterized by choosing $\mathbf{B} = \mathbf{L}^1(\mathbb{R})$, $G = \mathbb{R} \times \mathbb{R}^+$ with $\pi(x, \rho) = D_\rho^{(1)} T_x$, and as set of basic atoms $\Phi = \{f \in \mathbf{L}^2(\mathbb{R}) : \text{supp}(f) \subseteq [-\frac{1}{2}, +\frac{1}{2}], \|f\|_{\mathbf{L}^2} = 1, \int_{\mathbb{R}} f(x) dx = 0\} \subseteq \mathbf{L}^1(\mathbb{R})$ (see [5]).

The spaces $\mathbf{E}^q(\mathbb{R}^d)$ ($1 \leq q < \infty$) described in [10] are constructed from $\Phi = \{D_\rho^{(1)} f : f \in \mathbf{L}^q(\mathbb{R}^d), \|f\|_{\mathbf{L}^q} \leq 1, \text{supp}(f) \subseteq [-1, 1]^d\}$. They are dilation invariant Banach algebras under convolution, and have been useful in proving Tauberian theorems for functions of bounded \mathbf{L}^p -means on \mathbb{R}^d .

Möbius invariant spaces of functions on the unit disk and related spaces have been studied systematically by J. Peetre and others (see [1], [25]).

This list is by no means exhaustive, but only intended to give the reader an idea of the variety of such spaces.

3. THE SEGAL ALGEBRA $\mathbf{S}_0(\mathbb{R}^d)$

Definition. The space $\mathbf{S}_0(\mathbb{R}^d)$ is obtained by letting $\mathbf{B} = \mathbf{L}^2(\mathbb{R}^d)$, and choosing $\varphi = g_0$ and $G = \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ with the projective Schrödinger representation. This yields

$$\mathbf{S}_0(\mathbb{R}^d) = \left\{ \sum_n a_n M_{\xi_n} T_{x_n} g_0 : \sum_n |a_n| < \infty, (x_n, \xi_n) \in \mathbb{R} \times \widehat{\mathbb{R}}^d \right\}.$$

This space has been studied extensively in the literature, e.g., see [7], [8], [14], [15], [16]; furthermore in [17], where it appears as the modulation space M^1 . We only state those properties that we shall refer to later on. By the results from Section 2, the following properties are immediate from the definition of \mathbf{S}_0 .

Theorem 3.1. (i) \mathbf{S}_0 is a Banach space, continuously embedded in $\mathbf{L}^2(\mathbb{R}^d)$.

(ii) $\mathbf{S}_0(\mathbb{R}^d)$ is isometrically invariant under translation and modulation.

(iii) $\mathbf{S}_0(\mathbb{R}^d)$ is isometrically invariant under the Fourier transformation.

Originally, the space \mathbf{S}_0 was defined differently, namely, as a Wiener amalgam space (see [8]). The following result states that the definitions agree, and from this characterization of $\mathbf{S}_0(\mathbb{R}^d)$, various properties can be derived.

Theorem 3.2. ([11]) $\mathbf{S}_0(\mathbb{R}^d)$ coincides with the Wiener amalgam space $\mathbf{W}(\mathbf{A}, \ell^1)$, i.e., the spaces have the same elements and equivalent norms.

Corollary 3.3. (i) $\mathbf{A}_c = \mathbf{S}_{0,c}$ is a dense subspace of $\mathbf{S}_0(\mathbb{R}^d)$.

(ii) \mathbf{S}_0 is a dense subspace of $\mathbf{L}^2(\mathbb{R}^d)$.

(iii) \mathbf{S}_0 is (non-isometrically) invariant under dilation. Dilation operators within a compact parameter range are uniformly bounded (in the operator norm on \mathbf{S}_0).

(iv) Translation, modulation, and dilation act continuously on the elements of $\mathbf{S}_0(\mathbb{R}^d)$, i.e.,

$$\|D_\rho M_\xi T_x f - f\|_{\mathbf{S}_0} \rightarrow 0 \quad ((x, \xi, \rho) \rightarrow (0, 0, 1))$$

for all $f \in \mathbf{S}_0(\mathbb{R}^d)$.

Theorem 3.4. ([11]) In the construction of $\mathbf{S}_0(\mathbb{R}^d)$, any non-zero element of \mathbf{S}_0 can be used as basic atom, i.e.,

$$\mathbf{S}_0(\mathbb{R}^d) = \left\{ \sum_n a_n M_{\xi_n} T_{x_n} \varphi : \sum_n |a_n| < \infty, (x_n, \xi_n) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d \right\}$$

for any $\varphi \in \mathbf{S}_0(\mathbb{R}^d) \setminus \{0\}$, and the norms induced by different choices for φ are equivalent.

4. THE SPACE \mathcal{B}_0

The space we want to describe is obtained by using $\mathbf{B} = \mathbf{L}^2(\mathbb{R}^d)$; the affine Heisenberg group $G = (\mathbb{R}^d \times \widehat{\mathbb{R}}^d) \rtimes \mathbb{R}^+$, where $(x, \xi, \rho)(y, \eta, \lambda) = (\lambda x + y, \lambda^{-1} \xi + \eta, \rho \lambda)$; the projective representation $\pi(x, \xi, \rho) = D_\rho M_\xi T_x$ with $\sigma((x, \xi, \rho)(y, \eta, \lambda)) = e^{-2\pi i \lambda \eta x}$; and the Gauss function g_0 as basic atom. This setup yields the following.

Definition. For $\varphi \in \mathbf{L}^2(\mathbb{R}^d)$, we define

$$\mathcal{B}_\varphi(\mathbb{R}^d) = \left\{ f = \sum_{n \in \mathbb{N}} a_n D_{\rho_n} M_{\xi_n} T_{x_n} \varphi : \sum_{n \in \mathbb{N}} |a_n| < \infty, (x_n, \xi_n, \rho_n) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d \times \mathbb{R}^+ \right\}$$

with norm

$$\|f\|_{\mathcal{B}_\varphi(\mathbb{R}^d)} = \inf \left\{ \sum_{n \in \mathbb{N}} |a_n| : f = \sum_{n \in \mathbb{N}} a_n D_{\rho_n} M_{\xi_n} T_{x_n} \varphi \right\}.$$

For the special case where $\varphi = g_0$, the Gauss function, we write $\mathcal{B}_0(\mathbb{R}^d) := \mathcal{B}_{g_0}(\mathbb{R}^d)$.

By the results from Section 2, the space $\mathcal{B}_\varphi(\mathbb{R}^d)$ has the following properties.

Theorem 4.1. (i) $\mathcal{B}_\varphi(\mathbb{R}^d)$ is a Banach space, continuously embedded in $\mathbf{L}^2(\mathbb{R}^d)$.

(ii) $D_\rho M_\xi T_x$ is an isometry of $\mathcal{B}_\varphi(\mathbb{R}^d)$ for all $(x, \xi, \rho) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d \times \mathbb{R}^+$.

(iii) $\mathcal{B}_0(\mathbb{R}^d)$ is invariant under the Fourier transformation \mathcal{F} . Using the norm induced by $g_0 = \widehat{g}_0$, we even have that \mathcal{F} is an isometry on \mathcal{B}_0 .

4.1. Equivalent characterizations. In this section, we present a number of different, but equivalent atomic characterizations of $\mathcal{B}_0(\mathbb{R}^d)$.

Theorem 4.2. (i) $\mathcal{S}_0(\mathbb{R}^d)$ is continuously embedded in $\mathcal{B}_0(\mathbb{R}^d)$.

(ii) For $\varphi \in \mathcal{S}_0(\mathbb{R}^d) \setminus \{0\}$, we have $\mathcal{B}_\varphi = \mathcal{B}_0$.

Proof. (i) $\mathcal{S}_0(\mathbb{R}^d)$ is generated by the family of atoms $\{M_\xi T_x g_0 : \xi \in \widehat{\mathbb{R}}^d, x \in \mathbb{R}^d\}$, which is a subset of the family of atoms generating $\mathcal{B}_0(\mathbb{R}^d)$, so Lemma 2.2 applies.

(ii) By (i), $\varphi \in \mathcal{S}_0(\mathbb{R}^d)$ implies $\varphi \in \mathcal{B}_0(\mathbb{R}^d)$ and therefore, by Theorem 2.5.(ii), $\mathcal{B}_\varphi \subseteq \mathcal{B}_0$. Conversely, we know by Theorem 3.4 that $\{M_\xi T_x \varphi\}$ generates $\mathcal{S}_0(\mathbb{R}^d)$ also, and thus we have $g_0 \in \mathcal{S}_0(\mathbb{R}^d) \subseteq \mathcal{B}_\varphi(\mathbb{R}^d)$ which, again by Theorem 2.5.(ii), implies $\mathcal{B}_0 = \mathcal{B}_{g_0} \subseteq \mathcal{B}_\varphi$. \square

Remark. (i) It is not clear yet whether we actually do have $\mathcal{B}_\varphi = \mathcal{B}_0$ for all $\varphi \in \mathcal{B}_0 \setminus \{0\}$.

(ii) In the atomic representation of $\mathcal{S}_0(\mathbb{R}^d)$, it suffices to consider a lattice $a\mathbb{Z}^d \times b\mathbb{Z}^d \triangleleft \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, where a and b have to be chosen in accordance with the choice of the basic atom φ ; for details, e.g., see [16]. Furthermore, on $\mathcal{S}_0(\mathbb{R}^d)$, we do have uniform boundedness of the dilation operators on compact ranges of the scaling factor. Therefore, in analogy

to the \mathbf{S}_0 -case, it suffices to consider $H = \mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{Z}$ with $\pi(k, m, n) = D_{\rho^n} M_{ma} T_{kb}$ for appropriate $\varphi \in \mathbf{S}_0(\mathbb{R}^d) \setminus \{0\}$, matching $a, b > 0$ (ensuring that $\{M_{ma} T_{kb} \varphi\}$ generates \mathbf{S}_0), and arbitrary $\rho > 1$ to obtain $\mathbf{B}_{\pi(H)\varphi} = \mathbf{B}_0$. (Note, though, that H is just a discrete subset, but not a subgroup of the affine Heisenberg group).

The next result shows that in contrast to $\mathbf{S}_0(\mathbb{R}^d)$, the new space $\mathbf{B}_0(\mathbb{R}^d)$ can be generated from discontinuous functions. In particular, we see that $\mathbf{S}_0 \subsetneq \mathbf{B}_0$, since $\mathbf{S}_0 \subseteq \mathbf{C}(\mathbb{R}^d)$.

Theorem 4.3. *For $\varphi = \chi_{[0,1]}$ (and thus for the indicator function of any compact interval), we have that $\mathbf{B}_\varphi = \mathbf{B}_0$.*

Proof. By Theorem 2.5.(iii), it suffices to show that $\varphi \in \mathbf{B}_\psi$ and $\psi \in \mathbf{B}_\varphi$ for some $\psi \in \mathbf{B}_0(\mathbb{R})$ with $\mathbf{B}_\psi = \mathbf{B}_0$. By Theorem 4.2, we may choose the triangular function $\psi = \max\{0, 1 - |x|\}$, since it satisfies $\psi \in \mathbf{A}_c(\mathbb{R}) \subseteq \mathbf{S}_0(\mathbb{R})$.

Now φ and ψ satisfy

$$\varphi = 2^{-1/2} T_{1/2} D_2 \psi + \sum_{j=1}^{\infty} 2^{-(j+3)/2} T_{2^{-j-1}} D_{2^{j+1}} \psi + \sum_{j=1}^{\infty} 2^{-(j+3)/2} T_{1-2^{-j-1}} D_{2^{j+1}} \psi,$$

so $\|\varphi\|_{\mathbf{B}_\psi} \leq 2^{-1/2} + 2 \sum_{j=1}^{\infty} 2^{-(j+3)/2} = 2^{-1/2} \sum_{j=0}^{\infty} 2^{-j/2} < \infty$, and thus $\varphi \in \mathbf{B}_\psi$. Conversely,

$$\psi = \sum_{j=1}^{\infty} \sum_{k=1}^{2^{j-1}} \left(2^{-3j/2} T_{(2k-2)2^{-j}} D_{2^j} \varphi + 2^{-3j/2} T_{(1-2k)2^{-j}} D_{2^j} \varphi \right),$$

hence $\|\psi\|_{\mathbf{B}_\varphi} \leq \sum_{j=1}^{\infty} 2^{j-1} 2 \cdot 2^{-3j/2} = \sum_{j=1}^{\infty} 2^{-j/2} < \infty$, and thus $\psi \in \mathbf{B}_\varphi$ also. \square

Note that

$$M_{\xi_n} T_{x_n} D_{\rho_n} \varphi(x) = (b_n - a_n)^{-1/2} e^{2\pi i \xi_n x} \chi_{[a_n, b_n]}(x),$$

where $[a_n, b_n] = [x_n, x_n + \frac{1}{\rho_n}]$. Since $\|c_n e^{2\pi i \xi_n x} \chi_{[a_n, b_n]}\|_{L^2(\mathbb{R})} = |c_n| \sqrt{b_n - a_n}$, we obtain the following.

Corollary 4.4.

$$\mathbf{B}_0(\mathbb{R}) = \left\{ f = \sum_n c_n e^{2\pi i \xi_n x} \chi_{[a_n, b_n]} : \sum_n |c_n| \sqrt{b_n - a_n} < \infty \right\}$$

$$\text{and } \|f\|_{\mathbf{B}_0(\mathbb{R})} = \inf \left\{ \sum_n |c_n| \sqrt{b_n - a_n} : f = \sum_n c_n e^{2\pi i \xi_n x} \chi_{[a_n, b_n]} \right\}.$$

We observe the following dense inclusions in connection with \mathbf{B}_0 .

Corollary 4.5. (i) $\mathbf{S}_0(\mathbb{R}^d)$ is a dense subspace of $\mathbf{B}_0(\mathbb{R}^d)$.

(ii) $\mathbf{B}_{0,c}(\mathbb{R}^d)$ is a dense subspace of $\mathbf{B}_0(\mathbb{R}^d)$.

(iii) $\mathbf{B}_0(\mathbb{R}^d)$ is a dense subspace of $L^2(\mathbb{R}^d)$.

Proof. (i) Clear by Corollary 3.3.(iii) and Theorem 4.2.(ii).

(ii) Follows from Corollary 3.3.(i).

(iii) Follows from Corollary 3.3.(ii). □

Theorem 4.6. *The action of $G = (\mathbb{R}^d \times \widehat{\mathbb{R}}^d) \rtimes \mathbb{R}^+$ on the elements of $\mathcal{B}_0(\mathbb{R}^d)$ is strongly continuous, i.e., $D_\rho M_\xi T_x f \rightarrow f$ in $\mathcal{B}_0(\mathbb{R}^d)$ as $(x, \xi, \rho) \rightarrow (0, 0, 1)$ for all $f \in \mathcal{B}_0(\mathbb{R}^d)$.*

Proof. This is a straightforward consequence of the fact that G acts continuously on the elements of $\mathcal{S}_0(\mathbb{R}^d)$ (see Corollary 3.3.(iv)) and the density of this space in $\mathcal{B}_0(\mathbb{R}^d)$ according to Corollary 4.5.(i). □

Corollary 4.7. (i) $\mathcal{B}_0(\mathbb{R}^d)$ is a Banach module over $L^1(\mathbb{R}^d)$ with respect to convolution, i.e., $L^1 * \mathcal{B}_0 \subseteq \mathcal{B}_0$ with $\|g * f\|_{\mathcal{B}_0} \leq \|g\|_{L^1} \|f\|_{\mathcal{B}_0}$ for all $g \in L^1$, $f \in \mathcal{B}_0$.
(ii) $\mathcal{B}_0(\mathbb{R}^d)$ is a Banach module over the Fourier algebra $\mathbf{A}(\mathbb{R}^d) = \mathcal{F}(L^1(\widehat{\mathbb{R}}^d))$ with respect to pointwise multiplication, i.e., $\mathbf{A} \cdot \mathcal{B}_0 \subseteq \mathcal{B}_0$ with $\|g \cdot f\|_{\mathcal{B}_0} \leq \|g\|_{\mathbf{A}} \|f\|_{\mathcal{B}_0}$ for all $g \in \mathbf{A}$, $f \in \mathcal{B}_0$.

Remark. The two parts of the above corollary follow from the strong continuity of the actions of the group of translations and the group of modulations, respectively. A corresponding statement could be made with respect to the group of dilations.

4.2. \mathcal{B}_0 is a proper subspace of L^2 . In the following two sections, we exhibit some of the properties of \mathcal{B}_0 that are the reason for calling it an “exotic” space of functions. First, we show that it is properly contained in L^2 . We find this somewhat surprising: As already remarked before Proposition 2.3, using the unit sphere of any Banach space as family of atoms yields the whole Banach space. For $\mathcal{B} = L^1(\mathbb{R}^d)$, it suffices already to take $\pi(x, \rho) = D_\rho^{(1)} T_x$ and $\varphi = \chi_{[0,1]}$ in order to obtain $\mathcal{B}_\varphi = L^1$. So in our setup, one might expect to obtain all of L^2 due to the richness of atoms — but this is not the case, as we will see in a moment.

In Section 5, we examine the local and the global behavior of the elements of \mathcal{B}_0 , and we compare it to other L^p -spaces and to certain Wiener amalgam spaces.

Proposition 4.8. *Use $\varphi = \chi_{[0,1]}$, and let $f \in \mathcal{B}_{0,c}(\mathbb{R})$, say, $\text{supp } f \subseteq [a, b]$. Then it suffices to consider representations of f supported in $[a, b]$ to determine $\|f\|_{\mathcal{B}_0(\mathbb{R})}$ (and thus also to decide whether $f \in \mathcal{B}_0(\mathbb{R})$ or not).*

Proof. Since we use $\varphi = \chi_{[0,1]}$, we make use of the \mathcal{B}_0 -norm given in Corollary 4.4. It suffices to show that for an arbitrary representation of f , there is another one supported on $[a, b]$ such that the absolute coefficient sum of the latter representation is no larger than that of the former. But replacing each interval $[a_n, b_n]$ by the interval $[a_n, b_n] \cap [a, b]$ (or simply omitting the term if this intersection does not contain a proper interval) will yield just that. \square

Corollary 4.9. *Using the norm induced by $\varphi = \chi_{[0,1]}$, we have $\|f \cdot \chi_{[a,b]}\|_{\mathcal{B}_0} \leq \|f\|_{\mathcal{B}_0}$ for all $f \in \mathcal{B}_0(\mathbb{R})$, i.e., the functions $\chi_{[a,b]}$ are a bounded family of pointwise multipliers on $\mathcal{B}_0(\mathbb{R})$.*

Now we can construct an element of $L^2 \setminus \mathcal{B}_0$.

Theorem 4.10. *The following function is an element of $L^2 \setminus \mathcal{B}_0(\mathbb{R})$:*

$$f = \sum_{k=1}^{\infty} \frac{1}{k} \chi_{[4k^2, 4k^2+1]}.$$

Proof. Assume that $f \in \mathcal{B}_0(\mathbb{R})$. First, we want to show that given a representation of f in terms of the atoms generated by $\chi_{[0,1]}$, we can obtain a “cheaper” one by restricting it to $\bigcup_{k=1}^{\infty} [4k^2, 4k^2+1]$. I.e., we claim that

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \|c_n e^{2\pi i \xi_n x} \chi_{[a_n, b_n] \cap [4k^2, 4k^2+1]}\|_{L^2(\mathbb{R})} \leq \sum_{n=1}^{\infty} \|c_n e^{2\pi i \xi_n x} \chi_{[a_n, b_n]}\|_{L^2(\mathbb{R})}.$$

It suffices to show that for $[a, b] \subset \mathbb{R}$, we have

$$\sum_{k=1}^{\infty} \|\chi_{[a, b] \cap [4k^2, 4k^2+1]}\|_{L^2(\mathbb{R})} \leq \|\chi_{[a, b]}\|_{L^2(\mathbb{R})} = \sqrt{b-a}.$$

Note that $4(k+1)^2 - (4k^2+1) = 8k+3 \geq 11$, so if $b-a \leq 11$, then $[a, b] \cap [4k^2, 4k^2+1] \neq \emptyset$ for at most one k , and the claim is obvious. Furthermore, if $b \leq 4$, then the left hand side is zero, and there is nothing to show. If $a < 4 < b$, then $[a, b] \cap [4k^2, 4k^2+1] = [4, b] \cap [4k^2, 4k^2+1]$ for all k , and $\sqrt{b-4} \leq \sqrt{b-a}$, so we may assume that $4 \leq a$ and that $b-a \geq 11$.

Now $\|\chi_{[a,b] \cap [4k^2, 4k^2+1]}\|_{\mathbf{L}^2(\mathbb{R})}$ equals zero if $a \geq 4k^2+1$ or $b \leq 4k^2$, and at most one otherwise.

Therefore, writing $\sharp M$ for the number of elements in a set M , we have

$$\begin{aligned} \sum_{k=1}^{\infty} \|\chi_{[a,b] \cap [4k^2, 4k^2+1]}\|_{\mathbf{L}^2(\mathbb{R})} &\leq \sharp\{k : 4k^2 \leq b\} - \sharp\{k : 4k^2 \leq a-1\} \\ &\leq \frac{1}{2}\sqrt{b} - \left(\frac{1}{2}\sqrt{a-1} - 1\right) \\ &\leq \frac{1}{2}(\sqrt{b} - \sqrt{a} + 3) \leq \frac{1}{2}(\sqrt{b-a} + \sqrt{11}) \leq \sqrt{b-a}, \end{aligned}$$

as claimed.

Assume that $f \in \mathbf{B}_0(\mathbb{R})$. Then there exists an admissible representation

$$f = \sum_{n=1}^{\infty} c_n e^{2\pi i \xi_n x} \chi_{[a_n, b_n]} \quad \text{with} \quad \sum_{n=1}^{\infty} |c_n| \sqrt{b_n - a_n} < \infty.$$

Consequently, we know from the above that

$$\begin{aligned} f &= f \cdot \chi_{\bigcup_k [4k^2, 4k^2+1]} = \left(\sum_{n=1}^{\infty} c_n e^{2\pi i \xi_n x} \chi_{[a_n, b_n]} \right) \left(\sum_{k=1}^{\infty} \chi_{[4k^2, 4k^2+1]} \right) \\ &= \sum_{n=1}^{\infty} c_n e^{2\pi i \xi_n x} \sum_{k=1}^{\infty} \chi_{[a_n, b_n] \cap [4k^2, 4k^2+1]} \\ &= \sum_{m=1}^{\infty} f_m e^{2\pi i \eta_m x} \chi_{[d_m, e_m]} \quad \text{with} \quad \sum_{m=1}^{\infty} |f_m| \sqrt{e_m - d_m} \leq \sum_{n=1}^{\infty} |c_n| \sqrt{b_n - a_n} < \infty, \end{aligned}$$

where for each m , we have that $[d_m, e_m] \subseteq [4k^2, 4k^2+1]$ for some k . Due to the absolute convergence in $\mathbf{L}^2(\mathbb{R})$, we may reorder the series in the form

$$f = \sum_{k=1}^{\infty} \sum_{m \in J_k} f_m e^{2\pi i \eta_m x} \chi_{[d_m, e_m]} \quad \text{where} \quad J_k = \{m \in \mathbb{N} : [d_m, e_m] \subseteq [4k^2, 4k^2+1]\},$$

i.e., we must have $\sum_{m \in J_k} f_m e^{2\pi i \eta_m x} \chi_{[d_m, e_m]} = \frac{1}{k} \chi_{[4k^2, 4k^2+1]}$ for all $k \in \mathbb{N}$, and thus

$$\begin{aligned} \sum_{k=1}^{\infty} \sum_{m \in J_k} |f_m| \sqrt{e_m - d_m} &= \sum_{k=1}^{\infty} \sum_{m \in J_k} \|f_m e^{2\pi i \eta_m x} \chi_{[d_m, e_m]}\|_{\mathbf{L}^2(\mathbb{R})} \\ &\geq \sum_{k=1}^{\infty} \left\| \sum_{m \in J_k} f_m e^{2\pi i \eta_m x} \chi_{[d_m, e_m]} \right\|_{\mathbf{L}^2(\mathbb{R})} \\ &= \sum_{k=1}^{\infty} \left\| \frac{1}{k} \chi_{[4k^2, 4k^2+1]} \right\|_{\mathbf{L}^2(\mathbb{R})} = \sum_{k=1}^{\infty} \frac{1}{k} = \infty, \end{aligned}$$

which is a contradiction to the assumed admissibility of the representation of f . So we may conclude that $f \notin \mathbf{B}_0(\mathbb{R})$. \square

5. LOCAL AND GLOBAL PROPERTIES OF \mathcal{B}_0

Theorem 5.1. (i) $\mathcal{B}_0(\mathbb{R}) \not\subseteq \mathbf{L}^p(\mathbb{R})$ for $p \in [1, \infty] \setminus \{2\}$.

(ii) $\mathbf{L}^p(\mathbb{R}) \not\subseteq \mathcal{B}_0(\mathbb{R})$ for $p \in [1, \infty]$.

Proof. (i) For $p < 2$, let $c_n = 2^{n(p-2)/4p}$, so $(c_n)_{n \in \mathbb{N}} \in \ell^1(\mathbb{N})$. Then we have that

$$f = \sum_{n \in \mathbb{N}} c_n 2^{-n/2} \chi_{[2^n, 2^{n+1}]} \quad \text{is in } \mathcal{B}_0(\mathbb{R}),$$

$$\text{but } \|f\|_{\mathbf{L}^p(\mathbb{R})}^p = \sum_{n \in \mathbb{N}} |c_n|^p 2^{-np/2} 2^n = \sum_{n \in \mathbb{N}} 2^{n(2-p)/4} = \infty.$$

For $2 < p < \infty$, let $c_n = 2^{n(2-p)/4p}$, so $(c_n)_{n \in \mathbb{N}} \in \ell^1(\mathbb{N})$, and we have that

$$f = \sum_{n \in \mathbb{N}} c_n 2^{n/2} \chi_{[2^{-n}, 2^{-n+1}]} \quad \text{is in } \mathcal{B}_0(\mathbb{R}),$$

$$\text{but } \|f\|_{\mathbf{L}^p(\mathbb{R})}^p = \sum_{n \in \mathbb{N}} |c_n|^p 2^{np/2} 2^{-n} = \sum_{n \in \mathbb{N}} 2^{n(p-2)/4} = \infty.$$

These functions also show that $\mathcal{B}_0(\mathbb{R}) \not\subseteq \mathbf{L}^\infty(\mathbb{R})$.

(ii) Theorem 4.10 shows that $\mathbf{L}^2(\mathbb{R}) \not\subseteq \mathcal{B}_0(\mathbb{R})$. For the case $p \neq 2$, note that $\mathbf{L}^p(\mathbb{R}) \not\subseteq \mathbf{L}^2(\mathbb{R}) \supset \mathcal{B}_0(\mathbb{R})$. \square

5.1. Local properties. An elegant way to study examples of functions in $\mathcal{B}_0(\mathbb{R})$ is implied by Theorem 4.6. Since the action of G is strongly continuous, we may integrate $\pi(g)$ over G with respect to any bounded Radon measure on G to obtain new elements of \mathcal{B}_0 . For example, we know that for any f which can be written as

$$f = \int_{\mathbb{R}^+} g(\rho) D_{1/\rho} \varphi \frac{d\rho}{\rho}$$

for some $g \in \mathbf{L}^1(\mathbb{R}^+, \frac{d\rho}{\rho})$ and $\varphi \in \mathcal{B}_0$, we have $f \in \mathcal{B}_0$ with $\|f\|_{\mathcal{B}_0} \leq \|g\|_{\mathbf{L}^1(\mathbb{R}^+, \frac{d\rho}{\rho})} \|\varphi\|_{\mathcal{B}_0}$.

The particular choice $\varphi = \chi_{[0,1]}$ shows that for any $g \in \mathbf{L}^1(\mathbb{R}^+, \frac{d\rho}{\rho})$, the function

$$\begin{aligned} f(x) &= \int_0^\infty g(\rho) \rho^{-1/2} \chi_{[0,\rho]}(x) \frac{d\rho}{\rho} \\ &= \int_x^\infty g(\rho) \rho^{-3/2} d\rho \end{aligned}$$

belongs to $\mathcal{B}_0(\mathbb{R})$. Note that this is equivalent to $f'(x) = -g(x) x^{-3/2}$ on \mathbb{R}^+ . Since the same argument works on \mathbb{R}^- , we may conclude that the condition $\sqrt{|x|} f'(x) \in \mathbf{L}^1(\mathbb{R}, dx)$ is sufficient for $f \in \mathbf{L}^2(\mathbb{R})$ to be in $\mathcal{B}_0(\mathbb{R})$.

Consequently, $f = x^\alpha \chi_{(0,1]}$ with $\sqrt{|x|} f'(x) = \alpha x^{\alpha-1/2}$ is in $\mathcal{B}_0(\mathbb{R})$ if and only if $\alpha > -1/2$. (For $\alpha \leq -1/2$, we have $f \notin \mathbf{L}^2 \supseteq \mathcal{B}_0(\mathbb{R})$.) By considering $\alpha \in (0, 1]$, this shows

that an element of $\mathcal{B}_0(\mathbb{R})$ has to satisfy no Lipschitz condition at its points of continuity. Similarly, $f(x) = x^\alpha \chi_{[1,\infty)}$ is in $\mathcal{B}_0(\mathbb{R})$ if and only if $\alpha < -1/2$.

More generally, we may conclude that $C_c^1 \subseteq \mathcal{B}_0(\mathbb{R})$. This can be strengthened using the well-known result by Bernstein that $\text{Lip}_\alpha(\mathbb{T}) \subseteq \mathcal{A}(\mathbb{T})$ for $\alpha > 1/2$ (e.g., see [19], Thm.I.6.3). So if f has compact support and satisfies a Lipschitz condition with $\alpha > 1/2$, then $f \in \mathcal{A}_c \subseteq \mathcal{S}_0 \subseteq \mathcal{B}_0$.

On the other hand, consider the function

$$f = \frac{-1}{\sqrt{x} \log x} \chi_{(0,1/e]}.$$

It satisfies $f \in L^2(\mathbb{R})$, but $\sqrt{|x|} f' \notin L^1$. This leads us to conjecture that $f \in L^2 \setminus \mathcal{B}_0(\mathbb{R})$.

Another statement about the local behaviour can be made by comparing \mathcal{B}_0 with certain inhomogeneous Besov spaces.

Lemma 5.2. $B_{1,1}^s \subseteq \mathcal{B}_0(\mathbb{R})$ if and only if $s \geq \frac{1}{2}$.

For the proof, we make use of the following lemma.

Lemma 5.3. Letting $PW_\Omega \cap L^1(\mathbb{R}^d) = \{f \in L^1 : \text{supp}(\hat{f}) \subseteq [-\Omega, \Omega]^d\}$, we have that

$$PW_\Omega \cap L^1 \subseteq \mathcal{B}_0 \quad \text{with } \|f\|_{\mathcal{B}_0} \leq C \Omega^{1/2} \|f\|_{L^1}.$$

Proof. For $\Omega = \frac{1}{2}$, this is immediate from the fact that bandlimited elements of L^1 are in \mathcal{S}_0 and thus in \mathcal{B}_0 with $\|f\|_{\mathcal{B}_0} \leq \|f\|_{\mathcal{S}_0} \leq C \|f\|_{L^1}$ for all $f \in PW_{1/2} \cap L^1$. For general Ω , note that $\|f\|_{\mathcal{B}_0} = \|D_{1/\Omega} f\|_{\mathcal{B}_0} \leq C \|\Omega^{1/2} D_{1/\Omega} f\|_{L^1} = C \Omega^{1/2} \|f\|_{L^1}$. \square

Proof (of Lemma 5.2). Since we have $B_{1,1}^r \subseteq B_{1,1}^s$ for $r \geq s \geq 0$, it suffices to verify that $B_{1,1}^{1/2} \subseteq \mathcal{B}_0$. But this is an immediate consequence of the definition of the Besov spaces (see [26], p. 93) which describes $f \in B_{1,1}^{1/2}$ as a series of building blocks $f_n \in PW_{2^n} \cap L^1$. Lemma 5.3 shows that this series is absolutely convergent in \mathcal{B}_0 .

For $s < \frac{1}{2}$, we have $B_{1,1}^s \not\subseteq \mathcal{B}_0$ since $B_{1,1}^s \not\subseteq L^2$ (e.g., see [6]). \square

Example 5.4. The following example exhibits one of the problems in the practical usage of $\mathcal{B}_0(\mathbb{R})$. Consider the function $f = \chi_{[0,a] \cup [1-a,1]}$ for $a < 1/2$. There are two immediate representations of f using $\varphi = \chi_{[0,1]}$, namely, $f = \chi_{[0,a]} + \chi_{[1-a,1]}$ with absolute coefficient sum $2\sqrt{a}$, and $f = \chi_{[0,1]} - \chi_{[a,1-a]}$ with absolute coefficient sum $1 + \sqrt{1-2a}$. The two representations have the same ‘‘price’’ for $a = 4/9$. For $a < 4/9$, it is cheaper to use two separate blocks, while for $a > 4/9$, it is cheaper to ‘‘bridge the gap’’ and to subtract a

block in the center. Thus the optimal representation does not depend continuously on the function, a phenomenon which can also be observed in matching pursuit problems. See also the last paragraph of Section 6 for resulting consequences.

5.2. Global properties. Since \mathcal{B}_0 is a localizable space, Wiener amalgam spaces (see the introduction) allow us to get an idea of the global behaviour of elements in \mathcal{B}_0 .

Theorem 5.5. *Let $p \in [1, \infty]$.*

- (i) $\mathcal{B}_0(\mathbb{R}) \subseteq \mathbf{W}(\mathcal{B}_0, \ell^p)$ if and only if $p \in [2, \infty]$.
- (ii) $\mathbf{W}(\mathcal{B}_0, \ell^p) \subseteq \mathcal{B}_0(\mathbb{R})$ if and only if $p=1$.

Proof. Assume that $\|\chi_{[0,1]}\|_{\mathcal{B}_0} = 1$, and recall that the norm in \mathcal{B}_0 is invariant under translations and modulations.

(i) Assume $p < 2$. For the functions constructed in the proof of Theorem 5.1.(i), case $p < 2$, we have

$$\begin{aligned} \|f\|_{\mathbf{W}(\mathcal{B}_0, \ell^p)}^p &= \sum_{k \in \mathbb{Z}} \|f \chi_{[k, k+1]}\|_{\mathcal{B}_0}^p = \sum_{n \in \mathbb{N}} \sum_{m=0}^{2^n-1} \|2^{n(p-2)/4p} 2^{-n/2} \chi_{[2^n+m, 2^n+m+1]}\|_{\mathcal{B}_0}^p \\ &= \sum_{n \in \mathbb{N}} 2^n |2^{-n(p+2)/4p}|^p = \sum_{n \in \mathbb{N}} 2^{n(2-p)/4} = \infty. \end{aligned}$$

Since $\ell^2(\mathbb{Z}) \subseteq \ell^p(\mathbb{Z})$ for $p > 2$, it suffices to show the inclusion for $p=2$. We have for arbitrary $a, b \in \mathbb{R}$ and $\xi \in \widehat{\mathbb{R}}$ that

$$\begin{aligned} \|e^{2\pi i \xi x} \chi_{[a,b]}\|_{\mathbf{W}(\mathcal{B}_0, \ell^2)} &= \left(\sum_{n \in \mathbb{Z}} \|e^{2\pi i \xi x} \chi_{[a,b] \cap [n, n+1]}\|_{\mathcal{B}_0}^2 \right)^{1/2} \\ &= \left(\sqrt{[a]+1-a}^2 + \sum_{[a]+1}^{[b]-1} 1^2 + \sqrt{b-[b]}^2 \right)^{1/2} = \sqrt{b-a}. \end{aligned}$$

Therefore we have for an arbitrary $f = \sum_{n \in \mathbb{N}} c_n e^{2\pi i \xi_n x} \chi_{[a_n, b_n]} \in \mathcal{B}_0(\mathbb{R})$ that

$$\|f\|_{\mathbf{W}(\mathcal{B}_0, \ell^2)} \leq \sum_{n \in \mathbb{N}} |c_n| \|e^{2\pi i \xi_n x} \chi_{[a_n, b_n]}\|_{\mathbf{W}(\mathcal{B}_0, \ell^2)} = \sum_{n \in \mathbb{N}} |c_n| \sqrt{b_n - a_n} < \infty.$$

Since this holds for all representations of f , we even have $\|f\|_{\mathbf{W}(\mathcal{B}_0, \ell^2)} \leq \|f\|_{\mathcal{B}_0}$.

(ii) For $p=1$, the statement is trivially true since

$$\|f\|_{\mathcal{B}_0} = \left\| \sum_{n \in \mathbb{Z}} f \chi_{[n, n+1]} \right\|_{\mathcal{B}_0} \leq \sum_{n \in \mathbb{Z}} \|f \chi_{[n, n+1]}\|_{\mathcal{B}_0} = \|f\|_{\mathbf{W}(\mathcal{B}_0, \ell^1)}.$$

On the other hand, we have shown in Theorem 4.10 that $f = \sum_{k=1}^{\infty} \frac{1}{k} \chi_{[4k^2, 4k^2+1]}$ is not in $\mathcal{B}_0(\mathbb{R})$, but

$$\|f\|_{\mathbf{W}(\mathcal{B}_0, \ell^p)} = \left\| \left(\|f \chi_{[n, n+1]}\|_{\mathcal{B}_0} \right)_{n \in \mathbb{Z}} \right\|_{\ell^p(\mathbb{Z})} = \left\| \left(\frac{1}{k} \right)_{k \in \mathbb{N}} \right\|_{\ell^p(\mathbb{N})} < \infty \quad \text{for all } p \in (1, \infty].$$

□

6. IMPLICATIONS FOR MATCHING PURSUIT AND BASIS PURSUIT METHODS

In the literature on signal analysis, it is often argued that parsimonious representations are important for efficient compression or feature detection, where it is desirable to store only few coefficients. The prototype algorithm in this context is the matching pursuit algorithm suggested by Mallat and Zhang [22]. Although this method works well for “nice” signals (the meaning of the latter depending on the dictionary in use), it tends to result in a large number of fairly small coefficients (cf., [18] for such and other problems of matching pursuit). Therefore Donoho [3] suggested to minimize the ℓ^1 -norm of the coefficients by means of global optimization methods. This was one motivation for the definition of the space \mathcal{B}_0 .

Consequently, we emphasize the following two important observations. Firstly, compare the construction of \mathcal{B}_0 to ordinary Gabor expansions of \mathbf{L}^2 . It follows from standard frame theory that using time-frequency shifts of an appropriate basic atom and allowing ℓ^2 -coefficients yields all of \mathbf{L}^2 , while Theorem 4.10 shows that the restriction imposed by using ℓ^1 - instead of ℓ^2 -coefficients is not compensated by the use of a largely increased dictionary (namely, containing not only all translated and modulated, but also dilated versions of a given atom).

Secondly, Example 5.4 demonstrates that the mapping from $\mathbf{L}^2(\mathbb{R})$ to the coefficients is inherently unstable, even for simple signals, and consequently, its use for feature extraction purposes may suffer seriously in certain cases. However, one might argue from a somewhat more applied point of view that \mathcal{B}_0 is the appropriate domain for searching parsimonious representations.

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