Modulation Spaces on Locally Compact Abelian Groups

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This is a *literal reproduction* of the 1983 report [55] by Hans G. Feichtinger, with only the obvious typos being corrected, one additional section and minor extra footnotes. Only few symbols have been changed to more standard ones, e.g. for the translation operator (which was L_y, following Hans Reiter) has been replaced by T_y, and instead of $\mathcal{K}(G)$ we write $C_c(G)$ now. We hope that by adding comments about recent papers on modulation spaces and publications which have appeared in the mean-time, as well as updates to the bibliography the reader will find this "new edition" interesting. Of course the page numbers differ slightly from those in the original report (it was 52 pages long), but the numbering system of theorems and remarks has been preserved in the present version (so that one may refer to the results of this paper, which is now better accessible, in the same way as to the original report.)

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1 Introduction

The modulation spaces $M^s_{p,q}(\mathbf{R}^m), s \in \mathbf{R}, 1 \leq p,q \leq \infty$ to be discussed in this paper are Banach spaces of tempered distributions σ on $\mathbf{R}^m$, which are characterized by the behaviour of the convolution product $M_t g * \sigma$ in $L^p(\mathbf{R}^m)$, for $t \to \infty$ ($g \in \mathcal{S}(\mathbf{R}^m)$). As will be shown this family behaves (with respect to various properties) very much like the well known family of Besov spaces $B^s_{p,q}(\mathbf{R}^m)$ (cf. [32], [34], [41], [44], [47]) concerning duality, interpolation, embedding and trace theorems, or the Fourier transformation. Furthermore, the classical potential spaces $\mathcal{L}^s_2(\mathbf{R}^m) = H^s(\mathbf{R}^m)$ as well as the remarkable Segal algebra $S_0(\mathbf{R}^m)$ (see [18]) may be considered as particular modulation spaces.

In order to give a more precise definition of the modulation spaces $M^s_{p,q}(\mathbf{R}^m)$ fix any test function $g \in \mathcal{S}(\mathbf{R}^m)$, $g \neq 0$, and write $M_t g$ for the (oscillating) function $x \mapsto \exp(2\pi i\langle x,t\rangle)\ g(x)$, $x,t \in \mathbf{R}^m$. The convolution product $M_t g * \sigma$ is then well defined for any tempered distribution $\sigma \in \mathcal{S}'(\mathbf{R}^m)$ and we may set for $1 \leq q < \infty$:

$$M^s_{p,q}(\mathbf{R}^m) := \{\sigma \mid \sigma \in \mathcal{S}'(\mathbf{R}^m),\ M_t g * \sigma \in L^p(\mathbf{R}^m) \ \text{ for each } t \in \mathbf{R}^m, \text{ and}$$

$$\|\sigma|_{M^s_{p,q}}\| := \left[\int_{\mathbf{R}^m} \|M_t g * \sigma\|_p^q\ (1+|t|)^{sq} dt\right]^{1/q} < \infty\ \}.$$

The necessary modification for $q = \infty$ is obvious. It is then possible to show that different test functions define the same spaces and equivalent norms, and that one obtains a family of Banach spaces which is essentially closed with respect to duality and complex interpolation.

There is not only a formal similarity in the results concerning modulation spaces and Besov spaces. In fact, one can say that an element σ of a Besov space $B^s_{p,q}(\mathbf{R}^m)$ is characterized by the behaviour of $M_\rho g * \sigma$ in $L^p(\mathbf{R}^m)$, for $\rho \to \infty$, where now the "deformation" of the test function consists in a suitable dilation ($M_\rho g(x) := \rho^m g(\rho x),\ \rho > 0$).[1] Such characterizations can be found in the work of Calderon, Torchinsky and others (cf. [9], [43], [39], §8 and elsewhere). It is also possible to describe Besov spaces by dyadic decompositions of the Fourier transforms of their elements (the dyadic structure has to do very much with dilations). Such characterizations, going essentially back to Hörmander, have been used successfully by Peetre, Triebel and many others (cf. [7], [34], [41], [43] for the basic results, and [32], [41] for the "classical" characterizations).

Our approach to modulation spaces will be through "uniform" decompositions of the Fourier transforms of their elements. Since such decompositions correspond to "uniform" coverings, obtained by translation (and translation of $\hat{g}$ corresponds to multiplication of g with a character), these are in our situation the natural analogues to the dyadic decompositions mentioned above. The fact that Banach spaces of distributions characterized by uniform decompositions have been treated in detail in earlier papers by the author under the name of Wiener type spaces (cf. [20], [21], [23]), will be of great use here.

Moreover, since Wiener type spaces are well defined for a quite comprehensive class of Banach spaces of distributions on locally compact groups, it is possible to define modulation spaces for a class of solid BF-spaces $(B, \|\cdot\|_B)$ (including L^p, $1 \leq p \leq \infty$) on locally compact abelian groups (among them $\mathbf{R}^m$). In fact, the general modulation spaces $M(B, L^q_v)(G)$ consist of those (ultra) distributions σ on G, for which $t \to \|M_t g * \sigma\|_B,\ t \in \hat{G}$, satisfies a weighted q-integrability condition. Using a suitable general Fourier transform the relevant facts concerning modulation spaces (in the generality just described) can be drawn from corresponding properties of Wiener-type spaces of the form $W(\mathcal{F}_G B, L^q_v)\ (\hat{G})$. The results concerning the spaces $M^s_{p,q}(\mathbf{R}^m)$ can

[1] The notation is due to H. Reiter [38].

then be obtained as special cases of general results. The approach choosen is not only justified by the degree of generality obtained, but also by the fact that direct proofs for the spaces $M^s_{p,q}(\mathbf{R}^m)$ would not have been much shorter, but probably less transparent.

The paper is organized as follows. §2 contains the basic notations and facts, from harmonic analysis and concerning Banach spaces of distributions on locally compact abelian groups. In section 3 various information concerning Wiener-type spaces are collected, mainly for later use in the treatment of modulation spaces. In particular, weighted versions of the Hausdorff-Young theorem for Wiener-type spaces are derived, and some information concerning maximal functions are proved. The results of this section allow us to introduce modulation spaces in §4 in full generality. In this part the independence of modulation spaces from irrelevant parameters (or auxiliary expressions, such as the test functions involved) is shown, and various equivalent characterizations of these spaces (discrete and continuous versions of the norm, atomic representations, norms involving maximal functions) are given. Furthermore, several basic properties of modulation spaces, e.g. concerning the density of test function, duality, interpolation, convolution, are derived. In section 5 a general trace theorem is established. The last section gives information concerning the modulation spaces $M^s_{p,q}(\mathbf{R}^m)$ as described at the beginning. The facts concerning these spaces are obtained by specialization from the general principles to be found in sections 4 and 5, thus also illustrating the abstract results given in the earlier parts of this paper. The paper concludes with an outlook on further generalizations, related subjects and further possible applications.

2 Notations, Generalities.

In the sequel G denotes a lca (locally compact abelian) group, with the Haar measure dx. We shall be mainly interested in non-compact and non-discrete groups such as $\mathbf{R}^m$, $m \geq 1$. The Lebesgue spaces with respect to dx are denoted by $(L^p, \|\cdot\|_p)$ for $1 \leq p \leq \infty$, as usual. The translation operators T_y : $T_y f(x) := f(x-y)$ act isometrically on $(L^p, \|\cdot\|_p)$. For $1 \leq p \leq \infty$ the space $C_c(G)$ (of continuous, compactly supported complex-valued functions on G) is a dense subspace of $L^p(G)$, and the Banach dual of $L^p(G)$ can be identified with $L^{p'}(G)$, where $\frac{1}{p}+\frac{1}{p'}=1$. $(L^1(G),\|\cdot\|_1)$ is considered as a Banach algebra with respect to convolution, given by $f*g(x)=\int_G f(x-y)g(y)dy$ for $f,g\in C_c(G)$.

A strictly positive, locally bounded and measurable function w on G will be called a **weight function** if $w(x)\geq 1$ and $w(x+y)\leq w(x)w(y)$ for all $x,y\in G$. Then $L^1_w(G):=\{f \mid fw\in L^1(G)\}$ is a Banach algebra under convolution, called **Beurling algebra** (cf. [38] III, §7 and VI, §3), with the norm $\|f\|_{1,w}:=\|fw\|_1$. We restrict our attention to **symmetric** weights: $w(-x)=w(x)$.

The **dual group** $\hat{G}$ of a lca group consists of all continuous **characters** $t:G\to U$ (homomorphisms into the unit circle). We write $\langle t,x\rangle$ or $\langle x,t\rangle$ for $t(x)$. Recall that $\hat{G}$ is itself a lca group, and that G may always be identified with $(\hat{G})^\wedge$ in a natural way (due to the Pontrjagin duality theorem). Recall that $(\mathbf{R}^m)^\wedge\cong\mathbf{R}^m$ as a topological group, since the continuous characters on $\mathbf{R}^m$ are exactly of the form $x\mapsto \exp\left[2\pi i(\sum_{i=1}^m x_i t_i)\right]$, $x,t\in\mathbf{R}^m$.

The **Fourier transform** $\hat{f}$ of $f\in L^1(G)$ is given by

$$\hat{f}(t):=\int_G f(x)\overline{\langle x,t\rangle}dx$$

The Fourier transformation $\mathcal{F}_G : f\mapsto \hat{f}$ defines an injective, involutive Banach algebra homomorphism from $L^1(G)$ into $C^0(\hat{G})$ (considered as an involutive, pointwise algebra, with complex conjugation, i.e. $(\hat{f})^-=(f^*)^\wedge$, for $f^*(x)=\overline{f(-x)}$). Consequently, given any symmetric weight function w on G, the space $A_w(\hat{G}):=\{\hat{f}\mid f\in L^1_w(G)\}$ is a self-adjoint Banach algebra of continuous functions on $\hat{G}$ under pointwise multiplication, if it is endowed with the norm $\|\hat{f}\|_{A_w}:=\|f\|_{1,w}$.

We shall be **exclusively** interested in weights satisfying the so-called **BD-condition** (Beurling-Domar's non-quasianalyticity condition, cf. [38], VI, §3), i.e. only weights satisfying

$$(BD)\quad \sum_{n=1}^{\infty} n^{-2}\log w(nx)<\infty \ \text{ for all } \ x\in G$$

will be of interest for us. Typical examples of such weights on $\mathbf{R}^m$ are those of the form $w_s : x \mapsto (1 + |x|)^s$, $s \geq 0$ or $\bar{w}_{a,d}$: $x \mapsto \exp(a|x|^d)$, for $a \geq 0$, $d \in (0, 1)$ (cf. [11], [2], [28], [45], [14] for explanations concerning such weights). According to the fundamental work of Domar, the algebra $A_w(\hat{G})$ is a regular algebra of continuous functions under this condition. It is a even a Wiener algebra in the sense of Reiter (cf. [38], Chap. II).

Since $(M_t f)^\wedge = T_t \hat{f}$, and because the mapping $t \mapsto M_t f$ is continuous from $\hat{G}$ into $L^1_w(G)$ (as a consequence of the density of $C_c(G)$ in $L^1_w(G)$) it turns out that $(A_w, \| \cdot \|_{A_w})$ is a **'nice' Banach algebra** in the sense used in [6], [22]. Among others it is then possible to define Wiener type spaces $W(B, C)$ on $\hat{G}$ (cf. §3) for any Banach space $(B, \| \cdot \|_B)$ which is **in standard situation with respect to** $A_w(\hat{G})$, i.e. for spaces satisfying the following three conditions:

i) $(A_w)_0 \hookrightarrow B \hookrightarrow (A_w)_0'$
(here $(A_w)_0 := A_w \cap C_c(\hat{G})$ is considered as a topological vector space with respect to its natural inductive limit topology, and $(A_w)_0'$ denotes the topological dual; $\hookrightarrow$ indicates continuous embeddings).

ii) $(B, \| \cdot \|_B)$ is a Banach module (with respect to pointwise multiplication) over $A_w(\hat{G})$, i.e. $\|hf\|_B < \|h\|_{A_w} \|f\|_B$ for $h \in A_w(\hat{G})$, $f \in B$.

iii) $(B, \| \cdot \|_B)$ is a Banach module (with respect to convolution on $\hat{G}$) over a Beurling algebra $L^1_{\hat{w}}(\hat{G})$ (again we restrict our attention to weights $\hat{w}$ satisfying (BD)) [2].

Assumption i) as well as many typical examples on $\hat{G} = \mathbf{R}^m = G$ will justify our speaking of **Banach spaces of distributions** (on $\hat{G}$). We write $B^1 \hookrightarrow B^2$ for continuous embeddings of topological vector spaces. For Banach spaces in standard situation inclusions are automatically continuous by the closed graph theorem. Hence their (complete) norm is uniquely determined up to equivalence. If such a space is **translation** or **character invariant** (i.e. $T_y B \subseteq B$ for $y \in G$, or $M_t B \subseteq B$ for all $t \in \hat{G}$, respectively),

[2] The symbol for the weight function $\hat{w}$ on $\hat{G}$ is choosen to make clear that it is defined on the dual group. It should not be confused with the Fourier transform of w, which is not used anywhere in this paper (and would not make sense in the classical sense, but only as a distribution).

the operators T_y and M_t act boundedly on B, and we write $|||T_y|||_B$ and $|||M_t|||_B$ for the corresponding operators norms.

The most interesting class of such spaces for us will be the Fourier transforms of weighted, solid (e.g. rearrangement invariant) BF-spaces on G. Recall that a Banach space $(B, \| \ \|_B)$ is called a **BF-space** on G if it is continuously embedded into the space $L^1_{loc}(G)$ of locally integrable functions on G (endowed with the family of seminorms $f \mapsto \int_K |f(x)|dx$, $K \subseteq G$ compact). B is **solid** if $g \in L^1_{loc}(G)$, $f \in B$ and $|g(x)| \leq |f(x)|$ l.a.e. (locally almost everywhere) implies $g \in B$ and $\|g\|_B \leq \|f\|_B$ (equivalently: if B is a (pointwise) module over $L^\infty(G)$). B is called **rearrangement invariant** if $|\{x \mid |g(x)| \geq \alpha\}| = |\{x \mid |f(x)| \geq \alpha\}|$ for all $\alpha > 0$ (here the outer $|\cdot|$ indicates: Haar measure of the corresponding set) implies $\|g\|_B = \|f\|_B$. It is clear that such spaces are **isometrically translation invariant**, i.e. satisfy $\|T_y f\|_B = \|f\|_B$ for all $f \in B$, $y \in G$. Moreover, they have **continuous translation** (i.e. $\lim_{y \to 0} \|T_y f - f\|_B = 0$ for all $f \in B$) if $C_c(G)$ is a dense subspace of $(B, \| \cdot \|_B)$ (cf. [31], [15]).

In the sequel we shall be mainly interested in weighted L^p-spaces, or more generally in solid BF-spaces on G which are of the form $B_m := \{f \mid fm \in B\}$, with norm $\|f\|_{B,m} := \|fm\|_B$. Here we assume that B is a solid, isometrically translation invariant BF-space on G, containing $C_c(G)$ as a subspace, and that m is a **moderate**, strictly positive and continuous function on G, i.e. which satisfies $m(x+y) \leq w(y)m(x)$ for $x, y \in G$ and some weight function w. We shall call m **w-moderate** in this case, and consider again only weights satisfying (BD). For the norm of T_y on B_m we then have $|||T_y|||_{B_m} \leq w(y)$ for $y \in G$. If, furthermore, $(B, \| \cdot \|_B)$ contains $C_c(G)$ as a dense subspace, $C_c(G)$ is also dense in $(B_m, \| \cdot \|_{B,m})$, and therefore B_m has continuous translation in this case. Applying vector-valued integration one derives therefrom that B_m is a Banach convolution module over the Beurling algebra $L^1_w(G)$. In this case, $(B_m, \| \cdot \|_{B,m})$ is an **admissible** BF-space in the sense of §4 below.

For further generalities concerning harmonic analysis see [29], [40], [38]. For basic results on Euclidean Fourier analysis cf. [28], [41], [44], [2] et al.. For results on **homogeneous Banach** spaces, **quasimeasures**, multipliers, and the relevant (elementary) theory of **Banach modules** see [28], [30], [12], [22], [17], [6]. For generalities on **interpolation theory** see [1], [9], [44], [34].

Occasionally it will be convenient to write $\|f|B\|$ instead of $\|f\|_B$. Positive constants are denoted by C, C_1, C',... The same symbol may denote different constants at different places.

3 Some results on Wiener-type spaces

(Equivalent characterizations, dependence on test functions, a Hausdorff-Young theorem, a maximal function theorem)

Since many of the basic properties of modulation spaces to be discussed below are immediate consequences of the corresponding properties of Wiener-type spaces (as introduced in [20]), we shall recall shortly some facts about this family spaces. We shall also prove several new results on Wiener-type spaces in this section, which are of interest for themselves, but which will serve as auxiliary assertions for the main results of this paper.

Given a Banach space $(B, \|\cdot\|_B)$ of distributions on a locally compact group $\hat{G}$ which is in standard situation w.r.t. $A_w(\hat{G}) = \mathcal{F}[L^1_w(G)]$, and a continuous, moderate function v on $\hat{G}$ we can describe the Wiener-type space $W(B, L^q_v)(\hat{G})$ as follows:
Fixing any 'test function' $g \in A_w(\hat{G}) \cap C_c(\hat{G})$, $g \neq 0$, we have:

$$W(B, L^q_v) := \{f \mid f \in B_{loc},\ F^{(g)} : t \mapsto \|(T_t g)f\|_B \in L^q_v(\hat{G})\},$$

$$\text{and } \|f \mid W(B, L^q_v)\| := \left(\int_{\hat{G}} |F^{(g)}(t)|^q v^q(t)\right)^{1/q}, \quad \text{for } 1 \leq q < \infty$$

(or $\sup_{t\in\hat{G}} |F^{(g)}(t)|v(t)$ for $q = \infty$).

Here B_{loc} is the set of all distributions (members of the dual of $A_w(\hat{G}) \cap C_c(\hat{G})$) which belong to B locally, and hence $(T_t g)f$ belongs to B for all $t \in G$. Using this definition (slightly different but much more convenient than the original one given in [20]) one shows that different test functions g define the same space (for B, q, and v fixed) and equivalent norms. ([20], Remark 2). Moreover, there is an equivalent 'discrete' characterization.

Let us call a family $(\psi_i)_{i\in I}$ a **bounded, uniform partition of unity in** $A_w(\hat{G})$ if there exists some relatively compact set $\hat{Q} \subseteq \hat{G}$ such that

i) $\sup_{i \in I} \|\psi_i\|_{A_w(\hat{G})} < \infty$;

ii) $\sup \psi_i \subseteq t_i + \hat{Q}$ for $i \in I$;

iii) $\sup_{i \in I} |\{j \mid (t_i + \hat{Q}) \cap (t_j + \hat{Q}) \neq \emptyset\}| < \infty$.

Then $f \in B_{loc}$ belongs to $W(B, L^q_v)$ if and only if

$$\|f|D(\hat{Q}, B, l^q_v)\| := \left(\sum_{i \in I} \|f\psi_i\|^q_B v(t_i)^q \right)^{1/q} < \infty$$

(for $q = \infty$ one has to take $\sup_{i \in I} \|f\psi_i\|_B v(t_i)$).
Such discrete norms are already implicitly in [20] (cf. Theorem 2, and Remark 4) and they are treated in detail in [23]. There one can also find a (general) result concerning the equivalence of discrete and continuous (cf. [56]) norms, including the above assertion ([23], Theorem 4.3). Concerning Wiener-type spaces one has as special cases of theorems proved in [23] (e.g. Theorem 2.8) also the following assertion on duality:

If $q < \infty$ and if translation is continuous in B for any $f \in B$ with compact support, then $A_w(\hat{G}) \cap C_c(\hat{G})$ is dense in $B_A := \{f \mid f = \lim_\alpha f_\alpha \text{ in } B, \operatorname{supp} f_\alpha$ compact in $\hat{G}\}$, and $W(B, L^q_v) = W(B_A, L^q_v)$. Therefore $A_w(\hat{G}) \cap C_c(\hat{G})$ is dense in $W(B, L^q_v)$ (cf. [20], Theorem 1) and

$$W(B, L^q_v)' = W(B', L^{q'}_{1/v}).$$

Using either the discrete representation $f = \sum_{i \in I} f\psi_i$, or a continuous variant of it (described in [21]), one shows that these spaces behave nicely with respect to complex interpolation (c.f. [21], [23]).

Concerning the independence of the norm from the particular choice of g, $g \neq 0$, we give a slight extension of known results, which will be very useful for us later (cf. [16] for hints in this direction).

Proposition 3.1 *Given any $\hat{w}$-moderate, continuous function v on $\hat{G}$, any non-zero function $g \in W(A_w, L^1_{\hat{w}})$ is an* **admissible** *test function for $W(B, L^q_v)$ as above (i.e. the norm obtained by using g is still equivalent to the norms considered above).*

Proof. For $g, h \in W(A_w, L^1_{\hat{w}}) \subseteq A_w$ one has of course

$$F^{(hg)}(t) \leq \|h\|_{A_w} F^{(g)}(t)$$

for $t \in \hat{G}$, by the A_w-module structure of B. Given g it will be sufficient to choose $h \in A_w \cap C_c(\hat{G})$ in order to have $\|F^{(h_1)}\|_{q,v} \leq \|h\|_{A_w} \|F^{(g)}\|_{q,v}$ for $h_1 := hg \in A_w \cap C_c(\hat{G})$.

In order to obtain the converse estimate we may use the representation $g = \sum_{i \in I} g\psi_i$, for which we have

$$\sum_{i \in I} \|g\psi_i\|_{A_w} \hat{w}(t_i) < \infty.$$

Then one has for $h_2 \in A_w \cap C_c(\hat{G})$ satisfying $h_2(t) = 1$ on $\hat{Q}$: $g = \sum_{i \in I} g\psi_i$ $(T_{t_i} h_2)$. It follows therefrom

$$F^{(g)} \leq \sum_i \|g\psi_i\|_{A_w} F^{(T_{t_i} h_2)} \leq \sum_i \|g\psi_i\|_{A_w} T_{-t_i} F^{(h_2)}.$$

The norm of T_t on L^q_v being bounded by $\hat{w}(t)$ we obtain therefrom

$$\|F^{(g)}\|_{q,v} \leq \left(\sum_i \hat{w}(t_i) \|g\psi_i\|_{A_w} \right) \|F^{(h_2)}\|_{q,v}.$$

Since it is already known that the norms $f \mapsto \|F^{(h_1)}\|_{q,v}$ and $f \mapsto \|F^{(h_2)}\|_{q,v}$ are equivalent, the proof is complete (cf. [20], Theorem 1).

As a preparation for the general Hausdorff-Young type equalities (i.e. statements concerning the isomorphism of certain Wiener-type spaces related to L^p-spaces under the Fourier transformation) to be proved below we give the following lemma:

Lemma 3.2 *Let $(B, \|\cdot\|_B)$ be a solid, translation invariant BF-space on a lc group G (containing $C_c(G)$ as a dense subspace). Then one has:*

a) There exists a weight function $\bar{w}$ on G such that

$$W(C^0, L^1_{\bar{w}}) \hookrightarrow B \hookrightarrow W(M, L^\infty_{1/\bar{w}}).$$

If translation is isometric in B one may take $\bar{w} \equiv 1$.

b) If the following condition is satisfied (Beurling-Domar):

$$(BD) \qquad \sum_{n=1}^{\infty} n^{-2} \log |||T_{ny}|||_B < \infty \text{ for all } y \in G,$$

it follows therefrom that $\Lambda^K_{\bar{w}}(G) \hookrightarrow B \hookrightarrow [\Lambda^K_{\bar{w}}(G)]'$, *where*

$$\Lambda^K_{\bar{w}}(G) = \{f \mid f \in L^1_{\bar{w}}(G),\ \mathrm{supp}\hat{f} \text{ compact in } \hat{G}\},$$

endowed with the natural inductive limit topology.

Proof. By the assumptions we have $C_c(G) \subseteq B \subseteq L^1_{loc}(G)$. The closed graph theorem implies that we have: Given a compact set $Q \subseteq G$ there exists $C_Q > 0$ such that

$$C_Q^{-1}\|f\|_\infty \leq \|f\|_B \leq C_Q\|f\|_1 \text{ for } f \in C_c(G),\ \mathrm{supp} f \subseteq Q.$$

On the other hand, the density of $C_c(G)$ in B implies that translation is continuous. Therefore $y \mapsto |||T_y|||_B$ is an upper semicontinuous (hence measurable) and submultiplicative function on G. Consequently $\bar{w} : \bar{w}(x) := \max(1, |||L_x|||_B)$ defines a weight function on G (cf. [15]). Observing that any f has a representation $f = \sum_i T_{t_i}(T_{-t_i}(f\psi_i))$ with $\mathrm{supp} T_{-t_i}(f\psi_i) \subseteq Q$, one can see that assertion *a)* follows from the above estimate (for the 'atoms'). In order to prove *b)* suppose now that $\bar{w}$ satisfies (BD). Then $A_{\bar{w}}(\hat{G}) := \{\hat{f} \mid f \in L^1_{\bar{w}}(G)\}$ is a regular Banach algebra on $\hat{G}$, and $A_{\bar{w}}(\hat{G}) \cap C_c(\hat{G})$ is dense in $C_c(\hat{G})$ and $A_{\bar{w}}(\hat{G})$. Since Theorem 3 of [20] implies that $W(C^0, L^1_{\bar{w}})$ is a dense ideal in $L^1_{\bar{w}}$ one can derive therefrom that one has

$$A_{\bar{w}}(\hat{G}) \cap C_c(\hat{G}) \subseteq \mathcal{F}[W(C^0, L^1)], \text{ or } \Lambda^K_{\bar{w}} \hookrightarrow W(C^0, L^1_{\bar{w}})$$

(cf. Theorem 5 of [20]), as a dense subspace. Applying the duality formula (Theorem 2.8 of [23]) we obtain $W(M, L^\infty_{1/\bar{w}}) \hookrightarrow (\Lambda^K_{\bar{w}})'$, as was required for the second inclusion.

Remark 3.1 *It would have been sufficient as well (in the above lemma) to suppose (instead of density of $C_c(G)$ in B) that B is a closed subspace of a dual D' of a solid space D containing $C_c(G)$ as a dense subspace. In fact, as the supremum of a family of continuous functions, $y \mapsto |||T_y|||_B$ is still measurable in that case.*

Remark 3.2 *It will be relevant for obtaining some of the results below in their 'natural' generality to observe that it would have been sufficient above to suppose that B is a space in 'standard situation' with respect to some 'nice' homogeneous Banach algebra $(A, \|\cdot\|_A)$ on G, as considered in §3 of [6] (cf. also [23], [20]). Under this assumption one would obtain:*

$$W(A, L^1_{\bar{w}}) \hookrightarrow B \hookrightarrow W(A', L^\infty_{1/\bar{w}}).$$

Assuming again (BD) and a somewhat strengthened version of Theorem 5 of [20] gives us assertion b) from above as well.

Remark 3.3 *For $(B, \|\cdot\|_B)$ as in Lemma 3.2 the convolution of $f \in B$ with $g \in W(A, L^1_{\bar{w}})$ is well defined and gives a continuous function. In fact, Theorem 3 of [20] implies:*

$$g * f \ \in\ W(A, L^1_{\bar{w}}) \ * \ W(A', L^\infty_{1/\bar{w}}) \ \subseteq\ W(C^b, L^\infty_{1/\bar{w}}) \ =\ C^b_{1/\bar{w}}(G).$$

Since the choice $A = A_{\hat{w}}(G) = \{f \mid f = \hat{h},\ h \in L^1_{\hat{w}}(G)\} = \mathcal{F}L^1_{\hat{w}}(G)$ will be the most natural choice we shall now shortly collect some information concerning the Wiener-type space $W(\mathcal{F}L^1_w, L^1_{\hat{w}})(\hat{G})$. As will be seen these spaces are weighted variants of the Segal algebra $S_0(\hat{G})$ as treated in [18].

Proposition 3.3 *Let w and $\hat{w}$ be weight functions on G and $\hat{G}$ respectively, both satisfying the Beurling-Domar condition. Then the Wiener-type spaces $W(A_{\hat{w}}, L^1_w)(G)$ and $W(A_w, L^1_{\hat{w}})(\hat{G})$ are well defined and the Fourier transform $\mathcal{F}_G :\ L^1(G) \to A(\hat{G})$ establishes an isomorphism between these spaces.*

Proof. (a) The spaces $L^1_w(G)$ and $L^1_{\hat{w}}(\hat{G})$ are Banach convolution algebras, and as solid BF-spaces they may of course be considered as pointwise $A(G)$ or $A(\hat{G})$-modules respectively. Since the Fourier transformation interchanges the role of convolution and multiplication it is clear that $A_{\hat{w}}(G) = \mathcal{F}_{\hat{G}}\left(L^1_{\hat{w}}(\hat{G})\right)$ and $A_w(\hat{G}) = \mathcal{F}_G\left[L^1_w(G)\right]$ are pointwise (nice) Banach algebras (because (BD) is satisfied), as well as L^1-modules (i.e. homogeneous Banach spaces). Consequently they are in 'standard situation' over themselves and the corresponding Wiener-type spaces are well defined.

(b) In order to show the claimed isomorphism it will be sufficient (for reasons of symmetry) to verify the inclusion $\mathcal{F}_G\left(W(A_{\hat{w}}, L^1_w)\right) \subseteq W(A_w, L^1_{\hat{w}})$.

Using the discrete characterization of Wiener-type spaces we can find for any $f \in W$ a sequence $(f_n)_{n\geq 1}$ in $A_{\hat{w}}(G)$, a relatively compact set $Q \subseteq G$ and a sequence $(y_n)_{n\geq 1}$ in G such that $\mathrm{supp} f_n \subseteq Q$,

$$f = \sum_{n\geq 1} T_{y_n} f_n, \quad \text{and}$$

$$\sum_{n\geq 1} \|f_n\|_{A_{\hat{w}}} w(y_n) \leq C \|f \mid W(A_w, L^1_{\hat{w}})\|.$$

For the Fourier transform $\hat{f} = \mathcal{F}_G f$ of f this implies:

$$\begin{aligned} \|\hat{f} \mid W(A_w, L^1_{\hat{w}})\| &\leq \sum_n \|(T_{y_n} f_n)^\wedge \mid W(A_w, L^1_{\hat{w}})\| \\ &\leq \sum_n w(y_n) \|\hat{f}_n \mid W(A_w, L^1_{\hat{w}})\| =: (*). \end{aligned}$$

The last estimate follows from the formula $(T_y f)^\wedge = M_y \hat{f}$, where M_y denotes multiplication with the character $y \in (\hat{G})^\wedge = G$, and the fact that the boundedness of M_y on $W(A_w, L^1_{\hat{w}})$ follows from its boundedness on $A_w(\hat{G})$, where one has

$$\|M_y \hat{f}\|_{A_w} = \|T_y f\|_{1,w} \leq w(y) \|f\|_{1,w} = w(y) \|\hat{f}\|_{A_w}.$$

In order to continue the estimate let us choose some $g \in W(A_w, L^1_{\hat{w}})$, satisfying $\hat{g}(x) \equiv 1$ on Q. This is possible because $\hat{w}$ satisfies (BD), $W(A_w, L^1_{\hat{w}})$ is a dense Banach ideal in $L^1_{\hat{w}}(G)$. Consequently one has

$$\hat{f}_n = (\hat{g} \cdot f_n)^\wedge = g * \hat{f}_n.$$

Applying now the general convolution formula (Theorem 3 of [20]) for Wiener-type spaces one obtains:

$$\begin{aligned} (*) &\leq \sum_n w(y_n) \, \|\hat{f}_n * g \mid W(A_w, L^1_{\hat{w}})\| \\ &\leq \sum_n w(y_n) \, \|g \mid W(A_w, L^1_{\hat{w}})\| \, \|\hat{f}_n\|_{1,\hat{w}} \\ &\leq \|g \mid W(A_w, L^1_{\hat{w}})\| \sum_n w(y_n) \, \|f_n\|_{A_{\hat{w}}} \\ &\leq C^1 \, \|f \mid W(A_{\hat{w}}, L^1_w)\|. \end{aligned}$$

The proof is now complete.

By duality we obtain immediately the following result:

Corollary 3.4 *Let $w, \hat{w}$ be as above. Then the Fourier transform extends to an isomorphism between $W(\mathcal{F}(L^\infty_{1/\hat{w}}), L^\infty_{1/w})$ and $W(\mathcal{F}(L^\infty_{1/w}), L^\infty_{1/\hat{w}}))$.*

Proof. In view of the fact that the dual of a Wiener-type space may be obtained (under the assumption that the test functions are dense in the space, which is the case for $W(A_w, L^1_{\hat{w}})$, see [23], Theorem 2.8) by taking the dual space in each component one has $W(\mathcal{F}(L^1_w), L^1_{\hat{w}})' = W(\mathcal{F}(L^\infty_{1/w}), L^\infty_{1/\hat{w}}))$. The extended Fourier transform being defined by transposition, it is clear that the isomorphism of the preduals induced by the Fourier transformation gives rise (in fact: extends) to an isomorphism of the dual spaces. This gives the corollary.

Applying now complex interpolation we obtain the following general result:

Theorem 3.5 *Let $w, \hat{w}$ be weight functions on G and $\hat{G}$, both satisfying the Beurling-Domar condition. Furthermore, let $\alpha, \beta \in \mathbf{R}$, and $p \in (1, \infty)$ be given. Then the Fourier transform $\mathcal{F}_G$ extends to an isomorphism between $W(\mathcal{F}(L^p_{\hat{w}^\alpha}), L^p_{w^\beta})$ and $W(\mathcal{F}(L^p_{w^\beta}), L^p_{\hat{w}^\alpha})$.*

Proof. First we observe that the spaces involved are well defined (cf. [22], §2). In view of the above results and the characterization of interpolation spaces within the family of Wiener-type spaces (cf. [21]), it will be sufficient to show how the spaces in question are obtained by complex interpolation. Assuming for the moment that α is positive and that β is negative (the arguments for the other cases being similar). In fact, choosing $\theta = 1/p$ we have

$$W(\mathcal{F}L^p_{\hat{w}^\alpha}, L^p_{w^\beta}) = \left[W(\mathcal{F}(L^1_{\hat{w}^{\alpha p}}, L^1), W(\mathcal{F}L^\infty, L^\infty_{1/w^{\beta p'}})\right]_{[\theta]}$$

and consequently

$$\begin{aligned} \mathcal{F}_G W(\ldots) &= [\mathcal{F}_G W(\ldots), \mathcal{F}_G W(\ldots)]_{[\theta]} = \\ &= [W(\mathcal{F}L^1, L^1_{\hat{w}^{\alpha p}}), W(\mathcal{F}L^\infty_{1/w^{\beta p'}}, L^\infty)]_{[\theta]} = W(\mathcal{F}L^p_{w^\beta}, L^p_{\hat{w}^\alpha}), \end{aligned}$$

as was claimed. In fact, the use of the above results is justified since $w^{\alpha p}$ and $w^{\beta p'}$ are again weight functions satisfying (BD).

The case $p = 2$ of the above theorem will be of special interest, as it gives information concerning potential spaces and their pointwise multipliers (cf. [10], p. 8/9 for related assertions). Writing $\mathcal{L}^2_s$ for the space of Bessel potentials ($= \mathcal{F}^{-1}L^2_s(\mathbf{R}^m)$) we have:

Corollary 3.6 *For $s \in \mathbf{R}$ we have*

$$\mathcal{L}^2_s(\mathbf{R}^m) = W(\mathcal{L}^2_s, L^2)(\mathbf{R}^m).$$

For $s > m/2$ it follows that a (continuous) function h defines a pointwise multiplier if and only if $h \in W(\mathcal{L}^2_s, L^\infty)$.

Proof. We have for $\bar{w}_s$ with $\bar{w}_s(x) = (1 + |x|^2)^{s/2}$

$$\mathcal{L}^2_s \;= \mathcal{F}_{\mathbf{R}^m}[L^2_{\bar{w}_s}(\mathbf{R}^m)] = \mathcal{F}_{\mathbf{R}^m}[W(\mathcal{F}L^2, L^2_{\bar{w}_s})] = W(\mathcal{F}L^2_{\bar{w}_s}, L^2) = W(\mathcal{L}^2_s, L^2).$$

Since $L^2_{\bar{w}_s}(\mathbf{R}^m)$ is a Banach convolution algebra for $s > m/2$ (in this case $L^2_{\bar{w}_s} \hookrightarrow L^1(\mathbf{R}^m)$, cf. [15]) the second assertion follows from Cor.2.14 of [23].

Further complex interpolation between the result established above and the Hausdorff-Young inequality proved in [21] (Theorem 3.2) yields:

Theorem 3.7 *Let $w, \hat{w}, \alpha, \beta$ and p be as in Thm.3.5. Then for $r \in (1, p]$:*

$$\mathcal{F}_G\,[W(\mathcal{F}(L^p_{\hat{w}^\alpha}), L^r_{w^\beta})(G)] \quad \subseteq \quad W(\mathcal{F}(L^r_{w^\beta}), L^p_{\hat{w}^\alpha})(\hat{G}).$$

Proof. Starting with the observation that one has for $\theta \in (0, 1)$, satisfying $1/r = (1 - \theta)/p + \theta$.

$$W(\mathcal{F}(L^p_{\hat{w}^\alpha}), L^r_{w^\beta}) = [W(\mathcal{F}(L^p_{w_1}), L^p_{w_2}), W(\mathcal{F}L^p, L^1)]_{[\theta]},$$

where $w_1 = \hat{w}^{\alpha/(1-\theta)}$, and $w_2 = w^{\beta/\theta}$. The proof is straightforward.

We conclude this section with two typical results concerning Wiener-type spaces (based on methods developed already in [20]). The first one will be of importance in §5, and the second one will give the possibility of using certain maximal functions on arbitrary lca groups. We show the equivalence of certain norms, without making use of any direct analogue of the Hardy-Littlewood maximal function theorem.

Proposition 3.8 *For any admissible solid BF-space $(B, \| \cdot \|_B)$ on a lca group, and any compact set $\hat{Q} \subseteq \hat{G}$ there exists $C' = C'(\hat{Q}, B)$ such that for any $f \in B$ satisfying supp$\hat{f} \subseteq t + \hat{Q}$ for some $t \in \hat{G}$ the following is true:*

$$\|f|B\| \leq \|f|W(L^\infty, B)\| \leq C'\|f|B\|.$$

Proof. We observe that by the convolution theorem for Wiener-type spaces (i.e. by Theorem 3 of [20]) $W(C^0, L^1_w)$ is a dense Banach ideal in $L^1_w(G)$ (with respect to convolution; otherwise expressed: an abstract Segal algebra in $L^1_w(G)$). Since we have a symbolic calculus in Beurling algebras (i.e. local inversion is possible in $A_w(\hat{G})$, [11], or [38], Chap. VI, §3) the algebra $\mathcal{F}_G W(C^0, L^1_w)$ coincides locally with $A_w(\hat{G})$. The regularity of $A_w(\hat{G})$ now implies the existence of $g \in W(C^0, L^1_w)$, such that $\hat{g}$ has compact support and satisfies $\hat{g}(t) = 1$ for all $t \in \hat{Q}$. Consequently $(M_t g)^\wedge(\tilde{t}) = T_t\hat{g}(\tilde{t}) = 1$ for all $\tilde{t} \in t + \hat{Q}$, and thus $M_t g * f = f$ for f as in the lemma. Another applications of Theorem 3 of [20] implies then:

$$f = M_t g * f \in W(C^0, L^1_w) * B \subseteq W(C^0, L^1_w) * W(L^1, B) \subseteq W(C^0, B),$$

and

$$\|f|W(C^0, B)\| \leq \|M_t g|W(C^0, L^1_w)\| \; \|f|W(L^1, B)\| \leq \|g|W(C^0, L^1_w)\| \; \|f|B\|.$$

The first estimate being obvious the proof is complete.

We come now to the definition of maximal functions on lca groups:

Definition 3.1 *Given a continuous function f on G, and a continuous weight function w on G, we set:*

$$f^{\#(w)}(x) := \sup_{y \in G} \frac{|f(y)|}{w(x-y)}.$$

*$f^{\#(w)}$ will be called the w-***maximal function of f***.*

Theorem 3.9 *Let $(B, \|\cdot\|_B)$ be an admissible, solid BF-space on a lca group (which is a Banach module over $L^1_{\bar{w}}(G)$ for convolution), and let w be a continuous weight function on G, such that $\bar{w}w^{-1} \in L^1(G)$. Then there exists for any compact set $\hat{Q} \subseteq \hat{G}$ some $C = C(B, \hat{Q}, w) > 0$ such that one has for all $f \in B$, with* supp$\hat{f} \subseteq t + \hat{Q}$ *(for some $t \in \hat{G}$):*

$$\|f^{\#(w)}\|_B \leq C\|f\|_B.$$

Proof. Let $Q_0 \subseteq G$ be any symmetric ($Q_0 = -Q_0$), open and relatively compact subset of G, and let $(y_i)_{i\in I}$ be a discrete family in G, such that $Q := (y_i + Q_0)_{i\in I}$ is a uniform (hence admissible, cf. [23]) covering of G. The existence of such families is shown in [16]. Since $w^{-1}(z) \leq C_w w^{-1}(y_i)$ for $z \in y_i + Q_0$, and all $i \in I$, it is clear that one has

$$\begin{aligned} f^{\#(w)}(x) &\leq \sup_{i\in I} \max_{z\in y_i+Q_0} |f(x-z)|w^{-1}(z) \\ &\leq C_w \sup_{i\in I} w^{-1}(y_i) \max_{t\in Q_0} |T_{y_i} f(x-t)|. \end{aligned}$$

Writing f^∞ for the function, given by $f^\infty(x) := \max_{t\in Q_0} |f(x-t)|$ and observing that $(T_y f)^\infty = T_y(f^\infty)$ one has then

$$f^{\#(w)}(x) \leq C_w \sum_{i\in I} w^{-1}(y_i) T_{y_i} f^\infty(x).$$

Since the norm of T_y on B is less than $\bar{w}(y)$, we infer from the solidity of B:

$$\begin{aligned} \|f^{\#(w)}\|_B &\leq C_w \sum_{i\in I} w^{-1}(y_i) \|T_{y_i} f^\infty\|_B \\ &\leq C_w (\sum_{i\in I} w^{-1}(y_i)\bar{w}(y_i)) \|f^\infty\|_B. \end{aligned}$$

Observing now that the above sum is finite (e.g. by showing that the function $w' := \sum_{i\in I} w^{-1}(y_i)\bar{w}(y_i) c_{y_i} + Q_0$ is dominated by a scalar multiple of $w^{-1}\bar{w}$, which has been assumed to be integrable) we have by Proposition 3.7:

$$\begin{aligned} \|f^{\#(w)}\|_B &\leq C_w C''_w \|f^\infty\|_B \\ &= C_w C''_w \|f\|_{W(L^\infty,B)} \leq C_w C''_w C' \|f\|_B, \end{aligned}$$

and the proof of the theorem is complete.

Remark 3.4 *On $G = \mathbf{R}^m$ the weight functions w_s, given by $w_s(x) = (1 + |x|)^s$ are of course the most interesting ones. For rearrangement invariant spaces B on G, such as L^p-spaces or Lorentz- and Orlicz-spaces on $\mathbf{R}^m$ the above theorem is then applicable to show that*

$$\|f^{\#(w_s)}\|_B \leq C_s \|f\|_B,$$

for any $s > m$, and all $f \in B$ having a fixed (prescribed) diameter of their spectrum (i.e. of the support of their Fourier transforms, considered as a tempered distributions on $\mathbf{R}^m$). Taking the effect of dilations in $\mathbf{R}^m$ on the norm of f in $L^p(\mathbf{R}^m)$ and on the size of supp$\hat{f}$ into account one can also derive estimates for C as a function of the diameter of $\hat{Q}$ (for $B = L^p(\mathbf{R}^m)$, and $w = w_s$ as above).

Remark 3.5 *The above proof shows that at least some non-trivial estimates involving maximal functions (even on $\mathbf{R}^m$) can be obtained using neither estimates for some gradients nor the Hardy-Littlewood maximal functions. On the other hand, our results are available for arbitrary lca groups.*

4 Modulation spaces and basic properties

In this section the general definition of **modulation spaces** and some of their basic properties are given. We shall assume throughout this section that $(B, \|\cdot\|_B)$ is an **admissible**, solid BF-space on a lca group G, i.e. that there is a weight function w on G, satisfying the (BD)-condition, and such that $|||T_y|||_B \leq w(y)$ for $y \in G$, and such that $(B, \|\cdot\|_B)$ is a Banach convolution module over $L^1_w(G)$. This is the case, for example, if $C_c(G)$ is dense in a translation invariant, solid BF-space $(B, \|\cdot\|_B)$ and $\sum_{n=1}^{\infty} n^{-2} \log |||T_{ny}|||_B < \infty$ for any $y \in G$. For such spaces we define now:

Definition 4.1 *Let G be a locally compact Abelian group, and let w and $\hat{w}$ be continuous weight functions on G and $\hat{G}$ respectively, both satisfying (BD). Given $(B, \|\cdot\|)_B$ as described above, and a continuous $\hat{w}$-moderate function v on $\hat{G}$ we define for $1 \leq q \leq \infty$, fixing any*

$$k \in \Lambda^K_w(G) = \{f \mid f \in L^1_w(G), \ \mathrm{supp}\hat{f} \ \ \textit{compact in} \ \ \hat{G}\}, k \neq 0 :$$

$$\begin{aligned} M(B, L^q_v) \;\; := \quad & \{\sigma \in (\Lambda^K_w)', \;\; M_t k * \sigma \in B \;\; \textit{for all} \;\; t \in \hat{G}, \\ & \textit{and} \quad t \mapsto \|M_t k * \sigma\|_B \;\; \textit{belongs to} \;\; L^q_v(\hat{G})\}, \end{aligned}$$

and the expressions

$$\|f|M(B,L^q_v)\| := \left[\int_{\hat{G}} \|M_t k * \sigma\|^q_B v^q(t)dt\right]^{1/q}$$

for $1 \le q < \infty$*, and*

$$\|f|M(B,L^\infty_v)\| := \sup_{t\in\hat{G}} \|M_t k * \sigma\|_B v(t)$$

are the natural norms on $M(B,L^q_v)$*. The space* $M(B,L^q_v)$ *will be called* **modulation space***,* **derived from** B*,* **with degree of smoothness** L^q_v*.*

After the preliminaries given in sections 2 and 3 it is clear that we have the following basic results:

Theorem 4.1 *A) The spaces* $M(B,L^q_v)$*,* $1 \le q \le \infty$*, are Banach spaces (of distributions on G) with respect to their natural norms, and the embeddings* $\Lambda^K_w \hookrightarrow M(B,L^q_v) \hookrightarrow (\Lambda^K_w)' \hookrightarrow M(B,L^q_v)$ *are continuous (where each of these spaces being endowed with its natural topology).*

B) The spaces $M(B,L^q_v)$ *do not depend on the particular choice of the test functions* $k \in C_c(G) \cap A_{\hat{w}}(G)$ *(i.e. different test functions define the same space and equivalent norms). Furthermore these spaces do not depend on the weights* $w, \hat{w}$ *(satisfying BD) involved in the definition.*

C) For any $\hat{K} \subseteq \hat{G}$*, compact, the norms of B and these of* $M(B,L^q_v)$ *are equivalent on* $\{f \mid f \in B,\ \mathrm{supp}\hat{f} \subseteq \hat{K}\}$*.*

D) If $C_c(G)$ *is a dense subspace of* $(B, \|\cdot\|_B)$*, then* Λ^K_w *is a dense subspace of* $M(B,L^q_v)$ *for* $1 \le q < \infty$*, and translation is continuous in* $M(B,L^q_v)$ *in this case.*

E) For B and q as in D) above the dual space of $M(B,L^q_v)$ *can be identified with* $M(B', L^{q'}_{1/v})$*.*

Proof. A+B) Since $\hat{w}$ satisfies (BD) there exists $k \ne 0$, $k \in A_{\hat{w}}(G) \cap C_c(G)$. Applying the extended Fourier transform (which establishes an isomorphism between Λ^K_w and $(A_w)_0 = A_w \cap C_c(\hat{G})$, and between $(\Lambda^K_w)'$ and $(A_w)'_0$ respectively) it is clear that $\mathcal{F}_G[M(B,L^q_v)]$ coincides with the Wiener-type space

$W(\mathcal{F}B, L^q_v)$ on $\hat{G}$. In fact, by the convolution theorem one has $\mathcal{F}_G(M_t k * \sigma) = (T_t \hat{k})\hat{\sigma}$, where $\hat{k}$ belongs to

$$\mathcal{F}_G(A_w)_0 \subset \mathcal{F}_G(W(\mathcal{F}L^1_{\hat{w}}, L^1_w)) = W(\mathcal{F}L^1_w, L^1_{\hat{w}})$$

by Proposition 3.3. But then the norm of $t \mapsto \|(T_t\hat{k})\hat{\sigma}\|_{\mathcal{F}B} = \|M_t k * f\|_B$ in L^q_v actually defines a norm on $W(\mathcal{F}B, L^q_v)$, for any such k, and different choices give equivalent norms by Proposition 3.1 (and our assumption concerning v). The continuity of the embedding

$$\Lambda^K_w \hookrightarrow M(B, L^q_v) \hookrightarrow (\Lambda^K_w)'$$

then follows from the embeddings (cf. [20], Theorem 1)

$$(A_w)_0 \hookrightarrow W(\mathcal{F}B, L^q_v) \hookrightarrow (A_w)'_0.$$

In a similar way the completeness of $M(B, L^q_v)$ follows from that of Wiener type spaces.

The verification of C) is left to the reader. For the proof of D) observe that $A_{\hat{w}}(G) \cap C_c(G)$ is dense in $C_c(G)$ (by (BD)), and thus in B as well. It follows therefrom that $W(A_{\hat{w}}, L^1_w)$ is a dense subspace of B (apply remark 3.1, i.e. a modification of Lemma 3.2, to $\mathcal{F}B$, and use Proposition 3.3 then). But $W(A_{\hat{w}}, L^1_w)$ is a dense essential Banach ideal of $L^1_w(G)$, and therefore contains the space $\{f \mid f \in L^1_w, \ \mathrm{supp}\hat{f} \text{ compact } \} = \Lambda^K_w$ as a dense subspace (cf. [39], Chapter VI, §2.2 for related assertions concerning Segal algebras=dense, essential Banach ideals of $L^1(G)$, i.e. for the case $w \equiv 1$). Combining these facts with Theorem 1, iv) of [20] the density of Λ^K_w in $M(B, L^q_v)$ follows. Due to the density of Λ^K_w and the continuity of translation in $(\Lambda^K_w, \|\cdot\|_B)$ the proof of D) is complete.

Finally, the characterization of $M(B, L^q_v)'$ is a consequence of the density of Λ^K_w in $M(B, L^q_v)$ which allows (combined with the Fourier transform) to apply Theorem 2.8 of [23].

Remark 4.1 *It is clear that equivalent $\hat{w}$-moderate functions v and v' (i.e. having both quotients bounded) define the same space and equivalent norms (e.g. $(1+|x|)^s \sim (1+|x|^2)^{s/2}$ on $\mathbf{R}^m$).*

Remark 4.2 *If $(B, \| \cdot \|_B)$ as in D) above has isometric translation, then $M(B, L^q_v)$, $1 \leq q < \infty$ is a homogeneous Banach space of locally integrable functions. In particular, the spaces $M(L^1, L^q_v)$, $1 \leq q < \infty$, are Segal Algebras (related spaces are discussed in [24]).*

Remark 4.3 *Our definition is related to Calderon's characterization of Besov spaces (cf. [9], [43] and others). There is also some analogue to Calderon's representation formula, which reads in our setting (with a suitable interpretation of the vector-valued integral)*

$$\sigma = \int_{\hat{G}} M_t k * \sigma dt,$$

for k being suitably normed (cf. [21]). We shall not use this formula here.

There is also a discrete characterization of modulation spaces (very useful in practice), making use of a uniform decomposition of the Fourier transforms of their elements.

Corollary 4.2 *Given any bounded, uniform partition of unity $\Psi = (\psi_i)_{i \in I}$ in $A_w(\hat{G})$, (satisfying $\mathrm{supp}\psi_i \subseteq t_i + \hat{Q}_0$, for some relatively compact set $\hat{Q}_0 \subseteq \hat{G}$) we may characterize $M(B, L^q_v)$ as follows:*

$$M(B, L^q_v) \;\; = \;\; \{\sigma \mid \sigma \in (\Lambda^K_w)', \; \mathcal{F}_G^{-1}[\psi_i \mathcal{F}_G \sigma] \in B \;\; \text{for all} \;\; i \in I,$$

$$\|\sigma\| := [\sum_{i \in I} \|\mathcal{F}_G^{-1}[\psi_i(\mathcal{F}_G \sigma)]\|^q_B \, v^q(t_i)]^{1/q} < \infty\},$$

(with a supremum for $q = \infty$), and $\| \cdot \|$ defines an equivalent norm on $M(B, L^q_v)$.

Writing $\varphi_i := \mathcal{F}_G^{-1} \psi_i$ and applying (the maximal function) Theorem 3.9 we obtain another equivalent norm (cf. [46], §2.1.2 for a related result for Besov spaces).

Corollary 4.3 *Let $(\varphi_i)_{i \in I}$ be as above, then for any weight function $\bar{w}$ on G satisfying $\bar{w} v^{-1} \in L^1(G)$ the expression*

$$\left[\sum_{i \in I} \|(\varphi_i * \sigma)^{\#(\bar{w})}\|^q_B \, v^q(t_i) \right]^{1/q}$$

(or the corresponding supremum) defines an equivalent norm on $M(B, L^q_v)$.

There is also an 'atomic' characterization of $M(B, L^q_v)$, showing that modulation spaces on $\mathbf{R}$ might be considered as special cases of spaces as defined by Goldman ([26]). In fact, applying Corollary 2.6 of [23] we have:

Corollary 4.4 *Given any uniform covering $\hat{Q} = (t_i + \hat{Q}_0)_{i\in I}$ of $\hat{G}$, then one has $f \in M(B, L^q_v)$ $(1 \le q < \infty)$ if and only if there is a family $(f_i)_{i\in B}$ in B, such that* $\mathrm{supp}\hat{f}_i \subseteq t_i + \hat{Q}_0$*, $f = \sum_{i\in I} f_i$ (as a distribution), and*

$$\left[\sum_{i\in I} \|f_i\|_B^q \, v^q(t_i)\right]^{1/q} < \infty.$$

Moreover, the infimum over all such expressions defines an equivalent norm on $M(B, L^q_v)$ (again with obvious modifications for $q = \infty$).

The isomorphism to Wiener-type spaces given by the Fourier transform also allows to write down the result on complex interpolation immediately. as a typical application let us describe the case $B = L^p_m(G)$, for w-moderate, continuous functions m_i on G, $i = 1, 2$.

Theorem 4.5 *Given $1 \le p_1, q_1 < \infty$, $1 \le p_2, q_2 \le \infty$, and $\theta \in (0, 1)$ one has*

$$\Big(M(L^{p_1}_{m_1}, L^{q_1}_{v_1}), M(L^{p_2}_{m_2}, L^{q_2}_{v_2})\Big)_{[\theta]} = M(L^p_m, L^q_v),$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad m = m_1^{1-\theta} m_2^{\theta}, \quad v = v_1^{1-\theta} v_2^{\theta}.$$

Proof. cf. [21], or [23].

For the treatment of various questions (some of which are discussed in [6] and [19], [22] respectively) it is also of interest to check that modulation spaces have a double module structure. In fact, they are even Banach spaces in '**standard situation**' as treated in [6]:

Theorem 4.6 *A) For any solid, admissible BF-space on a lca group G and v as in Definition 4.1 and $1 \le q \le \infty$, the space $M(B, L^q_v)(G)$ is a Banach module over $A_{\hat{w}}(G)$ with respect to pointwise multiplication, as well as a Banach module over $L^1_w(G)$ with respect to convolution.*

B) Furthermore one has the inclusions

$$(A_{\hat{w}})_0 \hookrightarrow M(B, L^q_v) \hookrightarrow (A_{\hat{w}})'_0.$$

Therefore, both algebras having bounded approximate units (and all properties required further), $M(B, L^q_v)$ is a Banach space in standard situation in the sense of [6], §3.

C) If $C_c(G)$ is dense in $M(B, L^q_v)$ and if $q < \infty$, then $(A_{\hat{w}})_0$ is a dense subspace of $M(B, L^q_v)$.

Proof.

A) $(B, \|\cdot\|_B)$ being a Banach module over $L^1_w(G)$ it is clear that we have the following estimate:

$$\|M_t k * g * f\|_B \le \|g\|_{1,w} \|M_t k * f\|_B.$$

This implies that $M(B, L^q_v)$ is a module over $L^1_w(G)$ as well. The $C^0(G)$ (hence $A(G)$-) module structure corresponds to a $L^1(\hat{G})$-convolution structure on $\mathcal{F}B$. By Theorem 3 of [20] we have a $W(L^1, L^1_{\hat{w}}) = L^1_{\hat{w}}(\hat{G})$ convolution structure on $W(\mathcal{F}B, L^q_v)$. Applying the inverse Fourier transform the pointwise $A_{\hat{w}}$-structure on $M(B, L^q_v)$ follows.

B) The arguments for these assertions are very similar to those in the proof of Theorem 4.1 (cf. also [6]) and therefore left to the reader (see also Remark 3.1, for $W(\mathcal{F}B, L^q_v)$).

C) There are various ways of verifying the density of $(A_{\hat{w}})_0 = A_{\hat{w}} \cap C_c(G)$. Either it follows from the density of $W(\mathcal{F}L^1, L^1_{\hat{w}})$ in $W(\mathcal{F}B, L^q_v)$ for $1 \le q < \infty$ (observe that one has

$$\mathcal{F}^{-1}_G[W(\mathcal{F}L^1, L^1_w)] \cap C_c(G) = A_{\hat{w}} \cap C_c(G),$$

cf. Theorem 4.1), or it follows from the fact that $M(B, L^q_v)$ is an essential Banach module with respect to both actions defined (by Theorem 4.2 of [6]). That the actions are essential follows in turn from the continuity of $y \to T_y f$ and $t \to M_t f$ from G and $\hat{G}$ to $M(B, L^q_v)$ respectively, for all $f \in M(B, L^q_v)$, by vector-valued integration.

Corollary 4.7 (Compactness criterion) *Let $(B, \| \cdot \|_B)$ be a solid, admissible BF-space, containing $C_c(G)$ as a dense subspace. Then one has for $1 \leq q < \infty$:*

A bounded subset $S \subseteq M(B, L^q_v)$ is relatively compact if and only if for any $\varepsilon > 0$ there exist compact sets $K, \hat{K}$ in G and $\hat{G}$ respectively, such that for any $f \in M(B, L^q_v)$ there exists $k_1, k_2 \in M(B, L^q_v)$ such that $\mathrm{supp} k_1 \subseteq K$, $\mathrm{supp} \hat{k}_2 \subseteq \hat{K}$*, and*

$$\|(f - k_i) \mid M(B, L^q_v)\| < \varepsilon \;\; \text{for} \;\; i = 1, 2.$$

Proof. In view of Theorem 4.6 above the main result (Theorem 2.2) of [25] is applicable. The description of tightness and equicontinuity of S in $M(B, L^q_v)$ given above is justified by Propositions 2.3 and 2.4 of [22].

Remark 4.4 *If $C_c(G)$ is dense in $(B, \| \cdot \|_B)$ the closure of $A_{\hat{w}} \cap C_c(G)$ in $M(B, L^\infty_v)$ coincides with*

$$M(B, C^0_v) := \{\sigma \mid \sigma \in M(B, L^\infty_v), \; \lim_{t \to \infty} \|M_t k * \sigma\|_B \cdot v(t) = 0\}.$$

The dual of $M(B, C^0_v)$ coincides of course with $M(B', L^1_{1/v})$, cf. again [23], Theorem 2.8.

Remark 4.5 *At the end of this section we mention that the solidity of the original space $(B, \| \cdot \|_B)$ is by no means relevant (although certainly natural) for the possibility of constructing the modulation spaces $M(B, L^q_v)$ and showing their basic properties (as above). It would be sufficient to have a pointwise $A_{\bar{w}}(G)$-module (with $\bar{w}$ a weight on $\hat{G}$, satisfying (BD)) on a Banach space $(B, \| \cdot \|_B)$ in standard situation.*

Remark 4.6 *In view of the above remark it would be possible to ask about reiterations of the construction method yielding modulation spaces. At least for solid, admissible BF-spaces $(B, \| \cdot \|_B)$ we can state the following result (which is easily verified using the discrete description of Corollary 4.2):*

$$M(M(B, L^r_{v_1}), L^q_{v_2}) = M(B, L^q_{v_1 v_2}).$$

5 The trace theorem

We shall prove in this section a trace theorem for modulation spaces in the perhaps most general reasonable way, i.e. for spaces of the form $M(B_m, L^q_v)$, where B is a rearrangement-invariant solid BF-space. In fact, such spaces are defined on arbitrary measure spaces, and in particular $B(G)$ and $B(H)$ are related in a canonical way, if H is a closed subgroup of a locally compact group G (typical examples are of course the L^p-spaces, or Lorentz-spaces $L(p,q)$, defined for any group).

To be more precise, we assume that a lca group G is of the form $G = H_1 \times H_2$, where H_1, H_2 are considered as closed subgroups of G. Then the restriction operator $R_1 : C^b(G) \to C^b(H_1)$ (R_2 denoted restriction to H_2) maps $A_{\hat{w}} \cap C_c(G)$ into $A_{\dot{w}} \cap C_c(H_1)$, where $\dot{w}(z) := \inf_{y \in \hat{H}_2} \hat{w}(z,y)$ (this follows from [38], III, 7.13 via the Fourier transformation), and it will be shown to extend to a continuous and surjective linear mapping between suitable pairs of modulation spaces on G and H_1 respectively, if we suppose that the degree of smoothness of the domain is high enough (i.e. that $L^q_v(\hat{G})$ is small enough). In particular, we shall have to show that there exist suitable extension operators in order to establish the surjectivity in each case.

The proof of our main theorem will be obtained by a series of auxiliary results. The basic idea is to prove it first for (smooth) function having compact spectrum (supp $\hat{f}$) and of "pasting together" the results for these "atoms". Again, heavy use of methods from the theory of Wiener-type spaces will be made. But also ideas, developed for the study of the "canonical mapping" $T_H : L^1(H' \times H) \to L^1(H')$ (cf. [38], III and VIII) as operator acting on weighted L^p-spaces (cf. [14]) will be used at least indirectly.

The main theorem of this section reads as follows:

Theorem 5.1 (Trace theorem) *Assume that G is of the form $G = H_1 \times H_2$, for H_1, H_2 being locally compact abelian groups. Let m be a w-moderate function, and v a $\hat{w}$-moderate function on G and $\hat{G}$ respectively (with w, $\hat{w}$ satisfying (BD)). Furthermore, let $B(G)$ and $B(H)$ be rearrangement invariant, solid BF-spaces, corresponding to each other in a canonical way. Assume that $(\hat{R}_2 v)^{-1} \in L^{q'}(\hat{H}_2)$ ($\hat{R}_2$: restriction to $\hat{H}_2$), and define $v\langle q \rangle$ and*

m_1 by $m_1(z) := R_1 m(z) = m(z,0)$, and

$$v\langle q\rangle(t) = \left[\int_{\hat{H}_2} v(t,s)^{-q'} d_{\hat{H}_2} s\right]^{-1/q'} .$$

Then the restriction mapping $R_1 : C^b(G) \to C^b(H_1)$ $(R_1 f(x) := f(x,0))$ induces a continuous, surjective mapping from $M(B_m, L^q_v)(G)$ onto $M(B_{m_1}, L^q_{v\langle q\rangle})$. For $q = 1$ one has to set $v\langle 1\rangle(t) := \inf_{s\in\hat{H}_2} v(t,s)$.

Remark 5.1 *As usual, one must agree to obtain $R_1 f$ via regularizations, if f happens to be not continuous near $H_1 \subseteq G$. In our proof $R_1 f$ will arise as a convergent sum in $M(B_{m_1}, L^q_{v\langle q\rangle})$. It would be possible as well to set $R_1 f := \lim_\alpha R_1(e_\alpha * f)$ (the second R_1 being ordinary restriction), for a suitable, smooth approximation $(e_\alpha)_{\alpha\in I}$ to the identity (e.g. various classical 'means' for the applications).*

As already mentioned the general trace theorem will be based on a series of auxiliary results which we shall state first. In order not to interrupt the general flow of ideas we shall give the proof of Theorem 5.1 immediately after this series of lemmata, which will be proved afterwards.

Lemma 5.2 *For any compact set $\hat{Q} \subseteq \hat{G}$ there exists $C'' = C''(\hat{Q}, m) > 0$ such that*

$$\|R_1 f \mid B_{m_1}(H_1)\| \le C''\|f \mid B_m\|$$

for all $f \in B_m(G)$ satisfying $\operatorname{supp} \hat{f} \subseteq t + \hat{Q}$ for some $t \in \hat{G}$.

Lemma 5.3 *There exists some weight function w_2 on H_2, satisfying the Beurling-Domar condition, and $C_1 > 0$ such that $f \otimes g \in B_m(G)$ for $f \in B_{m_1}(H_1)$ and $g \in W(C^b, L^1_{w_2})(H_2)$ and*

$$\|f \otimes g \mid B_m(G)\| \le C_1\|f \mid B_{m_1}\| \, \|g \mid W(C^b, L^1_{w_2})\|$$

Lemma 5.4 *For any pair $(\hat{Q}_1, \hat{Q}_2)$ of open, relatively compact subsets of $\hat{H}_1$ and $\hat{H}_2$ respectively, there exists $C > 0$ such that for any $f \in B_{m_1}(H)$ with $\operatorname{supp}\hat{f} \subseteq t_1 + \hat{Q}$ for some $t_1 \in \hat{H}_1$ there exists an extension $F \in B_m(G)$ satisfying*

$$\|F \mid B_m(G)\| \le C\|f \mid B_{m_1}(H_1)\|$$

and

$$\mathrm{supp}\hat{F} \subseteq (t_1 + \hat{Q}_1) \times \hat{Q}_2.$$

Lemma 5.5 *Let $\hat{Q}_1$, $\hat{Q}_2$ be open, relatively compact subsets of $\hat{H}_1$ and $\hat{H}_2$ respectively, and let $(t_l)_{l\in L}$ and $(s_r)_{r\in R}$ be (uniformly discrete) families in $\hat{H}_1$ and $\hat{H}_2$ respectively, such that the families $(t_l + \hat{Q}_1)_{l\in L}$ and $(s_r + \hat{Q}_2)_{r\in R}$ are uniform (hence admissible) coverings (of bounded height) for $\hat{H}_1$ and $\hat{H}_2$ respectively. Then there exists for $q \in (1, \infty]$ a positive numbers $\gamma > 0$, such that for each $l \in L$ there is a finite set $E_l \subseteq R$, and $\delta_l \leq \gamma$, such that*

$$2\delta_l \left[\sum_{r\in R} v^{-q'}(t_l, s_r)\right]^{-1/q'} \geq \delta_l \left[\sum_{r\in E_l} v^{-q'}(t_l, s_r)\right]^{-1/q'} = v\langle q\rangle(t_l).$$

Proof of the Theorem. A) **(restriction theorem)**

We start by observing that the assumptions imply that $v\langle q\rangle$, as defined, is again a moderate function (on $\hat{H}_2$) and that the corresponding weight (we write $\hat{w}_1$) satisfies the condition (BD) as well. The same can be said concerning m_1 (and the corresponding weight $w_1 = R_1 w$). Therefore $M(B_{m_1}, L^q_{v\langle q\rangle})$ is at least well defined.

We start by observing that there are bounded, uniform partitions of unity $(\psi^1_l)_{l\in L}$ and $(\psi^2_r)_{r\in R}$ in $A_{w_1}(\hat{H}_1)$ and $A_{w_2}(\hat{H}_2)$ respectively, $(w_2 := R_2 w)$ satisfying

$$\text{supp } \psi^1_l \subseteq t_l + \hat{Q}_1 \text{ for all } l \in L \text{ and } \text{supp } \psi^2_r \subseteq s_r + \hat{Q}_2 \text{ for all } r \in R.$$

Then the family $\{\psi^1_l \otimes \psi^2_r\}_{(l,r)\in L\times R}$ defines a bounded, uniform partition in $A_w(G)$. It will be convenient to set $\psi_{l,r} := \psi_l \otimes \psi_r$, $\varphi_l := \mathcal{F}^{-1}_{H_1}\psi_l$, and $\varphi_{l,r} := \mathcal{F}^{-1}_G \psi_{l,r}$. Recall that we have then

$$\|f|M(B_m, L^q_v)(G)\| = [\sum_{l,r} \|\varphi_{l,r} * f|B_m(G)\|^q v^q(t_l, s_r)]^{1/q},$$

which is equal to the norm of $\alpha := (\alpha_{l,r})_{(l,r)\in L\times R}$, given by $\alpha_{l,r} := \|\varphi_{l,r} * f|B_m)(G)\|$, in a weighted l^q -space on $L \times R$ (this space is denoted by l^q_v in a suggestive way). What we have to do is to give an estimate for $\beta = (\beta_l)_{l\in L}$, given by $\beta_l := \|\varphi_l * R_1 f|B_{m_1}\|$. Since one has

$\mathcal{F}_{H_1}(R_1 g) = T_{\hat{H}_2}(\mathcal{F}_G g)$ for $g \in C_c(G)$ (cf. [38], V, 5.4) it is clear by a limiting argument that one has for $f \in M(B_m, L^q_v)(G)$

$$\varphi_m * R_1(\varphi_{l,r} * f) = 0 \ \text{ if } \psi_m \psi_l = 0,$$

i.e. only $m \in L(l) := \{n \mid n \in L,\ \psi_n \psi_l \neq 0\}$ is of relevance.

Therefore one has for each $l \in L$

$$\begin{aligned} \varphi_l * R_1 f \quad &= \varphi_l * R_1(\sum_{n,r} \varphi_{n,r} * f) = \sum_{n,r} \varphi_l * [R_1(\varphi_{n,r} * f)] \\ &= \sum_r \sum_{n \in L(l)} \varphi_l * [R_1(\varphi_{n,r} * f)]. \end{aligned}$$

Taking into account the uniform structure of our covering of $\hat{G}$ and the moderateness of v it is clear that there exists $C_1 > 0$ such that for all $y \in \hat{H}_2$, $l \in L$,

$$C_1^{-1} v(t_n, y) \leq v(t_l, y) \leq C_1 v(t_n, y) \ \text{ whenever } \ n \in L(l).$$

It follows therefrom that the weighted l^q-space l^q_v is regular (in the sense of [23], with respect to the uniform covering of $\hat{G}\{(t_l, s_r) + \hat{Q}_1 \times \hat{Q}_2\}$), and therefore we can give estimate of the form

$$\|\alpha^* | l^q_v\| \leq C_2 \|\alpha | l^q_v\| = C_2 \|f | M(B_m, L^q_v)\|,$$

given by $\alpha^*_{l,r} := \sum_{n \in L(l)} |\alpha_{n,r}|$. Applying now the norm of $B_{m_1}(H_1)$ to the above identity one obtains:

$$\begin{aligned} \beta_l \quad &\leq \quad \sum_{r \in R} \sum_{n \in L(l)} \|\varphi_l\|_{1,w_1} \|R_1(\varphi_{n,r} * f) | B_{m_1}(H_1)\| \\ &\overset{\text{by Lemma 5.2}}{\leq} \quad C_3 (\sup_{l \in L} \|\varphi_l\|_{1,w_1}) \sum_r \sum_{n \in L(l)} \|\varphi_{n,r} * f | B_m(G)\| \\ &\leq \quad C_4 \sum_{r \in R} \alpha^*_{l,r} \quad \text{and by Hölder's inequality} \\ &\leq \quad C_4 [\sum_{r \in R} \alpha^{*q}_{l,r} v^q(t_l, s_r)]^{1/q} [\sum_{r \in R} v(t_l, s_r)^{-q'}]^{1/q'} \\ &\overset{\text{by Lemma 5.5}}{\leq} \quad C_5 [\sum_{r \in R} \alpha^{*q}_{l,r} v^q(t_l, s_r)]^{1/q} v\langle q \rangle^{-1}(t_l). \end{aligned}$$

Consequently one has (by the estimate for α^*)

$$\begin{aligned}\|R_1 f | M(B_{m_1}, L^q_{v\langle q\rangle})\| &= [\sum_{l\in L} (\beta_l v\langle q\rangle(t_l))^q]^{1/q} \\ &\le C_5[\sum_{l\in L}\sum_{r\in R} (\alpha^{*q}_{l,r} v(t_l, s_r))^q]^{1/q} \\ &\le C_2C_5\|f|M(B_m, L^q_v)\|, \text{ q.e.d.}\end{aligned}$$

B) **(extension theorem)** It will be shown that an extension operator can be obtained by combining Lemma 5.5 with the corresponding 'atomic' extension given by Lemma 5.4. In fact, by Corollary 4.4 it will be sufficient to find, given $f \in M(B_{m_1}, L^q_{v\langle q\rangle})(H_1)$, a family $(F_{l,r})_{(l,r)\in L\times R}$ in the vector-valued sequence space $l^q_v(B_m(G))$, such that

$$\text{supp } \hat{F}_{l,r} \subseteq (t_l + \hat{Q}_1) \times (s_r + \hat{Q}_2), \quad \text{and} \quad \sum_{l,r} R_1 F_{l,r} = f$$

(the distribution $F := \sum_{l,r} F_{l,r}$ is then the required extension for f).

We start by observing that Lemma 5.4 gives for each $l \in L$ some $F_l \in B_m(G)$ satisfying $R_1 F_l = \varphi_l * f$, $\text{supp}\hat{F}_l \subseteq (t_l + \hat{Q}_1) \times \hat{Q}_2$ and $\|F_l|B_m\| \le C\|f|B_{m_1}\|$. Unfortunately the degree of smoothness of $H := \sum_l F_l$ is not good enough in general. However, for E_l, δ_l as in Lemma 5.5 one has

$$f * \varphi_l = R_1[v\langle q\rangle^{q'}(t_l)\delta_l^{-q'} \sum_{r\in E_l} v^{-q'}(t_l, s_r)F_l].$$

It is then possible to define

$$F_{l,r} := \delta_l^{-q'} v\langle q\rangle^{q'}(t_l) v^{-q'}(t_l, s_r) M_{(0,s_r)} F_l$$

for $r \in E_l$, and $F_{l,r} := 0$ for $r \notin E_l$. It is then clear that one has

$$\text{supp } \hat{F}_{r,l} = \text{supp } T_{(0,s_r)}\hat{F}_l = (0, s_r) + [(t_l + \hat{Q}_1) \times \hat{Q}_2],$$

as we claimed. It is also clear that $F := \sum_{l,r} F_{l,r}$ is well defined as a distribution, and that $R_1(M_{(0,s_r)}F_l) = R_1 F_l$ implies

$$R_1 F = \sum_{l,r} R_1 F_{l,r} = \sum_l R_1(F_l) = \sum_l \varphi_l * f = f.$$

Finally, the required norm estimate follows from Lemmas 5.4 and 5.5 (using the equality $q + q' = qq'$):

$$\begin{aligned}
\sum_{l,r} v^q(t_l, s_r)\|F_{l,r}|B_m\|^q &\le \sum_l \sum_{r\in E_l} v^q(t_l, s_r)[v^{-1}(t_l, s_r)v\langle q\rangle(t_l)\delta_l^{-1}]^{q'q} \cdot \|F_l|B_m\| \\
&= \sum_l [\sum_{r\in E_l} v^{-q'}(t_l, s_r)]v\langle q\rangle^{q'q}(t_l)\delta_l^{-q'q} \cdot \|F_l|B_m\| \\
&= \sum_l \delta_l^{q'} v\langle q\rangle^{-q'}(t_l)v\langle q\rangle^{q'+q}(t_l)\delta_l^{-q'-q} \cdot \|F_l|B_m\| \\
&\le \sum_l \delta_l^{-q} v\langle q\rangle^q(t_l)C^q\|f * \varphi_l|B_{m_1}\| \\
&\le (\gamma^{-1}C)^q \sum_l v\langle q\rangle^q(t_l)\|f * \varphi_l|B_{m_1}\| \\
&\le C_6\|f|M(B_{m_1}, L^q_{v\langle q\rangle}\|^q, \qquad \text{q.e.d.}
\end{aligned}$$

The necessary modifications for $q = 1$ are left to the reader (part B) is much easier in that case).

Proof of Lemma 5.2 Since $W(C^b, B_{m_1})(H_1) \hookrightarrow B_{m_1}(H_1)$ in view of the solidity of $B_{m_1}(H_1)$ it will be sufficient (in view of Proposition 3.8) to show that there exists $C^1 = C^1(\hat{Q}) > 0$ such

$$\|R_1 f|W(C^b, B_{m_1})(H_1)\| \le C^1\|f|W(C^b, B_m)(G)\|$$

for $f \in B_m(G)$ satisfying $\mathrm{supp}\hat{f} \subseteq t + \hat{Q}$ for some $t \in \hat{G}$. The easiest way to obtain this estimate will be that of describing the norms involved by the norms of $(R_1 f)^\infty$ and f^∞ in $B_{m_1}(H_1)$ and $B_m(G)$ respectively, where

$$(R_1 f)^\infty(x) := \max_{v\in x+Q_1} |R_1 f(v)|, \text{ and}$$

$$f^\infty(x, y) := \max_{z\in(x+Q_1, y+Q_2)} |f(z)|,$$

for compact subsets Q_1 and Q_2 in H_1 and H_2 respectively, which may be supposed to satisfy $\mu_{H_1}(Q_1) = 1 = \mu_{H_2}(Q_2)$, and $Q_2 = -Q_2$. It is then clear that $(R_1 f)^\infty(x) \ge \alpha$ for some $\alpha > 0$ and $x \in H_1$ implies $f^\infty(x, y) \ge \alpha$ for all $y \in Q_2$. The moderateness of m and the compactness of Q_2 imply that there exists $\delta > 0$ such that $m(x,y) \geq \delta m(x,0) = \delta m_1(x)$ for all $y \in Q_2$. Since $\mu_G(Q_1 \times Q_2) = 1$ we find

$$\mu_{H_1}\{x|x \in H_1, (R_1 f)^\infty(x) m_1(x) \geq \alpha\} \leq \mu_G\{(x,y)|\delta^{-1} f^\infty(x,y) m(x,y) \geq \alpha\}.$$

$B(G)$ and $B(H)$ being rearrangement invariant spaces, related in a canonical way, this implies

$$\|(R_1 f)^\infty | B_{m_1}(H_1)\| \leq \delta^{-1} \|f^\infty | B_m(G)\|, \text{ as required .}$$

Proof of Lemma 5.3 Given the w-moderate function m we write $m_1 : R_1 m$, $w_1 := R_1 w$, $w_2 := R_2 w$. Then it is clear that one has $m(x,y) \leq m_1(x) w_2(y)$ for all $(x,y) \in G$, and both weights w_1 and w_2 satisfy the (BD)-condition. For Q_2 given, there is a suitable family $(y_j)_{j\in J}$ in H_2, (such that $(y_j + Q_2)_{j \in J}$ is a uniform covering of H_2, such that the norm $\|g|W(C^0, L^1_{w_2})\ (H_2)\|$ may be replaced by the equivalent expression $\sum_{j\in J} \alpha_j\ w_2(y_j)$ whenever convenient, where one defines $\alpha_j := \sup_{y \in y_j + Q_2} |g(y)|$ (cf. [20], [16]). Since $g \leq \sum_{j \in J} \alpha_j\ c_{y_j + Q_2}$ one has

$$\mu_G\{z \mid f m_1 \otimes c_{y_j+Q_2}(z) \geq \alpha\} = \mu_{H_2}(Q_2) \mu_{H_1}\{x \mid f m_1(x) \geq \alpha\}.$$

One obtains therefrom (using the moderateness of w_2):

$$\|f \otimes c_{y_j+Q_2} | B_{m_1 \otimes w_2}(G)\| \leq C \|f | B_{m_1}(H_1)\| w_2(y_j)$$

for some $C > 0$. Finally one concludes

$$\begin{aligned} \|f \otimes g | B_m(G)\| &\leq \|f \otimes g | B_{m_1 \otimes w_2}(G)\| \\ &\leq (\sum_{j \in J} \alpha_j w_2(y_j)) C \|f | B_{m_1}(H_1)\| \\ &\leq C_1 \|f | B_{m_1}(H_1)\|\ \|g | W(C^0, L^1_{w_2})(H_2)\|. \end{aligned}$$

Proof of Lemma 5.4 Let $f \in B_{m_1}(H_1)$, and $\hat{Q}_2 \subseteq \hat{H}_2$, open, be given. Since w_2 satisfies the (BD)-condition we know that $A_{w_2}(\hat{H}_2)$ is a regular (pointwise) algebra, and it is possible to find $g \in L^1_{w_2}(H_2)$ such that supp $\hat{g}$

is a compact subset of $\hat{Q}_2$. Thus $g \in \Lambda^K_{w_2}(H_2)$ belongs to any dense ideal of $L^1_{w_2}(H_2)$, in particular to $W(C^0, L^1_{w_2})(H_2)$ (cf. [20], Theorem 3). Applying now Lemma 5.3 we have for $F := f \otimes g$:

$$\|F|M_m(G)\| \leq \Big(C_1 \|g|W(C^0, L^1_{w_2})(H_2)\|\Big)\, \|f|B_{m_1}(H_1)\|.$$

Furthermore, we may assume (after suitable renormalization) that $g(0) = 1$. Then $R_1 F = f$. Since

$$\text{supp } \mathcal{F}_G f \subseteq \text{ supp } (\mathcal{F}_{H_1} f) \times \text{ supp } (\mathcal{F}_{H_2} g) \subseteq (t_1 + \hat{Q}_1) \times \hat{Q}_2,$$

as was we required, the proof is complete.

Proof of Lemma 5.5 Since v^{-1} is a moderate function, and $\{(t_l, s_r) + (\hat{Q}_1 \times \hat{Q}_2)\} = \{Q_{1,r}\}$ is a uniform covering of bounded height one has

$$v^{-1} \sim \sum_{l,r} v^{-1}(t_l, s_r)\; c_{Q_{1,r}} \sim [\sum_{l,r} v^{-q'}(t_l, s_r)]^{-1/q'}.$$

Applying the definition of $v\langle q\rangle$ for $q > 1$ one obtains (for any $l_0 \in L$, fixed):

$$\begin{aligned} v\langle q\rangle^{-q'}(t_{l_0}) \quad &\sim \int_{\hat{H}_2} \sum_{l,r} v^{-q'}(t_l, s_r) c_{Q_{l,r}}(t_{l_0}, y) d_{\hat{H}_2} y \\ &\sim \int_{\hat{H}_2} \sum_{r} v^{-q'}(t_{l_0}, s_r) c_{Q_{l_0,r}}(t_{l_0}, y) d_{\hat{H}_2} y \\ &= \mu_{\hat{H}_2}(\hat{Q}_2) \sum_r v^{-q'}(t_{l_0}, s_r). \end{aligned}$$

The sum being convergent for each $l_0 \in L$ one can find for each $l \in L$ a finite set $E_l \subseteq R$ such that

$$\sum_{r \in E_l} v^{-q'}(t_l, s_r) \geq 2^{-q'} \sum_{r \in R} v^{-q'}(t_l, s_r).$$

and the proof is complete.

As a matter of fact the proof of Theorem 5.1 also gives information concerning the behaviour of weighted Wiener-type spaces such as $W(L^p, L^q)(H \times H')$ under the canonical mapping T_H, given by $T_H f(y) = \int_H f(x, y) dx$, $y \in H'$, for $C_c(H \times H')$ (cf. [38], and [14]). The corresponding result reads as follows:

Theorem 5.6 *Let m be a continuous, moderate function on a locally compact group $G = H \times H'$. Assume that $(R_H m)^{-1} \in L^{q'}(H)$ for some $q \in [1, \infty]$. Then T_H extends to a bounded, linear mapping from $W(L^p, L^q_m)(G)$, $1 \leq p \leq \infty$, onto $W(L^p, L^q_{m\langle q\rangle})(H')$, where $m\langle q\rangle$ is given by*

$$m\langle q\rangle(y) := \left(\int_H m^{-q'}(x, y)dx\right)^{-1/q'}$$

(appropriate modifications for $q = \infty$ are possible).

We don't give the proof of this result here because it is very similar (and even easier) than the proof of our trace theorem. In fact, one needs only results which replace the 'atomic' results, i.e. Lemma 5.2 and Lemma 5.4. The required assertions are:

i) Given a compact set $K \subseteq G$, and $p \in [1, \infty]$ there exists $C_K > 0$ and a compact set $K' \subseteq H' = G/H$, such that any $f \in L^p(G)$ with $\mathrm{supp} f \subseteq y + K$ for some $y \in G$ one has $\mathrm{supp} T_H f \subseteq y' + K'$ for some $y' \in H'$, and $\|T_H f | L^p(H')\| \leq C_K \|f | L^p(G)\|$.

ii) The converse of i) is true, i.e. any $f_1 \in L^p(H')$ with supp $f_1 \subseteq y' + K'$ for some $y' \in H' = G/H$ can be obtained in this way, i.e. $f_1 = T_H f$, for some f as above, satisfying $\|f | L^p(G)\| \leq C_{K'} \|f_1 | L^p(H')\|$ for some $C_{K'} > 0$.

Assertions i) and ii) can be obtained without difficulties by an application of the methods developed in [14].

Remark 5.2 *For $W^1(G) = W(C^0, L^1)$ the closure of $C_c(G)$ in $W(L^\infty, L^1)$ a corresponding result has been proved, using moderate (= functions of translation type) dominants by Bürger (see [8], Corollary 4.1).*

Remark 5.3 *By suitable modifications Theorem 5.6 can be even shown to hold true for arbitrary normal, closed subgroups H of G (cf. [38] VIII and [14] for the necessary modifications).*

6 Modulation spaces on Euclidean spaces

In this concluding section the general results given above are used to show that for a certain family of modulation spaces on Euclidean spaces one has many properties resembling those of the family of Besov spaces $B^s_{p,q}(\mathbf{R}^m)$ (e.g. concerning duality, interpolation, embeddings, traces,...). Our family of modulation spaces includes the classical spaces $\mathcal{L}^s_2(\mathbf{R}^m)$ of Bessel potentials, the remarkable Segal algebra $S_0(\mathbf{R}^m)$ (discussed in [18]), as well as the space of (distributions defining) multipliers on $S_0(\mathbf{R}^m)$ (cf. Lemma 6.3). In a sense it is even the smallest of Banach spaces which contains these three spaces and which is closed with respect to duality and complex interpolation.

For convenience we shall write $M_t f$ for the function $x \mapsto \exp(2\pi i\langle x, t\rangle)f(x)$ on $\mathbf{R}^m, t \in (\mathbf{R}^m)$, in this section. Furthermore, we write w_s for the moderate function $x \mapsto (1+|x|)^s$, $s \in \mathbf{R}^m$ (w_s is a weight function for $s \geq 0$), and L^q_s for $L^q_{w_s}(\mathbf{R}^m)$.

Definition 6.1 *Given $s \in \mathbf{R}$, $1 \leq p, \ q \leq \infty$ we define, fixing any $k \in \mathcal{S}(\mathbf{R}^m)$, $k \neq 0$:*

$$M^s_{p,q}(\mathbf{R}^m) := \ \{\sigma \mid \sigma \in \mathcal{S}'(\mathbf{R}^m),\ M_t k * \sigma \in L^p(\mathbf{R}^m) \ \text{for each} \ t \in \mathbf{R}^m,$$

$$\text{and} \ \ t \mapsto \|M_t k * \sigma\|_p \in L^q_s(\mathbf{R}^m)\}.$$

Writing $\sigma^{(p)}$ for the **control function of** *σ (***with respect to** *$L^p(\mathbf{R}^m)$), given by $\sigma^{(p)}(t) \ := \|M_t k * \sigma\|_p$, the natural norm on $M^s_{p,q}(\mathbf{R}^m)$ is given by:*

$$\|f|M^s_{p,q}(\mathbf{R}^m)\| := \left[\int_{\mathbf{R}^m} |\sigma^{(p)}(t)|^q w^q_s(t)dt\right]^{1/q} \ \text{for} \ \ 1 \leq q < \infty,$$

and

$$\|f|M^s_{p,\infty}(\mathbf{R}^m)\| : = \sup_{t\in\mathbf{R}^m} |\sigma^{(p)}(t)|w_s(t).$$

A first summary of basic results is given in the following theorem:

Theorem 6.1 *A) $M^s_{p,q}(\mathbf{R}^m)$ coincides with the modulation space $M(L^p, L^q_s)$ $(\mathbf{R}^m)$, and any two norms obtained from different test functions $k^1, k^2 \in \mathcal{S}(\mathbf{R}^m)$ define equivalent norms.*

B) There are continuous embeddings $\mathcal{S}(\mathbf{R}^m) \hookrightarrow M^s_{p,q}(\mathbf{R}^m) \hookrightarrow \mathcal{S}'(\mathbf{R}^m)$.

C) For $1 \leq p, q < \infty$, $s \in \mathbf{R}$ *the space* $\mathcal{S}(\mathbf{R}^m)$ *is dense in* $M^s_{p,q}(\mathbf{R}^m)$. *In that case one has*

$$(M^s_{p,q}(\mathbf{R}^m))' = M^{-s}_{p',q'}(\mathbf{R}^m).$$

For $s \geq 0$ *the spaces* $M^s_{p,q}(\mathbf{R}^m)$ *are homogeneous Banach spaces of quasi-measures.*

D) For $1 \leq p_1, q_1 < \infty$, $1 \leq p_2, q_2 \leq \infty$, $s_1, s_2 \in \mathbf{R}$ *and* $\theta \in (0,1)$ *one has*

$$(M^{s_1}_{p_1,q_1}(\mathbf{R}^m), M^{s_2}_{p_2,q_2}(\mathbf{R}^m))_{[\theta]} = M^s_{p,q}(\mathbf{R}^m),$$

with

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \ \frac{1}{q} = \frac{1-\theta}{p_1} + \frac{\theta}{q_2}, \ s = (1-\theta)s_1 + \theta s_2.$$

Proof. We discuss only A) and B), the other results being more or less immediate consequences of the corresponding results in §4. Due to the invariance of $\mathcal{S}$ and $\mathcal{S}'$ under the Fourier transform assertion A) may be equivalently described by the embeddings

$$\mathcal{S} \hookrightarrow W(\mathcal{F}L^p, L^q_s) \hookrightarrow \mathcal{S}'$$

and the admissibility of $\hat{k} \in \mathcal{S}(\mathbf{R}^m)$ in the definition of the norm of $W(\mathcal{F}L^p, L^q_s)$. Since w_s is a weight function on $\mathbf{R}^m$ for $s \geq 0$, it is not difficult to check that $W(\mathcal{F}L^p, L^p_s)$ is a Banach convolution module over $L^1_{|s|}$ (cf. [20], Theorem 3). By Proposition 3.1 any $\hat{k} \in W(\mathcal{F}L^1, L^1_{|s|})$ defines an equivalent norm on $W(\mathcal{F}L^p, L^q_s)$. An application of Proposition 3.3 yields the admissibility of any $k \in W(\mathcal{F}L^1_{|s|}, L^1)$ for the definition of $M^s_{p,q}(\mathbf{R}^m)$. Since it is also obvious that $W(\mathcal{F}L^1, L^1_{|s|}) \hookrightarrow W(\mathcal{F}L^p, L^q_s)$ for any $p, q \geq 1$, and $s \in \mathbf{R}$, it remains to show in a separate lemma bellow the dense inclusion $\mathcal{S} \hookrightarrow W(\mathcal{F}L^1_s, L^1_s)$ for any $s \geq 0$. In that fact, the density of $\mathcal{S}$ in $W(\mathcal{F}L^1, L^1_s)$ can be used to derive for any $s \in \mathbf{R}$:

$$W(\mathcal{F}L^p, L^q_s) \hookrightarrow W(\mathcal{F}L^\infty, L^\infty_{|s|}) = W(\mathcal{F}L^1, L^1_{|s|})' \hookrightarrow \mathcal{S}',$$

which gives the second required inclusion.

Lemma 6.2 *For any $s \in \mathbf{R}$ one has $\mathcal{S}(\mathbf{R}^m) \hookrightarrow W(\mathcal{F}L^1, L^1_s)(\mathbf{R}^m)$ is a dense subspace.*

Proof. (cf. [35] for $s = 0$). One has $\mathcal{S}(\mathbf{R}^m) \hookrightarrow L^1_s(\mathbf{R}^m)$ for $s \in \mathbf{R}$, and for any suitable partial differential operator D (satisfying $(1+\hat{D})^{-1} \in L^1(\mathbf{R}^m)$, for example $D = \partial^{2m}/\partial x_1^2 ... \partial x_m^2$)

$$\|f|\mathcal{F}L^1(\mathbf{R}^m)\| \le C_D(\|f\|_1 + \|Df\|_1) \text{ for any } f \in \mathcal{S}(\mathbf{R}^m)$$

Since (for any fixed $\varphi_0 \in \mathcal{D}(\mathbf{R}^m), \varphi_0 \neq 0$), one has

$$\begin{aligned}\|f|W(\mathcal{F}L^1, L^s_1)\| &\sim \int_{\mathbf{R}^m} \|(T_x\varphi_0)f\|_{\mathcal{F}L^1} w_s(x)dx \\ &\le \int_{\mathbf{R}^m} \|(T_x\varphi_0)f\|_1 w_s(x)dx + \int_{\mathbf{R}^m} \|D[(T_x\varphi_0)f]\|_1 w_s(x)dx,\end{aligned}$$

and since the first expression is equivalent to $\|f\|_{1,w_s}$ (cf. [20], Remark 3) one only has to care for an estimate of the second term. Applying the product rule one has (for suitably chosen sequences $(D^i_j)^k_{j=1},\ i = 1, 2$ of partial differential operators):[3]

Observing that $\varphi_j := D^1_j\varphi_0$ is a test function and that $D^2_j f \in \mathcal{S}(\mathbf{R}^m) \subseteq L^1_s(\mathbf{R}^m)$ for $1 \le j \le k$ one concludes

$$\int_{\mathbf{R}^m} \|(T_x\varphi_j)D^2_j f\|_1 w_s(x)dx \sim \|D^2_j f\|_{1,w_s} < \infty \text{ for } 1 \le j \le k,$$

which implies altogether the finiteness of $\|f|W(\mathcal{F}L^1, L^1_s)\|$. The continuity of translation in $W(\mathcal{F}L^1, L^1_s)$ (cf. [20], Theorem 1) implies that $f = \lim_\alpha e_\alpha * f$ for $f \in \mathcal{F}L^1 \bigcap C_c(\mathbf{R}^m)$, for any $L^1_{|s|}$ - bounded approximate unit $(e_\alpha)_{\alpha\in I}$ in $\mathcal{D}(\mathbf{R}^m)$. Since $\mathcal{D} * C_c(\mathbf{R}^m) \subseteq \mathcal{D} \subseteq \mathcal{S}$ and since $\mathcal{F}L^1 \bigcap C_c(\mathbf{R}^m)$ is dense in $W(\mathcal{F}L^1, L^1_s)$ the density of $\mathcal{S}(\mathbf{R}^m)$ is proved.

Before proceeding further let us mention that some of the spaces arising in our family of modulation spaces coincide with very useful spaces that can already be found in the literature (cf. [32], [41], [34], [44]-[47] and elsewhere).

[3] $D[(T_x\varphi_0)f] = \sum_{j=1}^{k} D^1_j(T_x\varphi_0)D^2_j f = \sum_{j=1}^{k} T_x(D^1_j\varphi_0)D^2_j f.$

Lemma 6.3 *a) The spaces $M^s_{2,2}(\mathbf{R}^m)$ coincide with the spaces $\mathcal{L}(\mathbf{R}^m) = H^s(\mathbf{R}^m)$ of Bessel potentials on $\mathbf{R}^m$;*

b) The spaces $M^0_{p,p}$ coincide with $W(\mathcal{F}L^p, L^p)$. They arise as complex interpolation spaces between $M^0_{1,1}(\mathbf{R}^m)$, the Segal algebra discussed in [18], and its dual $S_0'(\mathbf{R}^m) = M^0_{\infty,\infty}(\mathbf{R}^m)$ (cf. [17]).

c) The spaces of all distributions defining bounded convolution operators (=multipliers) on $S_0(\mathbf{R}^m)$ coincides with $M^0_{1,\infty}(\mathbf{R}^m)$.

Proof.

a) follows from the chain $\mathcal{F}\mathcal{L}^2_s = L^2_s = W(\mathcal{F}L^2, L^2_s)$;

b) is a consequence of the invariance of the spaces $W(\mathcal{F}L^p, L^p)$ under the Fourier transform (cf. [21], Theorem 3.2) and the definition of modulation spaces;

c) is a transcription of the assertion that pointwise multipliers of $\mathcal{F}S_0(\mathbf{R}^m) = S_0(\mathbf{R}^m)$ are exactly multiplications with elements of $W(\mathcal{F}L^1, L^\infty)$ (cf [23], Corollary 2.14).

The general trace theorem in §5, specialized to the family of modulation spaces considered in this section gives:

Theorem 6.4 *For $1 \leq p, q \leq \infty$, $k, n \in \mathbf{N}$ with $k < m$ and $s > k/q'$ one has*

$$R_{\mathbf{R}^{m-k}}(M^s_{p,q}(\mathbf{R}^m)) = M^{s-k/q'}_{p,q}(\mathbf{R}^m)$$

Proof. This result follows from Theorem 5.1, combined with the fact that $T_{\mathbf{R}^k} w_s \sim w_{s-k}$ (cf. [14], §3).

Remark 6.1 *From [14], combined with Theorem 5.1, one can also draw information concerning traces of more refined scales of modulation spaces defined by means of weights $w_{s,r} : w_{s,r}(x) = (1 + |x|)^s \log^r(1 + |x|)$. Here the critical index $s = k/q'$ (and r sufficiently large) is perhaps of particular interest. Another result on restrictions concerns the case $p = q = 1,\ s = 0$, proved in [18] (Theorem 7.C).*

Concerning embeddings between different metrics we can give the following result:

Proposition 6.5 *Given* $1 \leq p_1, p_2, q_1, q_2 \leq \infty$, *and* $s_1, s_2 \in \mathbf{R}$ *one has*

$$M^{s_1}_{p_1,q_1}(\mathbf{R}^m) \hookrightarrow M^{s_2}_{p_2,q_2} \text{ if and only if}$$

$$p_1 \leq p_2 \text{ and } \begin{cases} q_1 \leq q_2 & \text{and } s_1 \geq s_2, \text{ or} \\ q_1 > q_2 & \text{and } s_1 > s_2 + 1/q_2 + 1/q_1 (> s_2) \end{cases}$$

Each nontrivial inclusion is a proper one, i.e. two modulation spaces coincide if and only if the corresponding parameters are all equal (i.e. $p_1 = p_2, q_1 = q_2$ *and* $s_1 = s_2$*).*

Proof. The necessity of $p_1 \leq p_2$ follows from the fact that $\mathcal{F}L^{p_1}$ is locally contained in $\mathcal{F}L^{p_2}$ if and only if $p_1 \leq p_2$ (and strict if $p_1 < p_2$. cf. [25],[3]). The second set of conditions in turn describes the inclusions between weighted l^q spaces (involved in the discrete description of $M^s_{p,q}(\mathbf{R}^m)$, cf. Corollary 4.2, $v = w_s$). Since strict inclusion in the 'global' components gives strict inclusions for the corresponding Wiener-type spaces (hence the modulation spaces; cf. Corollaries 4.2 and 4.4) the last assertion can easily be verified.

Remark 6.2 *We do not give a comparison with the family of Besov spaces* $B^s_{p,q}(\mathbf{R}^m)$*, because such results will be given elsewhere in detail (cf. [27]). Comparisons with potential spaces are given below.*

That the modulation spaces under consideration behave quite nicely under the transformation (= convolution) by Bessel potentials is the content of the next result (cf. [46], [41], [31] for conventions):

Theorem 6.6 *For* $1 \leq p, \ q \leq \infty, \ s_1, s_2 \in \mathbf{R}$ *given, the Bessel potential with kernel* G_s, $s = s_2 - s_1$, *characterized by its Fourier transform* $\hat{G}_s(t) = (1 + 4\pi^2|t|^2)^{-s/2}$, *gives rise to an isomorphism between* $M^{s_1}_{p,q}(\mathbf{R}^m)$ *and* $M^{s_2}_{p,q}(\mathbf{R}^m)$. *More precisely*

$$M^{s_2}_{p,q}(\mathbf{R}^m) = G_s * M^{s_1}_{p,q}(\mathbf{R}^m), \text{ and}$$

$\|G_s * f|M^{s_2}_{p,q}\| \sim \|f|M^{s_1}_{p,q}\|$.

Proof. Although it would be possible to give a direct proof we prefer (for reasons of shortness) to make use of the corresponding 'lifting' property for Besov spaces; $G_s * B^r_{p,q} = B^{s+r}_{p,q}$, with equivalence of the corresponding norms (cf. [46], 2.6.2). Using this fact it follows that one has for any compact set $Q \subseteq \mathbf{R}^m$ and for any $f \in L^p(\mathbf{R}^m)$, $\mathrm{supp}\hat{f} \subseteq t + Q$:

$$\begin{aligned} \|G_s * f|M^{s_2}_{p,q}\| &\sim (1+|t|)^{s_2}\|G_s * f\|_p \sim \|G_s * f|B^{s_2}_{p,q}\| \\ &\sim \|f|B^{s_2-s}_{p,q}\| \sim (1+|t|)^{s_2-s}\|f\|_p \sim \|f|M^{s_1}_{p,q}\|. \end{aligned}$$

In view of Corollaries 4.2 or 4.4 the proof of the theorem is complete.

Remark 6.3 *Theorem 6.6 gives an alternative approach to the spaces* $M^s_{p,q}$ $(\mathbf{R}^m)$ *for* $s \geq 0$. *By defining first* $M^0_{p,q}(\mathbf{R}^m) := \mathcal{F}^{-1}(W(\mathcal{F}L^p, L^q))$ *one could define* $M^s_{p,q} := G_s * M^0_{p,q}$. *Observe, that it would be sufficient to take the Fourier transform in the sense of* $S_0'(\mathbf{R}^m)$ *(cf. [17]), because* $W(\mathcal{F}L^p, L^q) \subseteq S_0'$. *This opens the way to generalizations to lca groups.*

Proposition 6.7 *For* $q_1 \leq \min(p, p')$ *and* $q_2 \geq \max(p, p')$ *one has*

$$M^s_{p,q_1}(\mathbf{R}^m) \hookrightarrow \mathcal{L}^p_s(\mathbf{R}^m) \hookrightarrow M^s_{p,q_2}(\mathbf{R}^m).$$

Proof. In view of Theorem 6.6 it will be sufficient to check the result for $s = 0$. For $1 \leq p \leq 2$ we have by Theorem 3.2 of [21] (or Theorem 3.5 above) and the ordinary Hausdorff-Young inequality:

$$M^0_{p,q_1} = \mathcal{F}[W(\mathcal{F}L^p, L^p)] = W(\mathcal{F}L^p, L^p) \subseteq W(L^{p'}, L^p) \subseteq L^p,$$

and on the other hand one has $\mathcal{F}L^p \subseteq W(\mathcal{F}L^p, L^{p'})$ for $1 \leq p \leq 2$ (check first $p = 1, 2$ and interpolate), and therefore $L^p \subseteq \mathcal{F}W(\mathcal{F}L^p, L^{p'}) = M^0_{p,q_2}$. The result for $p \leq 2$ then follows by dualization.

There is also a variant of Sobolev's embedding theorem for modulation spaces (cf. [32], [41], V.2.2, [34] §4 or [44], §2.8).

Proposition 6.8 *For* $s > m/q'$ *one has* $M^s_{p,q}(\mathbf{R}^m) \hookrightarrow C^0(\mathbf{R}^m)$.

Proof. For $s > m/q'$ one has $L^q_s(\mathbf{R}^m) \hookrightarrow L^1(\mathbf{R}^m)$ by Hölder's inequality. Therefore (one has even) by Theorem 3.2 of [21]

$$M^s_{p,q} = \mathcal{F}[W(\mathcal{F}L^p, L^q_s)] \subseteq \mathcal{F}[W(\mathcal{F}L^p, L^1)] \subseteq W(\mathcal{F}L^1, L^p) \subseteq W(C^0, L^p) \subseteq C^0.$$

Some of the modulation spaces are also pointwise Banach algebras (for related assertions concerning Besov spaces or Sobolev spaces cf. for example [46], §2.6.2).

Proposition 6.9 *For $1 \leq p \leq 2$ and $s > m/q'$, or $s \geq 0$ and $q = 1$, the spaces $M^s_{p,q}(\mathbf{R}^m)$ are Banach algebras with respect to pointwise multiplication.*

Proof. For $s > m/q'$ we have $L^q_s(\mathbf{R}^m) \hookrightarrow L^1(\mathbf{R}^m)$. Since w_s is a weakly subadditive function on $\mathbf{R}^m$ this implies that $L^q_s(\mathbf{R}^m)$ is a Banach convolution algebra (see [14], Corollary 3.8).

Thus Theorem 3 of [20] implies

$$W(\mathcal{F}L^p, L^q_s) * W(\mathcal{F}L^p, L^q_s) \subseteq W(L^1, L^q_s) * W(\mathcal{F}L^p, L^p_s) \subseteq W(\mathcal{F}L^p, L^q_s).$$

The case $q = 1, s = 0$ is easily checked separately. Applying the Fourier transformation one obtains the required results.

Remark 6.4 *The above result even remains true for $2 \leq p \leq \infty$. In fact, the convolution of $\sigma_1, \sigma_2 \in \mathcal{F}L^\infty$ (well defined as a product of pseudomeasures) belongs to $\mathcal{F}L^\infty$. By interpolation with the corresponding result for $p = 2$ the general assertion follows.*

Another result involving pointwise multiplications concerns the compactness of pointwise multipliers between modulation spaces. We only give a typical result in this direction:

Proposition 6.10 *For $p \leq 2, s \in \mathbf{R}$, any $f \in M^{|s|}_{2,1}$ defines a compact (pointwise) multiplier from $M^s_{p,q}$ into $M^r_{1,q}$, for any $r < s$.*

Proof. Applying Theorem 3 of [20] and Plancherel's theorem (implying $L^2 * L^2 \subseteq \mathcal{F}L^1$) one obtains for $g \in M^s_{p,q}$:

$$\begin{aligned} \mathcal{F}(fg) = (\mathcal{F}f) * (\mathcal{F}g) &\in W(\mathcal{F}L^2, L^1_{|s|}) * W(\mathcal{F}L^p, L^q_s) \\ &\subseteq W(L^2, L^1_{|s|}) * W(L^2, L^q_s) \subseteq W(\mathcal{F}L^1, L^q_s). \end{aligned}$$

It follows therefrom that f induces a compact multiplication operator from $M^s_{p,q}$ into $M^s_{1,q}$. Since f can be approximated in $M^s_{2,1}$ by elements with compact support (cf. Theorem 4.6.C) the tightness of S in $M^s_{1,q}$,[4] for any bounded subset $S \subseteq M^s_{p,q}$ is obvious. Since the unit ball of $M^s_{1,q}$ is equicontinuous in $M^r_{1,q}$ for $r < s$ (the unit ball of $W(\mathcal{F}L^1, L^r_q)$ being tight in $W(\mathcal{F}L^1, L^r_q)$ for $r < s$) the assumptions of the compactness criterion (Corollary 4.7) are satisfied, and the proof is complete.

Remark 6.5 *The above result applies in particular to any* $f \in \mathcal{L}^2_t$, *with* $t > s + m/2$.

Concerning multipliers between modulation spaces we shall describe these cases where the space of multipliers itself coincides with some modulation space. Writing $H_G(B^1, B^2)$ for the space of all multipliers (i.e. bounded linear operators commuting with all translation operators $T_y, y \in G$) form B^1 to B^2 we have:

Theorem 6.11 *For* $1 \le p, r, t \le \infty$, *and* $s, s' \in \mathbf{R}$ *one has*

$$H_{\mathbf{R}^m}(M^s_{1,r}, M^{s'}_{p,t}) = M^{s'-s}_{p,q}, \text{ for } \frac{1}{q} = \max\left(0, \frac{1}{t} - \frac{1}{r}\right).$$

Proof. Via the Fourier transform we may identify the space of multipliers with the space of pointwise multipliers from $W(\mathcal{F}L^1, L^r_s)$ to $W(\mathcal{F}L^p, L^{s'}_t)$. Since $H_{\mathbf{R}^m}(L^1, L^p) \cong L^p(\mathbf{R}^m)$ for $1 < p \le \infty$, and since $H_{\mathbf{R}^m}(L^1, L^1) \cong M(\mathbf{R}^m)$ still implies that any Fourier multiplier ($h = \hat{\mu} \in \mathcal{F}M(\mathbf{R}^m)$) belongs locally to $\mathcal{F}L^1(\mathbf{R}^m)$, the required follows from Theorem 2.11 of [23] and simple results concerning 'coordinatewise' multipliers of weighted sequences spaces.

Remark 6.6 *One can even show for any* $s \in \mathbf{R}$ *that* $H_{\mathbf{R}^m}(M^s_{1,q}, L^1) \cong M^{-s}_{1,q'}$ *for* $1 \le q < \infty$, *and*

$$H_{\mathbf{R}^m}(L^1, M^s_{1,q}) \cong M^s_{1,q} \text{ for } 1 < q \le \infty \text{ or } q = 1 \text{ and } s > m.$$

[4] hence in $M^r_{1,q}$ for $r \ge s$.

Proofs will be given elsewhere.

We conclude this section with few results concerning the behaviour of modulation spaces under the Fourier transform. Again we state only typical special cases.

Proposition 6.12 *A) For any p, $1 \leq p \leq \infty$, the Fourier transform maps $M^0_{p,p}(\mathbf{R}^m)$ onto itself.*

B) For $1 \leq p \leq 2$ one has $\mathcal{F}[M^s_{p,q}(\mathbf{R}^m)] \subseteq W(L^{p'}, L^q_s)(\mathbf{R}^m)$.

Proof. Assertion A) is just Theorem 3.2 of [21], or a transcription of a special case of Theorem 3.5 above. B) follows easily from the ordinary Hausdorff-Young inequality which implies $W(\mathcal{F}L^p, L^q_s) \hookrightarrow W(L^{p'}, L^q_s)$ for $1 \leq p \leq 2$.

Remark 6.7 *Recall that A) is obtained from the invariance of $S_0(\mathbf{R}^m) = M^0_{1,1}(\mathbf{R}^m)$ by duality and interpolation. Theorem 3.5 also shows that the spaces $M(L^p_w, L^p_w)$ are invariant as well.*

Remark 6.8 *It is worth mentioning that assertions similar to A) are not available for Besov spaces. Otherwise expressed, modulation spaces show more symmetry with respect to the Fourier transformation than do Besov spaces.*

Concluding remarks

1) It is also possible to consider modulation spaces for $0 < p, q < 1$ (these are quasi-Banach spaces). Furthermore it is possible to consider not only 'uniform' variants of Besov spaces, but it makes also sense to consider 'uniform' variants of Triebel's $F^s_{p,q}$ -spaces. Some results in this direction are given in [48]. There also a trace theorem for F-spaces is given.

2) A general method of constructing Banach spaces of distributions defined by quite arbitrary decompositions of the Fourier transforms of their elements is given [23]. (cf. also [26]). The results of this paper give some idea of the joint aspects of Besov spaces and modulation

spaces. It also contains various examples, including a family of 'intermediate' smoothness, spaces in between these two 'extreme' functional classes. (cf. also [24]).

3) At least a good deal of the results of §6 also applies, suitable modified, to the corresponding spaces over $p-$ adic fields or Vilenkin-groups. The corresponding theory of Lipschitz- or Besov spaces and even potential spaces is described in Taibleson's book [42], and in the work of Bloom, Onneweer, Quek and Yap, for example (only to mention[4,5,33,36,37]). We even dare to say that modulation spaces are perhaps more natural as a mean of describing smoothness of functions on general lca groups, at least if no dilation is available.

4) The family of modulation spaces also seems to be a reasonable tool for the description of pseudodifferential operators. We hope to come back to this subject elsewhere.

5) The choice $v(t) = \exp(s|t|^d), 0 < d < 1, s \in \mathbf{R}$, leads to Banach spaces of ultradistributions on $\mathbf{R}^m$ (cf. [2], [45]) (i.e. tempered distributions are no more sufficient in that case). Furthermore, our presentation includes isotropic as well as anisotropic spaces, depending on the fact whether v is radially symmetric or not (e.g. $v = \otimes_{i=1}^m v_i$, which should lead to spaces related to Besov spaces with dominating mixed derivates) (see [32], [47]).

6) Although our general results are formulated for general lca groups we did not make use of structure theory for these groups. On the other hand, our approach makes clear that it is possible to introduce on $\mathbf{R}^m$ a reasonable family of Banach spaces of smooth functions or distributions without recurrence to the differentiability or the dilation structure of the Euclidean $n-$ space.

7 "Historical" comments, recent literature

Up to this paragraph the paper is the *unchanged copy* of the original report written in 1982/83, which was submitted twice, but not accepted for publication.[5] Only a few symbols where exchanged by those used by most authors nowadays. The motivation for publishing this report *now* was the fact that it has been cited at various places, hence should be available for the sake of future reference in that particular form. Moreover, there is generally increasing interest in modulation spaces (at least 4 of the contributions in [75] make explicit use of modulation spaces). In fact, different authors in the treatment of diverse mathematical problems repeatedly confirm that modulation spaces constitute the family of Banach spaces of distributions which frequently appear as the appropriate choice for the description of the time-frequency behaviour of functions, distributions or operators.

The reason why this report was not published earlier lies partially in the authors point of view that most of the properties of modulation spaces where subsumed already in the general theory of *coorbit spaces* as developed later together with K. Gröchenig (described in [65, 66] and subsequent papers). However, in retrospect one has to admit that the knowledge of modulation spaces in conjunction with the constructions of wavelets (occuring in the late 80's) was very important in order to develop the general *coorbit theory*. At this point the positive influence of the work of A. Grossmann (e.g., [91, 90]) has to be acknowledged. He already took a strong group-theoretical point of view, showing that the continuous wavelet transform and the short time Fourier transforms are special cases of (square) integrable group representations. In conjunction with the theory of *Wiener amalgam spaces* (see [20, 21, 93]) over the corresponding lc. groups [6] it was possible to develop a unified approach to atomic decomposition results (in [64, 65, 66]).

Moreover, thinking of the situation around 1986-89, let us recall that this was the time when the "wave of wavelets" took off, inspired by the construction of orthogonal wavelet systems by Y. Meyer and his coauthors (see [97, 98, 99, 101, 100, 102] for a list of very early publications). The author

[5] Most likely the main reason for these rejections was the fact that the manuscript was referring to various - at that time - unpublished papers (such as [2, 22, 23, 27, 48]).

[6] The term *Wiener-type spaces* as used in the "original part of this paper" has been replaced subsequently by the more specific term "Wiener amalgam spaces".

remembers a very inspiring, short visit to Yves Meyer in Paris, in February 1987, which was another strong motivation to work in that direction. Keeping up with those new developments appeared to be more exciting then rewriting old notes.

The fact that the 1983–version of this paper contains a substantial amount of still unpublished results (e.g. , the trace theorems) is more important reason for our aim to publish it than the "historical aspects" are. Only the paper [57] has been destilled from the results of the old report. It describes the generalized Fourier transform (in the context of [ultra-] distributions), proposing a collection of Fourier invariant Banach spaces of distributions. This level of generality subsequently becomes the basis for the generalization of irregular sampling algorithms to Banach spaces of band-limited functions over general lca. groups (cf. [73]).

From the very beginning modulation spaces have been treated in analogy to the corresponding theory of Besov spaces, essentially by replacing the dyadic decompositions of unity by *uniform* ones. One of the reasons to do this was the construction of "smoothness spaces" over general lca. groups which don't have dilation. This is of course possible by replacing dilation by "frequency shifts" (or modulation operators). Starting from this analogy it was natural to go first for an atomic approach corresponding to the Frazier-Jawerth expansions for Besov-Triebel-Lizorkin spaces (cf. [78, 79]). This was done in [58]. Once Daubechies, Jaffard and Journé published the construction of Wilson basis [54] it was become that they are unconditional bases for the family of modulation spaces (cf. [68] or [104]).

In the mid eighties the connection to time-frequency analysis, the role of the Schrödinger representation of the reduced Heisenberg group in this context, and the relation to Gabor's idea of a "series expansion of arbitrary function in terms of time-frequency shifted copies of a template" became transparent. At this point the author has to thank his colleagues from the technical university Vienna (in particular Franz Hlawatsch) for pointing out the relevance of Gabor's work [80] in the present context. As a consequence, contacts to A.J.E.M. Janssen has intensified very much over the years. He was the first (e.g. in [95]) to analyze the informal classical approach suggested by D. Gabor in a strict mathematical way by using (tempered) distributions. Of course, the work of I. Daubechies (on localization operators) also has to be acknowledged here (see [53]).

With the appearance of wavelets, which where shown to be *unconditional bases* for the whole family of Besov-Triebel-Lizorkin spaces (cf. [82]) it became immediately clear that there must be a strong analogy between those function spaces and their unconditional bases and what is nowadays known as *Banach frames* of *Gabor type* for modulation spaces. As already mentioned the unifying approach of the group theoretical framework, covering both cases (as well as others) was developed in the late eighties by Feichtinger and Gröchenig, who introduced *coorbit spaces* with respect to a given irreducible, integrable group representation (cf. [64, 65, 66, 83]). Only subsequently it has been realized that the atomic decompositions which are obtained in the case by using the reduced Heisenberg group and the Schrödinger representation, or in the terminology of engineers a collection of time-frequency shifts of a given "father wavelet" (or atom, such as the Gaussian), is exactly what D.Gabor has proposed in his paper of 1946, at least in spirit (see [67] for a translation between the two contexts).

Nowadays one may recommend K. Gröchenig's book [85], maybe in conjunction with G. Folland's book [76], as an introduction to this branch of mathematical analysis. Gröchenig's book offers - to mathematicians at least - an optimal (self-contained) introduction into the field, covering the fine details of *Gabor Analysis* and the use of *modulation spaces* in this context, up to pseudo-differential operators. The relevance of modulation spaces in this context became evident in [86] where Gröchenig and Heil demonstrate that the modulation spaces can be used in the generalization of the classical Calderon-Vaillancourt theorem. A whole stream of publications is presently following this direction, starting from the pioneering work of Tachizawa [109, 110, 107], up to the recent papers by Heil and Gröchenig [87] in this volume giving also the overview over the present state. There is also work concerning pseudodifferential operators on modulation spaces by Bényi and Okoudjou [49], Hogan and Lakey [94], Labate [96], Pilipovic and Teofanov [104, 106, 105] and Toft [113], for example. Again one can point to Gröchenig's book [85], Chap.14, as a suitable introduction to the field.

Given the interest in modulation spaces it is natural that there is an increasing interest in embedding results between modulation spaces and other (maybe more well known) Banach spaces of functions. The first sharp results in this direction appeared already in PhD thesis of Peter Gröbner (Vienna, 1992, [27]). More recent results are obtained by K. Okoudjou [103] and Toft [112, 113], but one should also mention [86] and [94] in this context.

Modulation spaces are also the "appropriate class of function spaces" when it comes to the description of the boundedness properties of Gabor multipliers (depending on the ingredients), cf. [72] and the references given there. They also take their role in the description of quantization of TF-expansions ([115]) or (in an ongoing project) to describe functions of variable band-width.

Final remark: With the time going on (and the report reproduced above having been not available to many authors in the past) there has been some diffuseness in the use of the word *"modulation spaces"* recently. The two main interpretations are either that modulation spaces are defined as "Wiener amalgams on the Fourier transform side" (which was the original viewpoint) or (what we suggest to become the standard interpretation for the future) to call "modulation space" a space which arises as coorbit space with respect to the Schrödinger representation of the reduced Weyl-Heisenberg group, which is equivalent to say that members of a modulation space are characterized by the fact that their short-time Fourier transforms (say with respect to the Gaussian window) belong to some solid and translation invariant Banach space of measurable functions over the time-frequency plane (a continued discussion on such matters is planned on the authors Web-pages, cf. [61]).

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