

ERROR ESTIMATES FOR IRREGULAR SAMPLING OF BAND-LIMITED FUNCTIONS ON A LOCALLY COMPACT ABELIAN GROUP

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Abstract

It is known that band-limited functions can be recovered by means of iterative methods, if only the sampling density is high enough. In this paper we present an error analysis for these methods, for the typical sources of error, i.e., jitter error, truncation error, aliasing error, quantization error, and their combinations. The error analysis applies uniformly to a variety of norms, such as weighted L^p -spaces over locally compact Abelian groups. In contrast to earlier papers we do not make use of any (relative) separation condition on the sampling sets, or polynomial growth of the weights over Euclidean spaces. Thus even for the case of regular sampling, i.e. sampling along lattices in G , the results are new in this generality.

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1 Introduction and Notations

In a previous paper by the authors ^[FP02] [9] it has been shown that the qualitative theory of irregular sampling based on iterative reconstruction as developed by Feichtinger and Gröchenig extends to *locally compact Abelian groups*, if properly reformulated. At the same time the class of admissible weight functions has been enlarged to its natural limit, i.e., to *weight functions* which are *moderate* with respect to submultiplicative weight functions, satisfying the so-called Beurling-Domar (*non-quasianalyticity*) condition. Such a condition is indeed required in order to ensure the existence of non-zero band-limited functions.

In the present paper we study various types of error estimates for these reconstruction methods. Our theorems extend the corresponding earlier results in the literature, given for the Euclidean n -space only, as in (cf. ^[FG-ERR] [8]). We will give qualitative estimates for the jitter error, for the aliasing error, or truncation errors and combined errors. These error estimates apply uniformly to large families of function spaces. Even for the Euclidean case our results apply to cases which have not been covered previously. For example, sampling sets which allow arbitrary clustering (as long as they are sufficiently dense) are now included, or subexponential weight functions (such as $x \rightarrow \exp(c|x|^\gamma)$ for some $\gamma \in (0, 1)$), which may grow faster than any polynomial. The results are new even for the case of regular sampling over locally compact Abelian (LCA) groups, i.e. for the case that the sampling values are taken from some lattice in G (a discrete subgroup with compact quotient). Thus to our knowledge they provide for the first time a detailed error analysis for Kluvanek's sampling theorem, which corresponds to Shannon's sampling theorem in the context of LCA groups (cf. ^[Klu69] [14]).

In order to fix notations let us summarize the situation. We shall consider functions f over a *locally compact Abelian* (LCA) group G , the group law being written additively. As usual the *dual group* \hat{G} consists of the "pure frequencies" (or characters), i.e. continuous mappings from G into the complex numbers, satisfying $\chi(x+y) = \chi(x)\chi(y)$ and $|\chi(x)| = 1$ for all $x, y \in G$, endowed with pointwise multiplication (and uniform convergence over compact sets). We write dx and $d\gamma$ for the Haar measures on G and \hat{G} respectively. The spaces $L^p(G)$ are defined as usual with respect to this Haar measure. We refer to the books of Rudin ^[Rud67] [16], Folland ^[Fol194] [11], or Reiter ^[RS00] [15] for generalities on the theory of LCA groups and their Haar measures.

For a strictly positive function m we define the Banach space L_m^p by

$$L_m^p(G) = \left\{ f \mid fm \in L^p \right\} = \left\{ f \mid \|f\|_{p,m} = \left(\int_G |f(x)|^p m^p(x) dx \right)^{1/p} < \infty \right\}.$$

For $p = \infty$ the usual modification takes place. This way of generating weighted L^p -spaces is more appropriate than the usual one (through a change of the measure), in order to group families of weighted spaces as done below.

The space $C_c(G)$ of all continuous, complex-valued functions on G with compact support is contained in any of these spaces, and is dense if and only if $p < \infty$. With the usual rule (obtaining functionals via integration over G) the Banach space for $L_m^p(G)$ is just $L_{m'}^{p'}(G)$, with $1/p + 1/p' = 1$ and $m' = 1/m$. We write $C^0(G)$ for space of (bounded) continuous functions which vanish at infinity. It is a Banach space with respect to the sup-norm, and again $C_c(G)$ is dense in $C^0(G)$. Its dual may be identified with $M(G)$, the bounded measures on G .

We shall consistently use the symbol w only for *Beurling weight functions* $w : G \rightarrow [1, \infty)$ which are assumed to be continuous and *submultiplicative*, i.e., $w(x+y) \leq w(x)w(y)$ for all $x, y \in G$. For such weights the Banach space $(L_w^1(G), \|\cdot\|_{1,w})$ is a commutative Banach algebra with respect to convolution, a so-called *Beurling algebra* ([15]), with $f * g(x) = \int_G g(x-y)f(y)dy$ for $f, g \in C_c(G)$ being the usual convolution product over G .

A general weight m is called *w-moderate*, if it satisfies $m(x+y) \leq w(x)m(y)$ for all $x, y \in G$. Both $m = w$ itself and $m = 1/w$ are *w-moderate* weights, but also w^α , for any $\alpha \in [-1, 1]$.

The spaces L_m^p with moderate weights are translation invariant, i.e., each of the translation operators L_z , defined as $L_z f(x) = f(x-z)$, is bounded on L_m^p and $\|L_z f\|_{p,m} \leq w(z)\|f\|_{p,m}$. Due to the pointwise estimate $|(f * g)m| \leq |f|w * |g|m$ one also has $L_w^1 * L_m^p \subseteq L_m^p$. Altogether this implies that the spaces L_m^p (for a fixed w) will be typical examples of the Banach spaces $(B, \|\cdot\|_B)$ to which our error analysis applies.

As in [9],^{FP02} we make sure that our statements apply uniformly for the collection \mathcal{B}_w of all Banach spaces of functions $(B, \|\cdot\|_B)$ on G which satisfy the following six conditions with respect to some fixed Beurling weight w .

- (B1) $(B, \|\cdot\|_B)$ is a Banach space, continuously embedded into $L_{loc}^1(G)$, i.e., for every compact set $K \subseteq G$, there exists a constant $C_K > 0$ such that $\int_K |f(x)|dx \leq C_K \|f\|_B$ for all $f \in B$.

(B2) $(B, \|\cdot\|_B)$ is *solid*, i.e., if $f \in L^1_{loc}(G)$ (or f continuous for the case $B = C^0$) satisfies $|f(x)| \leq |g(x)|$ almost everywhere on G for some $g \in B$, then $f \in B$ and $\|f\|_B \leq \|g\|_B$.

(B3) B is invariant under translations, i.e., $L_x f \in B$, for all $f \in B, x \in G$.

(B4) The weight function w controls the operator norm of L_x on B , i.e., $\|L_x\|_B \leq w(x)$, or more explicitly

$$\|L_x f\|_B \leq w(x) \|f\|_B, \quad \forall x \in G, f \in B. \quad (1) \quad \boxed{\text{shift-norm}}$$

(B5) $(B, \|\cdot\|_B)$ is a Banach convolution module over $L^1_w(G)$, i.e.,

$$L^1_w * B \subseteq B, \quad \text{and } \|g * f\|_B \leq \|g\|_{1,w} \|f\|_B, \quad \text{for } g \in L^1_w(G), f \in B.$$

(B6) $C_c(G)$ is dense in $(B, \|\cdot\|_B)$.

For the sake of simplicity we assume $w(-x) = w(x)$. Note that (B5) above is actually a consequence of (B4) and (B6) (via vector-valued integration), but some of our statements are valid without (B6) as well.

In order to ensure the existence of a generalized Fourier transform as well as non-zero band-limited functions in any $B \in \mathcal{B}_w$ we have to assume that the weight function w satisfies the so-called *Beurling-Domar condition* (BD)

$$\sum_n \log(w(nx))/n^2 < \infty \quad \text{for all } x \in G,$$

see [\[RS00\]](#). As explained in [\[FP02\]](#), for any such $(B, \|\cdot\|_B)$ the following definition makes sense. For a given compact set $\Omega \subset \hat{G}$ with nonvoid interior we define the space of *band-limited functions* in B (with spectrum in Ω) by

$$B^\Omega = \{f \in B \mid \text{spec}(f) := \text{supp}(\hat{f}) \subseteq \Omega\},$$

where \hat{f} denotes the Fourier transform of f in the distributional sense (as tempered distributions or ultradistributions, cf. [\[FP02\]](#) and [\[F-Can\]](#) for details).

For any $(B, \|\cdot\|_B)$ as above also *Wiener amalgam spaces* such as $W(C^0, B)$ are well defined, with the natural norm for this space being the (global) B -norm of local control-function $x \mapsto \|L_x k \cdot f\|_\infty$ for some non-zero $k \in C_c(G)$, cf. [\[F-Can\]](#). The Banach space $W(C_0, B)$ is continuously embedded into both $(B, \|\cdot\|_B)$ and $C_0(G)$, hence convergence in $W(C_0, B)$ implies B -norm and uniform convergence. The reader can find the definition of Wiener amalgam

spaces (previously called *Wiener-type spaces*, cf. [\[4\]](#)^{F-Hom} such as $W(L^p, l^1) = W(L^p, L^1)$ (with local L^p -norm and global L^1 -behavior) in [\[5\]](#)^{F-Can}, as well as a listing of their basic properties, but also [\[12\]](#)^{H90} or [\[13\]](#)^{Hei02} and [\[10\]](#)^{FW02}.

In order to avoid confusion let us note that the description of those amalgam spaces requires the use of BUPUs (bounded uniform partitions of unity) which are supposed to be fixed, while the partitions of unity given below are adapted to the sampling families. The constants given below are given for a fixed norm on those amalgam spaces (using e.g. a smooth BUPU).

2 Partitions of Unity on G and Operators

We summarize in this section various operators and (pointwise) estimates which are relevant for the error analysis carried out later on.

Let $X = (x_i)_{i \in I}$ be a discrete set of sampling points and U a neighborhood of identity in G . It is called *U-dense* if $\bigcup_{i \in I} (x_i + U) = G$. $\Psi = (\psi_i)_{i \in I}$ is called a (non-negative) *partition of unity of size U* associated with X if

- (i) ψ_i is measurable and $0 \leq \psi_i(x) \leq 1$ for all $i \in I$,
- (ii) $\text{supp}(\psi_i) \subseteq (x_i + U)$, for all $i \in I$, (uniform size).
- (iii) $\sum_{i \in I} \psi_i(x) = 1$ for all $x \in G$.

Of course (ii) and (iii) imply that X has to be a *U-dense family* in G , cf. [\[4\]](#)^{F-Hom}.

Following Feichtinger and Gröchenig [\[8\]](#)^{FG-ERR}, we define for a fixed compact (and symmetric) neighborhood U_0 of the identity the *local maximal function*

$$f^\#(x) = \sup_{y \in U_0} |f(x + y)|, \tag{2} \quad \boxed{\text{sharp-def}}$$

and furthermore for $U \subseteq U_0$ and the *local U-oscillation* by

$$\text{Osc}_U f(x) = \sup_{z \in U} |f(x - z) - f(x)|. \tag{3} \quad \boxed{\text{OSC-def}}$$

We will make use of the space CB defined by

$$CB = \{f \in B \mid f \text{ continuous, } f^\# \in B\} \tag{4} \quad \boxed{\text{CB-def}}$$

with the norm $\|f\|_{CB} = \|f^\#\|_B$. For the sake of comparison with other notations let us mention that $CB = W(C^0, B)$ (with equivalent norms).

The following pointwise estimates are easily verified:

$$(f * h)^\#(x) \leq (|f| * h^\#)(x) \tag{5} \quad \boxed{\text{estimate1}}$$

$$Osc_U(f * h)(x) \leq (|f| * Osc_U h)(x) \tag{6} \quad \boxed{\text{estimate2}}$$

For a given partition of unity Ψ of size U we define the operator

$$Sp_\Psi f = \sum_{i \in I} f(x_i) \psi_i,$$

which may be regarded as an irregular spline approximation of f . It is an important fact for our analysis that it makes only use of the given sampling values ($f(x_i)$). We need them only for sufficiently dense sampling sets, so it is reasonable to assume from now on that $U \subseteq U_0$ and consequently $|Sp_\Psi f(x)| \leq f^\#(x)$ for all $x \in G$, and thus by the solidity of B

$$\|Sp_\Psi f\|_B \leq \|f^\#\|_B. \tag{7} \quad \boxed{\text{estimate3}}$$

One more pointwise estimate is of great use for our estimates:

$$|f - Sp_\Psi f|(x) \leq Osc_U f(x). \tag{8} \quad \boxed{\text{estimate8}}$$

The main results of [\[9\]](#) ^{FPO2} show that it is possible to recover a band-limited function $f \in B^\Omega$ from its sampled values on any $U(\Omega)$ -dense discrete subset $X = (x_i)_{i \in I}$ of G by means of a series representation of the form

$$f = \sum_{i \in I} f(x_i) e_i, \tag{9} \quad \boxed{\text{series-exp1}}$$

where the family (e_i) is in $L_w^1(G) \cap B$, with joint spectrum, i.e., $supp(\hat{e}_i) \subseteq \Omega_0$ for all $i \in I$. Indeed, such a family (e_i) can be constructed given only the weight w , the sampling family $X = (x_i)$ and the spectral set Ω .

From now on, for a given compact set Ω in \hat{G} we fix the neighborhood of the identity in G , say U_0 , such that the expression [\(9\)](#) ^{series-exp1} holds true for any U -dense discrete subset $X = (x_i)_{i \in I}$ of G with $U \subset U_0$.

As we are going to show the robustness of this kind of reconstruction we have to recall the definition of certain approximation operators. They are the key ingredients for our iterative reconstruction methods. They all grew out of attempts to discretize a convolution relation of the form $f = f * h$ in various different ways, trying to make use of the sampling values of f only (instead of all of f), resp. to make use of "atoms" (shifted copies of h) only.

Let $X = (x_i)_{i \in I}$ and $Y = (y_j)_{j \in J}$ be two families of points in G , and $\Psi = (\psi_i)_{i \in I}$ and $\Phi = (\phi_j)_{j \in J}$ two partitions of unity of size U associated with X and Y respectively. At this point we assume that $h \in L_w^1(G)$ is band-limited with $\text{spec}(h) = \Omega_0$. It follows that $h^\# \in L_w^1(G)$. As in [\[7\]](#), we consider the following approximations to the convolution operator $C_h : f \mapsto h * f$.

$$\begin{aligned} A_1 f &= (Sp_\Psi f) * h = \left[\sum_{i \in I} f(x_i) \psi_i \right] * h. \\ A_2 f &= (D_\Psi f) * h = \sum_{i \in I} \langle f, \psi_i \rangle L_{x_i} h. \\ A_3 f &= (D_\Psi^+ f) * h = \sum_{i \in I} \left[f(x_i) \cdot \int \psi_i(x) dx \right] L_{x_i} h \\ A_4 f &= [D_\Phi(Sp_\Psi f)] * h = \sum_{j \in J} \left[\sum_{i \in I} f(x_i) \int (\psi_i \phi_j)(x) dx \right] L_{y_j} h. \end{aligned}$$

As $C_c(G)$ is dense in $(B, \|\cdot\|_B)$ by assumption (B6), the solidity of $(B, \|\cdot\|_B)$ ensures that the partial sums of the series $\sum_{i \in I} f(x_i) \psi_i$ are (unconditional) norm convergent in $(B, \|\cdot\|_B)$ and even in CB . As a consequence the operators are well defined at least on CB as convergent series in $(B, \|\cdot\|_B)$.

The reconstruction methods described in [\[7\]](#) and [\[9\]](#) are based on the fact that the approximation operator A (which may be A_1, A_3 or A_4 , depending on the user's preference) satisfies $\|A - C_h\|_{B^{\Omega_0}} < 1$ (where $\|T\|_{B^{\Omega_0}}$ stands for the operator norm on B^{Ω_0}), as long as the sampling set $X = (x_i)_{i \in I}$ is sufficiently dense. Hence an operator D defined as the geometric sum based on $(A - C_h)$ can be formed. Under the additional assumption that $\hat{h} = 1$ on Ω - which *will be assumed* throughout this article - the iterations given by

$$\phi_0 = Af$$

$$\phi_{n+1} = h * \phi_n - A\phi_n = (C_h - A)\phi_n$$

lead to the following representation of f (as described in detail in [\[9\]](#)).

$$f = \sum_{n=0}^{\infty} \phi_n = \left(\sum_{n=0}^{\infty} (C_h - A)^n \right) Af =: D Af,$$

as absolutely convergent series (due to the geometric convergence rate) with

$$\sum_{n=0}^{\infty} \|\phi_n\|_B \leq \left(\sum_{n=0}^{\infty} \|C_h - A\|_{B^{\Omega_0}}^n \right) \cdot \|Af\|_B < \infty.$$

As long as for some sequence $(c_i)_{i \in I}$ the function $f_c = \sum_{i \in I} c_i \psi_i \in B$ is well defined, one may apply D to $f_c * h$ and arrives at $D(f_c * h) \in B^{\Omega_0}$, just

by starting the iteration with f_c instead of Af . As a matter of fact, a typical building block e_{i_0} as it is used in [\(9\)](#) is obtained by choosing as sequence \mathbf{c} the unit vector of i_0 , i.e., with 1 at i_0 and 0 elsewhere. Alternatively, e_{i_0} arises as limit of the iteration procedure, starting with ψ_{i_0} instead of Af .

We are going to discuss different kinds of errors that may occur in applications, and will give error estimates which are *independent* not only from the individual f under consideration, the sampling sets (only their density matters), and the partitions of unity $\Psi = (\psi_i)_{i \in I}$ and $\Phi = (\phi_j)_{j \in J}$ in use, but even from the space $(B, \|\cdot\|_B)$ to which f belongs (or by which the error is measured), as long as it belongs to the class \mathcal{B}_w , i.e. satisfies conditions (B1 – B6) for some fixed w . A typical universal constant arising in this context will be the L_w^1 -norm of some function $h \in L_w^1(G)$ satisfying $\hat{h}(\omega) \equiv 1$ for all $\omega \in \Omega$, clearly indicating that one will have larger constants for larger sets Ω and stronger weights w .

3 Quantization Error

When it comes to the reconstruction from sampling values often the exact sampling values are not available. Thus typically additive noise may occur, or only quantized version of such data are available. Therefore we study the effect of quantization on the reconstruction result.

We write $\tilde{f}(x_i)$ for the quantized (or noisy) versions of the precise sampling value $f(x_i)$, and consequently the reconstruction formula [\(9\)](#) yields an approximate reconstruction $\sum_{i \in I} \tilde{f}(x_i)e_i$. The *round-off error* $E_{RO}f$ is defined by

$$E_{RO}f = f - \sum_{i \in I} \tilde{f}(x_i)e_i.$$

Since for the estimate of the *quantization error* the sup-norm is the natural one, we use it in the following form.

th1 **Theorem 3.1** *Given a compact set Ω in G and $h \in L^1(G)$ with $\hat{h} = 1$ on Ω , there exists a constant $d_h > 0$ such that*

$$\|E_{RO}f\|_\infty = \left\| \sum_{i \in I} [f(x_i) - \tilde{f}(x_i)] e_i \right\|_\infty \leq d_h \sigma,$$

for any $f \in C_0(G)$ with $\text{spec}(f) \subseteq \Omega$ and sampling values that are uniformly quantized with step width $\leq \sigma$.

The proof of the above theorem depends on the following lemmas. The first one has been proved in [9]. Both statements make use of the pointwise estimates ^{estimate1}(5) and ^{estimate2}(6) and the solidity of $(B, \|\cdot\|_B)$.

In the subsequent lemma we write D_1 for the series expansion which is derived from the approximation operator A_1 .

lem2 **Lemma 3.2** *Assume $B \in \mathcal{B}_w$.*

(a) *For any compact subset Ω of \hat{G} , there exists a constant $C = C(\Omega)$ such that for all $f \in B^\Omega$*

$$\|f^\#\|_B \leq C(\Omega)\|f\|_B. \quad (10) \quad \text{EST1}$$

(b) *For a given $\eta > 0$ there exists $U_1 \subseteq U_0$ such that for all $f \in B^{\Omega_0}$*

$$\|Osc_{U_1} f\|_B \leq \eta\|f\|_B. \quad (11) \quad \text{EST2}$$

(c) *For a given $\eta > 0$ there exists $U \subseteq U_0$ such that for all $f \in B^\Omega$*

$$\|Osc_U^\# f\|_B \leq \eta \cdot \|f\|_B. \quad (12) \quad \text{EST3}$$

lem1 **Lemma 3.3** *Assume $B \in \mathcal{B}_w$, and consider any approximation operator of the form A_1 for some sufficiently dense set $X = (x_i)_{i \in I}$. Then there exists a positive constant $d_h > 0$ such that for any $(\lambda_i)_{i \in I}$ with $\sum \lambda_i \psi_i \in B$, the series $\sum_{i \in I} \lambda_i e_i$ is norm convergent in B and*

$$\left\| \sum_{i \in I} \lambda_i e_i \right\|_B = \left\| D_1 \left(\sum_{i \in I} \lambda_i \psi_i * h \right) \right\|_B \leq d_h \left\| \sum_{i \in I} \lambda_i \psi_i \right\|_B < \infty.$$

For d_h we can take $d_h = \|D_1\|_{B^{\Omega_0}} \|h\|_{1,w}$, where $D_1 = \sum_{n=0}^{\infty} (C_h - A_1)^n$.

Proof. For $f \in B^{\Omega_0}$, estimate ^{estimate8}(8) gives

$$\|(C_h - A_1)f\|_B = \|(f - Sp_\Psi f) * h\|_B \leq \|Osc_U f * |h|\|_B \leq \eta\|f\|_B \|h\|_{1,w}.$$

Choosing $\eta < \|h\|_{1,w}^{-1}$, we have $\|(C_h - A_1)f\|_B \leq \gamma\|f\|_B$ for some $\gamma < 1$ and all $f \in B^{\Omega_0}$. As in Lemma 4.1 of [9] ^{FP02}this implies the operator norm estimate

$$\|C_h - A_1\|_{B^{\Omega_0}} \leq \gamma < 1.$$

Hence the geometric series $D_1 = \sum_{n=0}^{\infty} (C_h - A_1)^n$ is a well defined operator on B^{Ω_0} . Setting

$$e_i = D_1(\psi_i * h),$$

we obtain

$$\begin{aligned} \left\| \sum_{i \in I} \lambda_i e_i \right\|_B &= \left\| \sum_{i \in I} \lambda_i D_1(\psi_i * h) \right\|_B = \left\| D_1 \left(\sum_{i \in I} \lambda_i (\psi_i * h) \right) \right\|_B \\ &\leq \|D_1\|_{B^{\Omega_0}} \|h\|_{1,w} \sum_{i \in I} \lambda_i \|\psi_i\|_B = d_h \left\| \sum_{i \in I} \lambda_i \psi_i \right\|_B. \end{aligned}$$

Thus the lemma is proved.

By choosing $(B, \|\cdot\|_B) = (C^0(G), \|\cdot\|_\infty)$ we obtain the following corollary.

Corollary 3.4 *There exists $d_h > 0$ (depending only on the choice of h , which in turn depends only on w and Ω) such that for any bounded sequence $(\lambda_i)_{i \in I}$*

$$\left\| \sum_{i \in I} \lambda_i e_i \right\|_\infty \leq d_h \|\lambda_i\|_\infty < \infty.$$

Proof of Theorem ^{th1}3.1. Setting $\lambda_i = [f(x_i) - \tilde{f}(x_i)]$, we obtain $\|\lambda\|_\infty \leq \sigma$. Applying Lemma ^{lem1}3.3, for $B = C^0(G)$ we have

$$\|E_{R0} f\|_\infty \leq d_h \left\| \sum_{i \in I} [f(x_i) - \tilde{f}(x_i)] \psi_i \right\|_\infty \leq d_h \sigma \sum_{i \in I} \psi_i(x) \leq d_h \sigma < \infty.$$

4 The Jitter Error

The *jitter error* arises if the function is sampled at the incorrect instant x'_i instead of x_i , or if the sampling positions are not precisely known, and the samples $f(x'_i)$ are used as the input in the reconstruction algorithm. Then the resulting reconstruction takes the form

$$\tilde{f} = \sum_{i \in I} f(x'_i) e_i$$

and the jitter error is the deviation of \tilde{f} from the correct reconstruction

$$E_J f = f - \tilde{f} = \sum_{i \in I} [f(x_i) - f(x'_i)] e_i = D_1 \left(\sum_{i \in I} [f(x_i) - f(x'_i)] (\psi_i * h) \right).$$

Assuming that the deviation from the correct sampling positions is not too large in a uniform sense, one may hope that it is possible to control the

influence of jitter error with respect to the B -norm, for all band-limited functions f in B with a given spectrum. The following theorem shows that this is indeed the case. Note that the sequences $X' = (x'_i)_{i \in I}$ don't have to be reconstruction sequences for the following theorem to be valid. On the other hand the result implies that uniform closeness of a family X' to a family with reconstruction property inherits this property as well.

th2 **Theorem 4.1** *Let $X = (x_i)_{i \in I}$ be a U -dense set of sampling points in G . Then, for any neighborhood $V \subset U$ there exists a constant $C = C_h(V, \Omega_0)$ such that for any family $X' = (x'_i)_{i \in I}$ which is V -close to X , i.e., $x_i - x'_i \in V$ for all $i \in I$,*

$$\|E_J f\|_B \leq C_h(V, \Omega_0) \|f\|_B, \quad \forall f \in B^\Omega.$$

More importantly, these constants $C_h(V, \Omega_0)$ tend towards zero as $V \rightarrow \{0\}$.

Remark. Using appropriate modifications the same statements can be made for the other reconstruction operators instead of D_1 . We leave it to the interested reader to fill in the necessary details.

Because two of the most important reconstruction algorithms described in [9] start with spline approximations as a first step let us analyze the error arising in this step first.

lemma42 **Lemma 4.2** *Let $X = (x_i)_{i \in I}$ be a U -dense set of sampling points in G . For any neighborhood $V \subseteq U$ there exists some constant $C = C(V)$ such that for any $X' = (x'_i)_{i \in I}$ satisfying $x_i - x'_i \in V$ for all $i \in I$,*

$$\left\| \sum_{i \in I} [f(x_i) - f(x'_i)] \psi_i \right\|_B < C(V) \|f\|_B$$

for all $f \in B^\Omega$ and all $B \in \mathcal{B}_w$. Moreover, $C(V) \rightarrow 0$ for $V \rightarrow \{0\}$.

Proof. We are going to estimate the expression on the left side by the function $Sp_\Psi(Osc_V(f))$. Indeed, it is obvious that the assumption imply

$$|f(x_i) - f(x'_i)| \leq Osc_V f(x_i), \quad \forall i \in I.$$

Applying now the solidity assumption (B2) and estimate (estimate3) we obtain

$$\|Sp_\Psi(Osc_V(f))\|_B \leq \|Osc_V^\# f\|_B.$$

It is therefore sufficient to ensure that $\|Osc_V^\# f\|_B < \varepsilon \|f\|_B$ for all $f \in B^{\Omega_0}$.

Proof of Theorem 4.1. For given Ω we choose a band-limited function h in $L_w^1(G)$ such that $\hat{h}(\gamma) = 1$ for $\gamma \in \Omega$. Then $f = f * h$ for all $f \in B^\Omega$. Using also Lemma 4.2 the jitter error can then be estimated as follows.

$$\begin{aligned} \|E_J f\|_B &= \|D_1 \left(\sum_{i \in I} [f(x_i) - f(x'_i)] \psi_i * h \right)\|_B \\ &\leq \|(\text{osc}_V^\# f) * h\|_B \|D_1\|_{B^{\Omega_0}} \leq \|\text{osc}_V^\# f\|_B \|h\|_{1,w} \|D_1\|_{B^{\Omega_0}} \\ &\leq \eta \cdot \|h\|_{1,w} \|D_1\|_{B^{\Omega_0}} \cdot \|f\|_B \end{aligned}$$

Thus the theorem holds true for $C(V, \Omega_0) = \eta \cdot \|h\|_{1,w} \|D_1\|_{B^{\Omega_0}}$.

5 The Truncation Error

This type of error appears when only local information is available. Hence we assume that only the sampling values of f at the points x_i contained in a compact set $K \subset G$ are available. Using them we obtain a band-limited function $f_K \in B^{\Omega_0}$, given by the (usually finite) sum

$$f_K = \sum_{x_i \in K} f(x_i) e_i.$$

The *truncation error* is then given by $E_T f = f - f_K$ or

$$E_T f = f - \sum_{x_i \in K} f(x_i) e_i = \sum_{x_i \notin K} f(x_i) e_i.$$

We prove two theorems concerning the truncation error. Showing that - independent of the area of interest K - the relative truncation error over K can be made arbitrarily small for all band-limited functions in B , with respect to their B -norm and uniformly over all considered B -norms, if only samples from some “smeared” version of K are used. The amount of smearing expressed by a compact set W can again be chosen independently from the individual function of the Banach space $B \in \mathcal{B}_w$. In [17], one can find explicit estimates for the one-dimensional L^2 -case.

Theorem 5.1 *For any $\varepsilon > 0$ there exists a compact set W such that for any compact set K and for any U -dense family $X = (x_i)_{i \in I}$, we have*

$$\| [f - \sum_{x_i \in K+W} f(x_i) e_i] \mathbf{1}_K \|_B \leq \varepsilon \|f\|_B \quad \forall f \in B^\Omega.$$

Proof. Let K be any compact subset in G . We write

$$\phi_{K,W} = \sum_{x_i \notin K+W} f(x_i)\psi_i.$$

Then by Lemma [3.2\(a\)](#) we see that

$$\|\phi_{K,W}\|_B \leq \|f^\#\|_B \leq C(\Omega_0)\|f\|_B < \infty \quad (13) \quad \boxed{\text{estimate4}}$$

holds true for all K and W . Thus, for the global truncation error, we have

$$\begin{aligned} E_T f &= \sum_{x_i \notin K+W} f(x_i)e_i = \sum_{x_i \notin K+W} f(x_i)D_1(\psi_i * h) \\ &= D_1\left(\sum_{x_i \notin K+W} f(x_i)\psi_i * h\right) = D_1(\phi_{K,W} * h). \end{aligned}$$

Now we have to determine W so that the local truncation error on K , i.e., $E_T f \cdot \mathbf{1}_K$, is small and independent of K . By assumption on X , the geometric series

$$D_1 = \sum_{n=0}^{\infty} (C_h - A_1)^n$$

is absolutely convergent, thus for any given $\varepsilon > 0$ there exists $s \geq 1$ such that the finite sum

$$D_1^s = \sum_{n=0}^s (C_h - A_1)^n$$

satisfies the condition

$$\|D_1 - D_1^s\|_{B^{\Omega_0}} < \frac{\varepsilon}{3C(\Omega_0)\|h\|_{1,w}}.$$

Then, the local truncation error on K is given by

$$\begin{aligned} E_T f \mathbf{1}_K &= D_1(\phi_{K,W} * h) \cdot \mathbf{1}_K = \\ &= \underbrace{\left[(D_1 - D_1^s)(\phi_{K,W} * h)\right] \mathbf{1}_K}_{I_1} + \underbrace{D_1^s(\phi_{K,W} * (h - k)) \mathbf{1}_K}_{I_2} + \underbrace{D_1^s(\phi_{K,W} * k) \mathbf{1}_K}_{I_3}. \end{aligned}$$

We consider I_1 for the given s and use estimate [\(13\)](#) in order to compute

$$\begin{aligned} \|I_1\|_B &= \|(D_1 - D_1^s)(\phi_{K,W} * h)\mathbf{1}_K\|_B \\ &\leq \|D_1 - D_1^s\|_B \cdot \|\phi_{K,W} * h\|_B \\ &\leq \varepsilon[3C(\Omega_0)\|h\|_{1,w}]^{-1}C(\Omega_0)\|h\|_{1,w}\|f\|_B \\ &= \|f\|_B \cdot \varepsilon/3. \end{aligned}$$

For I_2 we recall that D_1^s is well defined for all $f \in B$ with $f^\# \in B$ and

$$\|D_1^s f\|_B = \left\| \sum_{n=0}^s (C_h - A_1)^n f \right\|_B.$$

Since f and $f^\#$ both belong to B , we obtain

$$\begin{aligned} (C_h f - A_1 f)^\#(x) &= (f * h - Sp_{\Psi} f * h)^\#(x). \\ &\leq (2f^\# * |h|)^\#(x) \leq (2f^\# * h^\#)(x), \quad \forall x \in G, \end{aligned}$$

which ensures that

$$\|(C_h f - A_1 f)^\#\|_B \leq 2\|f^\#\|_B \|h^\#\|_{1,w}.$$

Repeating this process s times, we obtain

$$\|D_1^s\|_B \leq 2^s \|f^\#\|_B \|h^\#\|_{1,w}^s = C_s \|f^\#\|_B. \quad (14) \quad \boxed{\text{estimate5}}$$

Combining [\(13\)](#) and [\(14\)](#), we have

$$\begin{aligned} \|I_2\|_B &= \|D_1^s[\phi_{K,W} * (h - k)]\mathbf{1}_K\|_B \\ &\leq C(\Omega_0) \|f\|_B \|(h - k)\|_{1,w} \\ &\leq \|f\|_B \cdot \varepsilon/3 \end{aligned}$$

for all $f \in B^\Omega$ and $(h - k)$ sufficiently small, and independent of K and W .

Finally, we discuss I_3 . We write

$$T(f) = [f - \sum_{i \in I} f(x_i) \psi_i] * k,$$

where $k \in L_w^1(G)$. We have

$$T(f)^\#(x) \leq (2f^\# * |k|)^\#(x) \leq (2f^\# * k^\#)(x),$$

hence

$$\|Tf\|_B \leq 2\|k\|_{1,w} \cdot \|f^\#\|_B$$

which implies that

$$\|(T^j f)^\#\|_B \leq 2^j \|k\|_{1,w}^j \|f^\#\|_B = C_1^j \|f^\#\|_B$$

for all f and $f^\#$ in B . Furthermore,

$$(C_h f - A_1 f - Tf)^\# = [(f - \sum_{i \in I} f(x_i) \psi_i) * (h - k)]^\#$$

$$\leq (Osc_W f)^\# * |h - k|.$$

As in the Lemma [5.2\(b\)](#), we get

$$\begin{aligned} \|(C_h f - A_1 f - T f)^\#\|_B &\leq \|(Osc_W f)^\#\|_B \|(h - k)\|_{1,w} \\ &\leq \eta \|f\|_B \|(h - k)\|_{1,w} \end{aligned}$$

for all $f \in B^\Omega$.

For $\phi, \phi^\# \in B$, the operator identity

$$\begin{aligned} &(C_h - A_1)^m - T^m \\ &= (C_h - A_1 - T)[(C_h - A_1)^{m-1} T^0 + \cdots + (C_h - A_1)^0 T^m] \\ &= \sum_{j=0}^{m-1} T^j (C_h - A_1 - T) (C_h - A_1)^{m-j-1} \end{aligned}$$

holds and the above estimates provide

$$\begin{aligned} \|(C_h - A_1)^m \phi - T^m \phi\|_B &= \left\| \sum_{j=0}^{m-1} T^j (C_h - A_1 - T) (C_h - A_1)^{m-j-1} \right\|_B \\ &\leq \sum_{j=0}^{m-1} C_1^j \|(C_h - A_1 - T) (C_h - A_1)^{m-j-1} \phi^\#\|_B \\ &\leq C(\Omega_0) \|(h - k)\|_{1,w} \sum_{j=0}^{m-1} C_1^j \|(C_h - A_1)^{m-j-1} \phi\|_B \\ &\leq C(\Omega_0) \sum_{j=0}^{m-1} C_1^j C_1^{m-j-1} \|(h - k)\|_{1,w} \cdot \|\phi^\#\|_B. \end{aligned}$$

Setting $T_s = \sum_{j=0}^s T^j$, we have

$$\begin{aligned} \|D_1^s \phi - T_s \phi\|_B &= \left\| \sum_{j=0}^s (C_h - A_1)^j \phi - T^j \phi \right\|_B \\ &\leq C \|(h - k)\|_{1,w} \cdot \|\phi^\#\|_B \end{aligned}$$

where C is a general constant, covering all constants of the previous calculations. Combining these estimates we see that

$$\|D_1^s(\phi_{K,W} * k) \cdot \mathbf{1}_K\|_B \leq \|(D_1^s - T_s)(\phi_{K,W} * k)\|_B + \|T_s(\phi_{K,U} * k) \mathbf{1}_K\|_B$$

$$\begin{aligned}
&\leq C\|(h-k)\|_{1,w} \cdot \|\phi_{K,W} * k\|_B + \|T_s(\phi_{K,U} * k)\mathbf{1}_K\|_B \\
&\leq C\|(h-k)\|_{1,w} d(W, \Omega_0) \|f\|_B \|k^\#\|_{1,w} + \|[T_s(\phi_{K,U} * k)]\mathbf{1}_K\|_B.
\end{aligned}$$

Now, choosing k as the restriction of h to a compact set in G such that

$$\|(h-k)\|_{1,w} \leq \frac{\varepsilon}{3C\|k^\#\|_{1,w}C(\Omega_0)}$$

we obtain

$$\|I_3\| \leq \varepsilon\|f\|_B/3 + \|[T_s(\phi_{K,W} * k)]\mathbf{1}_K\|_B.$$

With $\text{supp}(k) \subseteq W$, it can be seen that

$$T_s[\phi_{K,W} * k]\mathbf{1}_K = 0.$$

This completes the proof.

Theorem 5.2 *Let K be a compact set in G and $f \in [L_w^p]^\Omega(G)$, then*

$$\|E_T f\|_p \leq d_h \|f\|_{p,w} \cdot \sup_{x \notin K} w^{-1}(x),$$

where

$$d_h = \| \|D_1\| \|_{p,w} \cdot \|h\|_{1,w}.$$

Proof. Since the support of $\sum_{i \notin I_K} f(x_i)\psi_i$ is contained in

$$V = \{x \in G : x = x_i + u, x_i \notin K, u \in U\},$$

we obtain by means of Lemma 5.2

$$\begin{aligned}
\|E_T f\|_p &= \left\| \sum_{i \notin K} f(x_i)e_i \right\|_p \\
&= \left\| \sum_{i \notin K} f(x_i)D_1(\psi_i * h) \right\|_p = \|D_1 \sum_{i \notin K} f(x_i)(\psi_i * h)\|_p \\
&\leq \| \|D_1\| \|_{B^{\Omega_0}} \|h\|_{p,w} \cdot \left\| \sum_{i \notin K} f(x_i)\psi_i \right\|_p \leq d_h \left\| \sum_{i \notin K} f(x_i)\psi_i \right\|_p \\
&\leq d_h \|f^\#\mathbf{1}_V\|_p = d_h \|f^\#\mathbf{1}_V w w^{-1}\|_p \leq d_h \|f^\#\|_{p,w} \sup_{x \notin V} w^{-1}(x),
\end{aligned}$$

which completes the proof.

6 Aliasing Error

If the reconstruction algorithm is applied to the samples $f(x_i)$ of a non-band-limited, but at least a continuous function f it generates a band-limited function

$$f_A = \sum_{i \in I} f(x_i) e_i$$

with spectrum in Ω_0 . Then the difference $E_A f = f - f_A$ is called *aliasing error*. In order to estimate E_A we use again the space $CB = W(C_0, B)$ defined earlier. We will further make use of the following result which is new even for the Euclidean case.

th3 **Theorem 6.1** *Assume that \hat{G} is σ -compact. Then there exists a bounded sequence $h^{(k)}$ of band-limited functions in $L^1(G)$ (with $\|h^{(k)}\|_1 \leq 2$) and a sequence $U^{(k)}$ of neighborhoods of the identity such that for any choice of a $U^{(k)}$ -BUPU $\Psi^{(k)}$, the sequence of reconstruction operators $D^{(k)}A^{(k)}$ satisfies*

$$\|f - D^{(k)}A^{(k)}f\|_{p'} \rightarrow 0,$$

whenever $\hat{f} \in W(L^p, l^1)$, for some $p \in [1, 2]$.

Remarks.

1. For general dual groups \hat{G} which are not σ -compact one has to use a bounded net of reconstruction operators instead of the sequence $(D^{(k)}A^{(k)})_{k \geq 0}$.
2. If $f \in L^p_v(\hat{G})$, for some weight function v with $1/v \in L^p(\hat{G})$, then $\hat{f} \in W(L^p, l^p_v) \subseteq W(L^p, l^1)$. In particular Theorem 6.1 applies for any $p \in [1, 2]$ if $G = \mathbb{R}^d$, $|\hat{f}(x)| \leq C(1 + |x|)^\alpha$, for some $\alpha > d$.

The proof of Theorem 6.1 depends on the following lemmas.

lem3 **Lemma 6.2** *For any given weight w satisfying the BD-condition, there exists $C_w > 0$, such that for any compact set Ω in \hat{G} , there exists some band-limited $h \in L^1_w(G)$ satisfying $\|h\|_{1,w} \leq 2C_w$ and $\hat{h}(\omega) = 1$ on Ω .*

Proof. The proof follows ^{BPR}[2], using the following facts.

1. There exists $\phi \in L_w^1(G)$, $\hat{\phi}(\omega) \stackrel{\text{RS00}}{=} 1$ on Ω , because \mathcal{FL}_w^1 is a Wiener algebra in the sense of Reiter [15].
2. There exist bounded approximate units $(\rho_\beta)_{\beta \in J}$ in $L_w^1(G)$, of band-limited functions, i.e., $\sup_\rho \|\rho_\beta\|_{1,w} \leq C_w$. Since \hat{G} is σ -compact one may assume that this is a sequence.

Due to these observations, we can find β_0 such that $\|\rho_{\beta_0} * \phi - \phi\|_{1,w} < C_w$. Setting $\rho = \rho_{\beta_0}$ and following [2], we have for

$$h = \rho + \phi - \rho * \phi$$

$\hat{h}(x) = 1$ on Ω , since $(1 - \hat{h})(\omega) = (1 - \hat{\phi})(\omega)(1 - \hat{\rho})(\omega) = 0$ on Ω , and

$$\|h\|_{1,w} \leq \|\rho\|_{1,w} + \|\phi - \rho * \phi\|_{1,w} < C_w + C_w = 2C_w.$$

Remark. Clearly one can choose $C_w = 2$ (even $C_w = 1 + \epsilon$, for any $\epsilon > 0$) for the case that $w(x) = 1$.

lem4 **Lemma 6.3** *If $C_c(G)$ is dense in B and $B \subseteq C^0(G)$, then for every $f \in CB$, $Osc_U f \in CB$ for a sufficient small $U \in \mathcal{U}(e)$ and $\|Osc_U f\|_{CB} \rightarrow 0$ for $U \rightarrow \{e\}$.*

Proof. Since $|Osc_U f(x)| \leq 2f^\#(x)$ for $U \subseteq U_0$, we have $(Osc_U f)^\# \leq 2f^{\#\#} \in B$, by the translation invariance and solidity of B , hence $Osc_U f \in CB$ for $f \in CB$. Since $C_c(G)$ is dense in B , for a given $\eta > 0$ there exists some compact set $K \subseteq G$ such that

$$\|f^{\#\#} \mathbf{1}_{K^c}\|_B < \eta/4,$$

hence

$$\|(Osc_U^\# f) \mathbf{1}_{K^c}\|_B < \eta/2. \quad (15) \quad \text{estimate6}$$

Furthermore, $f \in B \subseteq C^0(G)$ is uniformly continuous and therefore one can find $U \subseteq U_0$ such that $\|Osc_U f\|_\infty < \eta/(2\|\mathbf{1}_K\|_B)$, and

$$\|(Osc_U^\# f) \mathbf{1}_K\|_B \leq \|Osc_U f\|_\infty \|\mathbf{1}_K\|_B < \eta/2. \quad (16) \quad \text{estimate7}$$

From estimate6 and estimate7, we infer that for any sufficiently small $U \subseteq U_0$

$$\|Osc_U^\# f\|_B \leq \|(Osc_U^\# f) \mathbf{1}_{K^c}\|_B + \|(Osc_U^\# f) \mathbf{1}_K\|_B < \eta.$$

Proof of Theorem ^{th3}6.1 Let Ω^k be an increasing sequence of compact subsets in \hat{G} with $\hat{G} = \cup \Omega^k$. We choose $h^{(k)}$ according to Lemma ^{lem3}6.2 such that $\hat{h}^{(k)} = 1$ on Ω^k and $\|h^{(k)}\|_1 \leq 2$. For $h^{(k)}$ we choose $U^{(k)}$ such that for every $U^{(k)}$ -BUPU $\Psi^{(k)}$ we have for $A^{(k)}f = Sp_{\Psi^{(k)}}f * h^{(k)}$, (where $A = A_1$ as in ^{FPQ2}[9]) $\|C_{h^{(k)}} - A^{(k)}\|_{p'} < 1/2$. Then $D^{(k)} = \sum_{n=0}^{\infty} (C_{h^{(k)}} - A^{(k)})^n$ satisfies $\|D^{(k)}\|_{p'} \leq 2$ and

$$\|D^{(k)}A^{(k)}g\|_{p'} \leq 3\|g\|_{p'}, \quad \forall g \in L^{p'}(G),$$

because

$$\begin{aligned} \|A^{(k)}\|_{p'} &\leq \|C_{h^{(k)}}\|_{p'} + \|C_{h^{(k)}} - A^{(k)}\|_{p'} \\ &\leq \|h^{(k)}\|_1 + 1/2 \leq 2 + 1/2 < 3. \end{aligned}$$

For any $\epsilon > 0$ and f with $\hat{f} \in W(L^p, l^1)$, we may choose some k such that

$$\|\hat{f} - \hat{f} \mathbf{1}_{\Omega^k}\|_{W(L^p, l^1)} < \epsilon/6,$$

and define f_k by $\hat{f}_k = \hat{f} \mathbf{1}_{\Omega^k}$. For given samples of f , we get $f_a = D^{(k)}A^{(k)}f$. Using the assumption $B \hookrightarrow W(C, l^{p'})$ (e.g. if $\hat{f} \in W(L^p, l^1)$ for all $f \in B$), we compute

$$\begin{aligned} \|E_A f\|_B &= \|f - f_a\|_B = \|f_k + (f - f_k) - D^{(k)}A^{(k)}f_k - D^{(k)}A^{(k)}(f - f_k)\|_B \\ &\leq \|f - f_k\|_B + \|D^{(k)}\| \cdot \|A^{(k)}\| \cdot \|f - f_k\|_B \end{aligned}$$

because $f_k = D^{(k)}A^{(k)}f_k$. Combining these estimates we obtain

$$\|E_A f\|_{p'} \leq 6\|f - f_k\|_B < \epsilon.$$

7 Combined Errors

Apparently, in typical applications the errors described in the previous sections may occur simultaneously. Thus, it will be necessary to deal with the possible combinations of these errors. In order to avoid multiple, but very similar arguments we describe in this section the combination of jitter and aliasing errors on a locally compact Abelian group. We prove the following:

th4 **Theorem 7.1** *If $f \in CB$ is sampled at points $x'_i, i \in I$, instead of x_i and the reconstruction algorithm yields*

$$\tilde{f}(x) = \sum_{i \in I} f(x'_i) e_i,$$

then

$$\|\tilde{f} - f\|_B \leq \|f - f * h\|_B + d_h \|f - f * h\|_{CB} + C(U, \Omega_0) \|f\|_B,$$

where $U, h \in L_w^1(G)$ and constants are taken from the previous theorems.

Proof. Combining the jitter and aliasing errors for the sample points $x'_i, i \in I$, we get

$$\begin{aligned} \tilde{f} - f &= \underbrace{\sum_{i \in I} [f(x'_i) - (f * h)(x'_i)] e_i}_{I_1} \\ &+ \underbrace{\sum_{i \in I} [(f * h)(x'_i) - (f * h)(x_i)] e_i}_{I_2} \\ &+ \underbrace{\left(\sum_{i \in I} (f * h)(x_i) e_i \right) - f}_{I_3}. \end{aligned}$$

We have

$$\begin{aligned} \|I_1\|_B &= \left\| \sum_{i \in I} [f(x'_i) - (f * h)(x'_i)] e_i \right\|_B \\ &= \left\| \sum_{i \in I} [f(x'_i) - (f * h)(x'_i)] D_1(\psi_i * h) \right\|_B \\ &\leq d_h \left\| \sum_{i \in I} [f(x'_i) - (f * h)(x'_i)] \psi_i \right\|_B \\ &\leq d_h \|(f - f * h)^\# \|_B = d_h \|(f - f * h)\|_{CB}. \end{aligned}$$

We also have

$$\|I_3\|_B = \|(f - f * h)\|_B.$$

Finally, we see that I_2 is a jitter error of $f * h$. Hence, as in the proof of Theorem [4.1](#), we have

$$\begin{aligned} \|I_2\|_B &= \|E_J(f * h)\|_B = \left\| \sum_{i \in I} [(f * h)(x'_i) - (f * h)(x_i)] e_i \right\|_B \\ &= \left\| D \left(\sum_{i \in I} [(f * h)(x'_i) - (f * h)(x_i)] (\psi_i * h) \right) \right\|_B \\ &\leq C_h(U, \Omega_0) \|f * h\|_B \leq C_h(U, \Omega_0) \|h\|_{1,w} \|f\|_B. \end{aligned}$$

Combining these estimates completes the proof of the theorem. **Remarks.**

1. Even CB -norm estimates for $B = L^{p'}(G)$ and f as in Theorem [7.1](#) are possible.
2. The corresponding error estimates using the operators A_2, A_3 and A_4 can be obtained by similar arguments as for the theorems involving A_1 .

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