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# Non-uniform Sampling: Exact Reconstruction From Non-uniformly Distributed Weighted-Averages

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In this article, we discuss the problem of reconstructing a function  $f$  in a lattice-invariant subspace of  $L^p_\nu(\mathbb{R}^d)$  from a family of non-uniformly distributed, weighted-averages  $\{\langle f, \psi_{x_j} \rangle : j \in J\}$  using an approximation-projection iterative algorithm.

## 1 Introduction

The central topic of this article is a summary of some sufficient conditions under which a function  $f$  satisfying some a priori conditions, expressed in term of certain function spaces, can be completely reconstructed from a collection of local averages of the form  $\{\langle f, \psi_{x_j} \rangle = \int_{\mathbb{R}^d} f(x)\psi_{x_j}(x)dx : j \in J\}$ .

When  $\{\psi_{x_j}, : j \in J\}$  are Dirac delta distributions, then the data are the exact sample values of  $f$  at the sampling points  $x_j$ , and the problem has a long history already, under various assumptions, such as band-limitedness, or membership of  $f$  in some spline-type space.

However, in applications, one needs to consider the case where  $\{\psi_{x_j} : j \in J\}$  are not Dirac delta distributions but functions that reflect the characteristic of the more general sampling devices. For this case, the set  $\{\langle f, \psi_{x_j} \rangle : j \in J\}$  consists of weighted average sample values of  $f$ . Another case that includes the particular situation where the data are of the form  $\{f(x_j) : j \in J\}$  is when  $\{\psi_{x_j} = \mu_{x_j} : j \in J\}$  is a set of non-negative bounded measures with compact support. For example, grouped data are

preferred over exact sampling values due to the fact that they have a better signal-to-noise ration (cf. [19] for statistical background in this direction).

Obviously, to reconstruct a function from the discrete data, we have to make some general assumptions about the function  $f$ . Clearly these assumptions must be sufficiently flexible to accommodate a large number of possible models for  $f$ . Also, the standard assumption of band-limitedness should be a special case or a limit case of the model spaces. It turns out that lattice-invariant spaces are a sufficiently large family of possible model spaces which appear to be adequate for our problem. They can be described as:

$$V_\nu^p(\phi) = \left\{ \sum_{k \in \mathbb{Z}^d} c_k \phi(\cdot - Lk) : c \in \ell_\nu^p \right\}, \quad (1)$$

where  $\phi$  is a suitable generator,  $L$  is a  $d \times d$  non-singular matrix,  $1 \leq p < \infty$ , and  $\nu$  is a weight that controls the growth or decay of the functions in the space  $V_\nu^p(\phi)$ . The matrix  $L$  transforms the lattice  $\mathbb{Z}^d$  to the lattice  $\Lambda$ , and when  $L$  is the identity matrix, we obtain the standard shift-invariant spaces. That a combination of atoms  $\phi$  (e.g. radial symmetric ones) with suitable lattices  $\Lambda$  (related to sphere packing) is a good alternative for the usual voxel representation of volume data has been observed also in another context (cf. [18]). As a matter of fact such spaces can also be considered over LCA groups [11], where applications to the theory of Gabor multipliers are given.

Let us therefore come back to our main problem: generally we are given a set of data  $\{\langle f, \psi_{x_j} \rangle : j \in J\}$  and only an approximate information about the function  $f$ , e.g., we may know that  $f$  is continuous, and we may know the generator  $\phi$  approximately, but we may not know the value of  $p$  or the rate of growth or decay of the function  $f$ . Moreover, even if we had the exact information about the model, the data are typically corrupted by noise. Thus our goal is to describe algorithms with the following properties:

1. In the ideal case, i.e., when the model for  $f$  is exact and the data are not corrupted by noise, the algorithm must reconstruct the function  $f$  exactly, and must do so “fast”.
2. In the non-ideal case, e.g., when our information about  $f$  is partial or when the data are corrupted by noise, the algorithm must converge and must be able to “guess” the missing information.

In this paper we describe a family of Approximation-Projection algorithms (AP algorithms) for recovering the function  $f \in V_\nu^p(\phi)$  exactly from

the family of weighted averages, i.e.  $\{\langle f, \psi_{x_j} \rangle : j \in J\}$ . Such AP algorithms perform exact reconstruction, as long as the data are obtained from a function within the family of spaces (irrespective of  $\nu$  or the value of  $p$ ). Thus, after reconstruction we can inspect whether  $f$  belongs to a certain space or not. Moreover, the reconstruction automatically converges in the right norm without feeding this a priori information into the algorithm. Thus the algorithm is adaptive and always converges in the appropriate space. In contrast, the use of a frame algorithm usually is said to require some a priori estimates on the size of the frame bounds, and claims about convergence apply only to the Hilbert space norm.

When the model of  $f$  is not exact, i.e., the assumption that  $f$  belongs to a lattice-invariant space generated by  $\phi$  is only a rough approximation, or when the sample data are corrupted by noise, then the AP algorithm still converges to a function  $f_\infty$  in some appropriate  $V_\nu^p(\phi)$ , and the reconstructed function  $f_\infty$  is “best” in some sense.

Since the results for weighted average sampling when  $\nu = 1$  are well-known [1, 3, 5], and since the proof for general  $\nu$  is similar to the proofs in [1, 2, 3, 5], we will omit them. Instead, our aim will be mostly tutorial and qualitative. Related results on reconstruction from averages can be found in [14, 15, 21, 22, 23]. For more references on sampling and reconstruction see [3]. More references on sampling and applications can be found in [8].

## 2 Notation

Although we are mainly interested in lattice-invariant spaces (or spline-type like spaces) for some weighted  $L^p$ -spaces, the Wiener amalgam [12] are important auxiliary tools for the development of the theory. A function  $f$  belongs to the weighted space  $L_\nu^p$  if and only if  $f\nu$  belongs to  $L^p$ . We will use two types of weight functions: 1) The weights denoted by  $w$  are continuous, strictly positive ( $w > 0$ ), symmetric ( $w(x) = w(-x)$ ), and submultiplicative ( $w(x+y) \leq w(x)w(y)$ ) (The assumptions on the submultiplicative weight  $w$  imply that  $w \geq 1$ ); and 2) the  $w$ -moderate weights  $\nu$  that satisfy  $\nu(x+y) \leq C\nu(x)w(y)$ , and that are also continuous, symmetric and positive. By choosing a  $w$  that grows at infinity, we control the decay of the functions in  $L^p$ . On the other hand the moderate weights  $\nu$  are not restricted to be greater than 1. Thus, for an  $w$ -moderate  $\nu$  that decays to zero at infinity, the space  $L_\nu^p$  contains functions that grow at infinity.

A measurable function  $f$  belongs to the Wiener amalgam space  $W_\nu^p$ ,  $1 \leq p < \infty$ , if it satisfies

$$\|f\|_{W_\nu^p} = \left( \sum_{k \in \mathbb{Z}^d} \text{ess sup}\{|f(x+k)|^p \nu(k)^p : x \in [0, 1]^d\} \right)^{1/p} < \infty. \quad (2)$$

If  $p = \infty$ , a measurable function  $f$  belongs to  $W_\nu^\infty$  if it satisfies

$$\|f\|_{W_\nu^\infty} = \sup_{k \in \mathbb{Z}^d} \{\text{ess sup}\{|f(x+k)| \nu(k) : x \in [0, 1]^d\}\} < \infty. \quad (3)$$

The space  $W_\nu^p$  is a Banach space, and the subspace of continuous functions  $W_{\nu,0}^p \subset W_\nu^p$  is a closed subspace of  $W_\nu^p$  [12].

### 3 AP algorithm for exact reconstruction from weighted averages

For the spaces  $V_\nu^p(\phi)$  in (1) to be well defined, we require that the generator  $\phi$  belongs to the space  $W_{w,0}^1$ , and satisfies the condition

$$A \leq \sum_{j \in \mathbb{Z}^d} |\hat{\phi}(\xi + (L^{-1})^T j)|^2 \leq B \quad (4)$$

where  $(L^{-1})^T$  is the transpose of the matrix inverse of  $L$ . These two requirements on  $\phi$  imply that  $V_\nu^p(\phi)$  is a closed subspace of  $W_{\nu,0}^p \subset L_\nu^p$  for all  $w$ -moderate weights  $\nu$  and for  $1 \leq p \leq \infty$  [3]. Moreover, under the same conditions on  $\phi$ , the dual function  $\tilde{\phi}$  is a well defined function in  $W_1^1$  and can be expressed as

$$\tilde{\phi}(\cdot) = \sum_{k \in \mathbb{Z}^d} b_k \phi(\cdot - Lk) \quad (5)$$

where the coefficients  $b_k$  are determined explicitly by the Fourier series

$$\sum_{k \in \mathbb{Z}^d} b_k e^{2\pi i k \xi} = \left( \sum_{k \in \mathbb{Z}^d} |\hat{\phi}(\xi + (L^{-1})^T k)|^2 \right)^{-1},$$

i.e.,  $(b_k)$  is the inverse Fourier transform of  $\left( \sum_{k \in \mathbb{Z}^d} |\hat{\phi}(\xi + (L^{-1})^T k)|^2 \right)^{-1}$ . In addition, the operator

$$P : f \rightarrow \sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\phi}(\cdot - Lk) \rangle \phi(\cdot - Lk), \quad (6)$$

is a bounded projection from  $L_\nu^p$  onto  $V_\nu^p(\phi)$  for all  $p, 1 \leq p \leq \infty$  and all  $w$ -moderate weights  $\nu$  [3].

Obviously, to reconstruct a function  $f \in V_\nu^p(\phi)$ , the sampling density must be sufficiently large. One way to measure the sampling density of a set  $X = \{x_j, j \in J\}$  is to find a number  $\gamma_0$  that allows us to cover  $\mathbb{R}^d$  with balls of radius  $\gamma > \gamma_0$  that are centered on  $x_j$ , i.e.,  $\mathbb{R}^d = \cup_j B_\gamma(x_j)$  for every  $\gamma > \gamma_0$ . A set  $X$  with this covering property will be called  $\gamma_0$ -dense.

Before describing the AP algorithm we first define the quasi-reconstruction operator  $A_X$  used in the reconstruction scheme:

$$A_X f = \sum_{j \in J} \langle f, \psi_{x_j} \rangle \beta_j. \quad (7)$$

Here  $\{\beta_j\}_{j \in J}$  is any partition of unity which is associated with the balls  $\{B_\gamma(x_j)\}_{j \in J}$ , and satisfies

1.  $0 \leq \beta_j \leq 1, \forall j \in J$ ;
2.  $\text{supp } \beta_j \subset B_\gamma(x_j)$ ; and
3.  $\sum_{j \in J} \beta_j = 1$ .

A special case for  $A_X$  is the nearest neighbor interpolation, also known as the Voronoi interpolation scheme. However, in practice condition (3) on  $\beta_j$  can be relaxed to  $\sum_{j \in J} \beta_j \leq 1$  as long as  $\sum_{j \in J} \beta_j$  is close to one “most of the time”, i.e., the regions where such deviations occur are thin and spread out and do not cover large regions. The use of such “alternative systems” will slightly deteriorate the speed of convergence (have a slightly negative effect on the rate of convergence), but may have the big advantage that the computational costs per iteration are much lower, so that the overall approximation quality achieved at fixed computational cost can be increased significantly in some cases.

We can now use the quasi-reconstruction operator  $A_X$  in an iterative scheme to reconstruct the function  $f \in V_\nu^p(\phi)$  exactly from the data  $\langle f, \psi_{x_j} \rangle$  as described by the theorem below. The only assumptions are that  $\psi_{x_j} \in L^1$  is such that  $\int_{\mathbb{R}^d} \psi_{x_j} = 1$ ,  $\text{supp } \psi_{x_j}(\cdot + x_j) \subset [-a, a]^d$ , and  $\int_{\mathbb{R}^d} |\psi_{x_j}| \leq M$ , where  $a > 0, M > 0$  are independent of  $x_j$  in the case of averaging functions, or that  $\mu_{x_j}$  are non-negative bounded measures with support in  $[-a, a]^d$  and total mass 1 in the case  $\psi_{x_j} = \mu_{x_j}$ .

**Theorem 3.1** *Let the weight  $w$ , the Lattice  $\Lambda$ , and the generator  $\phi$  be given. Then there exists a density  $\gamma = \gamma(\phi) > 0$  and  $a_0 > 0$  such that any  $f \in V_\nu^p(\phi)$  can be recovered from the data  $\{\langle f, \psi_{x_j} \rangle : j \in J\}$  on any  $\gamma$ -dense set  $X = \{x_j : j \in J\}$  and for any  $0 < a < a_0$ , by the following iterative algorithm :*

$$\begin{cases} f_1 = & PA_X f \\ f_{n+1} = & PA_X(f - f_n) + f_n, \end{cases} \quad (8)$$

where,  $P$  is the operator in (6). In this case, the iterate  $f_n$  converges to  $f$  in the  $W_\nu^p$ -norm, hence both in the  $L_\nu^p$ -norm and uniformly over compact sets. If furthermore  $\nu(x) \geq 1$  for all  $x \in \mathbb{R}^d$ , then  $L_\nu^p \subset L^p$  and one has uniform convergence.

The convergence is geometric, that is,

$$\|f - f_n\|_{L_\nu^p} \leq \|f - f_n\|_{W_\nu^p} \leq C_1 \alpha^n \|f - f_1\|_{W_\nu^p}$$

for some  $\alpha = \alpha(\gamma, a, \phi) < 1$  and  $C_1 < \infty$ .

Let us make the following remarks.

1. Reconstruction from weighted averages in lattice-invariant spaces for the  $L^2$  case is discussed in [1]. The  $L^p$  case and when the lattice-invariant space is constructed from multiple generators is presented in [5]. Reconstruction from perfect sampling (i.e.,  $\psi_{x_j} = \delta(\cdot - x_j)$ ) in lattice-invariant spaces using AP algorithms is discussed in [2, 3]. The bandlimited case is worked out in [13]. The general uniform sampling cases are discussed in [6, 7].
2. Theorem 3.1 does not rule out sampling point clustering. Thus in principle, algorithm 8 still works even in the presence of clustering. However, if the sampling set  $X$  is separated, i.e.,  $\inf |x_j - x_l| \geq \delta > 0$ , then we have the following norm equivalence

$$c \|f\|_{L_\nu^p} \leq \left( \sum_{x_j \in X} |\langle f, \psi_{x_j} \rangle|^p |\nu(x_j)|^p \right)^{1/p} \leq C \|f\|_{L_\nu^p} \quad (9)$$

where  $c$  and  $C$  are positive constants independent of  $f$ ,  $p$ , or  $\nu$ . However,  $c$  and  $C$  may depend of  $\delta$  and the submultiplicative weight  $w$ .

This norm equivalence remains valid if  $X$  is relatively separated, i.e.,  $X$  is the finite union of separated sampling sets  $X_i, i = 1, \dots, N$ .

3. Note that the algorithm (8) does not require the a priori knowledge of  $p$  or  $\nu$ .
4. Although in lattice-invariant spaces  $f(\cdot) \in V_\nu^p(\phi)$  implies that  $f(\cdot - Lk) \in V_\nu^p(\phi)$  for all  $k \in \mathbb{Z}^d$ , it does not imply that  $V_\nu^p(\phi)$  is translation invariant, in general, e.g.,  $f \in V_\nu^p(\phi)$  does not imply that  $f(\cdot - \tau) \in V_\nu^p(\phi)$  for all  $\tau \in \mathbb{R}^d$ . Thus, knowledge of the origin in  $V_\nu^p(\phi)$  is assumed for exact reconstruction. Uncertainty in the origin will introduce error in the reconstruction. This is equivalent to a fixed jitter error in the sampling process and can be analyzed using jitter error methods.
5. A generator  $\phi$  which is continuous and has compact support always satisfies the assumptions of Theorem 3.1.
6. If the assumptions on  $\phi$  in Theorem 3.1 are satisfied, i.e., when the generator  $\phi$  belongs to the space  $W_{w,0}^1$  and satisfies condition 4, then the projector  $P$  in (6) is uniformly bounded for all  $p$  with  $1 \leq p \leq \infty$ . The case  $\phi = \text{sinc}$  (i.e., the bandlimited case) does not satisfy the assumption of Theorem 3.1, because of the slow decay of the *sinc*-function. However, this case is covered by the results in [13, 20]. For the bandlimited case, the  $P$  in (6) is bounded for all  $p$  with  $1 < p < \infty$ . However in this case,  $P$  is not uniformly bounded and  $\|P\| \rightarrow \infty$  as  $p \rightarrow 1$  or  $p \rightarrow \infty$ .
7. The projector (6) can be implemented using convolutions. As a consequence, every iteration of algorithm (8) has a complexity of order  $N$  for data of length  $N$ . Recently, Gröchenig and Schwab have produced reconstruction algorithms in lattice-invariant spaces that are order  $N$  [16].
8. In the special case where  $\psi_{x_j} = \psi(\cdot - x_j)$  (e.g., same sampling device everywhere), the average sampling problem in  $V_\nu^p(\phi)$  can be viewed as ideal sampling in the space  $V_\nu^p(\phi * \psi)$ .
9. If the function  $f \in W_\nu^p$  does not belong to  $V_\nu^p(\phi)$  (e.g., our model of  $f$  is not exact), then the iterate  $f_n$  in algorithm (8) still converges to a function  $f_\infty$  such that  $P(A_X f_\infty - A_X f) = 0$ .

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