GABOR ANALYSIS OVER FINITE ABELIAN GROUPS

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Abstract. The topic of this paper is (multi-window) Gabor frames for signals over finite Abelian groups, generated by an arbitrary lattice within the finite time-frequency plane. Our generic approach covers both multi-dimensional signals as well as non-separable lattices. The main results reduce to well-known fundamental facts about Gabor expansions of finite signals for the case of product lattices, as they have been given by Qiu, Wexler-Raz or Tolimieri-Orr, Bastiaans and Van-Leest, among others. In our presentation a central role is given to spreading function of linear operators between finite-dimensional Hilbert spaces. Another relevant tool is a symplectic version of Poisson's summation formula over the finite time-frequency plane. It provides the Fundamental Identity of Gabor Analysis. In addition we highlight projective representations of the time-frequency plane and its subgroups and explain the natural connection to twisted group algebras. In the finite-dimensional setting these twisted group algebras are just matrix algebras and their structure provides the algebraic framework for the study of the deeper properties of finite-dimensional Gabor frames.

1. Introduction

In the last two decades a new branch of time-frequency analysis, called *Gabor analysis*, has found many applications in pure and applied mathematics. The connection between time-variant systems in communication theory and Gabor analysis has turned out to be of great importance for the application of Gabor analysis to real-world problems such as the transmission of signals between cellular phones. In modern digital communication there is an ongoing trend towards FFT-based multi-carrier modulation: popular wireline systems for the internet access such as ADSL are based on discrete multitone-modulation (DMT) and important wireless systems such as WLAN, UMTS, WIMAX make use of OFDM (orthogonal frequency division multiplex)-type modulation. In mathematical terminology both DMT and OFDM are essentially Weyl-Heisenberg group structured Riesz bases. Consequently all the mathematical machinery developed for Gabor frame theory can be exploited for the advanced design of so-called pulse-shaped multicarrier systems, [40]. All these applications lead in a natural manner to the discussion of Gabor frames for finite-dimensional Hilbert spaces. Due to the large potential of the material to the above mentioned applications of Gabor frames in signal analysis we want to address this note not only to mathematicians but also to engineers who are interested in the deeper mathematical background which is likely to provide a sound basis for further applications. For the same reason we have tried to present the results in a self-contained, although very general form. Finally we want to mention that many researchers have contributed to the construction of frames for finite-dimensional

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Hilbert spaces, e.g. [3, 4, 41]. The case of Gabor frames for finite-dimensional Hilbert spaces was especially treated by [1, 2, 9, 10, 15, 13, 27, 28, 31, 33, 34, 39], e.g. the Zibulski-Zeevi representation of a Gabor frame operator in the finite-dimensional setting was discussed in [9] and in her Ph.D. thesis [**?**] Matusiak has presented an approach based on representation theory of the Heisenberg group for elementary locally compact abelian groups. Finally, we want to mention Kaiblinger's work on the computation of dual Gabor windows of a continuous signal with the help of finte-dimensional Gabor frames in [25].

Let $\mathcal{H}$ be a finite-dimensional Hilbert space of dimension N with inner-product $\langle .,. \rangle_{\mathcal{H}}$. Then a *frame* for $\mathcal{H}$ is a **spanning family** of elements $\mathcal{G} = \{\mathbf{g}_\mathrm{i}\}_{\mathrm{i}\in\mathrm{I}}$ for an index set I of finite cardinality M. Obviously, the number of frame elements M has to be at least equal to the dimension N of $\mathcal{H}$.

For a frame $\mathcal{G} = \{\mathbf{g}_\mathrm{i}\}_{\mathrm{i}\in\mathrm{I}}$ for $\mathcal{H}$ there are certain operators which are naturally associated with a frame $\mathcal{G}$. Namely, the *analysis operator* $\mathrm{C}_\mathcal{G}$ is given by

$$\mathbf{f} \mapsto (\langle \mathbf{f}, \mathbf{g}_\mathrm{i} \rangle_\mathcal{H})_{\mathrm{i}\in\mathrm{I}} \in \mathbb{C}^\mathrm{M},$$

which maps $\mathbf{f} \in \mathcal{H}$ into $\mathbb{C}^\mathrm{M}$ and the *synthesis operator* $\mathrm{D}_\mathcal{G}$ which assigns to $\mathbf{c} = (\mathrm{c}_\mathrm{i})_{\mathrm{i}\in\mathrm{I}} \in \mathbb{C}^\mathrm{M}$ an element of $\mathcal{H}$, i.e.

$$\mathbf{c} \mapsto \sum_{\mathrm{i}\in\mathrm{I}} c_\mathrm{i} \mathbf{g}_\mathrm{i}.$$

An elementary computation shows that the analysis operator $\mathrm{C}_\mathcal{G}$ is the adjoint operator of the synthesis operator $\mathrm{D}_\mathcal{G}$, i.e. $\mathrm{C}_\mathcal{G}^\star = \mathrm{D}_\mathcal{G}$. We have that for $\mathbf{f} \in \mathcal{H}$ and $\mathbf{c} \in \mathbb{C}^\mathrm{M}$:

$$\langle \mathrm{C}_\mathcal{G}\mathbf{f}, \mathbf{c} \rangle_\mathcal{H} = \langle \mathbf{f}, \mathrm{D}_\mathcal{G}\mathbf{c} \rangle_{\mathbb{C}^\mathrm{M}}$$

The most important operator associated to $\{\mathbf{g}_\mathrm{i}\}_{\mathrm{i}\in\mathrm{I}}$ is the *frame operator* $\mathrm{S}_\mathcal{G}$ given by

$$\mathrm{S}_\mathcal{G}\mathbf{f} = \sum_{\mathrm{i}\in\mathrm{I}} \langle \mathbf{f}, \mathbf{g}_\mathrm{i} \rangle_\mathcal{H} \, \mathbf{g}_\mathrm{i}.$$

It can be used to characterize frames for $\mathcal{H}$ in the following manner: An indexed family $\mathcal{G} = \{\mathbf{g}_\mathrm{i}\}_{\mathrm{i}\in\mathrm{I}}$ is a frame if and only if the kernel of $\mathrm{C}_\mathcal{G}$ is the nullspace, i.e. $\ker(\mathrm{C}_\mathcal{G}) = \{0\}$. Since the kernel $\ker(\mathrm{C}_\mathcal{G})$ of $\mathrm{C}_\mathcal{G}$ and the kernel $\ker(\mathrm{S}_\mathcal{G})$ of $\mathrm{S}_\mathcal{G}$ are equal. The function $\mathbf{f} \mapsto \sum_{i\in I} |\langle \mathbf{f}, \mathbf{g}_i \rangle_\mathcal{H}|^2$ is continuous and non-zero on the compact unit-sphere $\{\mathbf{f}, \|\mathbf{f}\| = 1\}$ of $\mathcal{H}$, one concludes that $\{\mathbf{g}_\mathrm{i}\}_{\mathrm{i}\in\mathrm{I}}$ is a frame for $\mathcal{H}$ if (and only if) there exist some constants $\mathrm{A}, \mathrm{B} > 0$ such that for all non-zero $\mathbf{f} \in \mathcal{H}$

$$\mathrm{A}\|\mathbf{f}\|_\mathcal{H}^2 \le \|\mathrm{C}_\mathcal{G}\mathbf{f}\|_{\mathbb{C}^\mathrm{M}}^2 = \sum_{\mathrm{i}\in\mathrm{I}} |\langle \mathbf{f}, \mathbf{g}_\mathrm{i} \rangle_\mathcal{H}|^2 \le \mathrm{B}\|\mathbf{f}\|_\mathcal{H}^2.$$

If $\mathrm{A} = \mathrm{B}$ in the preceding inequalities, then $\{\mathbf{g}_\mathrm{i}\}_{\mathrm{i}\in\mathrm{I}}$ is called a *tight frame* for $\mathcal{H}$. Some authors call tight frames with $\mathrm{A} = \mathrm{B} = 1$ *normalised tight frames* or *Parseval* *frames*. Any frame $\mathcal{G}$ provides natural expansion of arbitrary elements $\mathbf{f} \in \mathcal{H}$:

$$\begin{aligned}
\mathbf{f} &= S_{\mathcal{G}} S_{\mathcal{G}}^{-1}\,\mathbf{f} = \sum_{i\in I} \langle \mathbf{f}, S_{\mathcal{G}}^{-1}\mathbf{g}_i\rangle_{\mathcal{H}}\mathbf{g}_i \\
&= S_{\mathcal{G}}^{-1} S_{\mathcal{G}}\,\mathbf{f} = \sum_{i\in I} \langle \mathbf{f}, \mathbf{g}_i\rangle_{\mathcal{H}}\, S_{\mathcal{G}}^{-1}\mathbf{g}_i \\
&= S_{\mathcal{G}}^{-1/2} S_{\mathcal{G}} S_{\mathcal{G}}^{-1/2}\,\mathbf{f} = \sum_{i\in I} \langle \mathbf{f}, S_{\mathcal{G}}^{-1/2}\mathbf{g}_i\rangle_{\mathcal{H}}\, S_{\mathcal{G}}^{-1/2}\mathbf{g}_i.
\end{aligned}$$

These formulas also show how the reconstruction of f from the frame coefficients $(\langle \mathbf{f}, \mathbf{g}_i\rangle_{\mathcal{H}})_{i\in I}$ is possible. In general the family $\{\mathbf{g}_i\}_{i\in I}$ is not linear independent which implies the non-uniqueness of the frame decompositions of $\mathbf{f}$.

The paper is organized as follows: In Section 2 we recall some basic facts about group algebras and finite-dimensional matrix algebras which we will apply to the study of twisted group algebras for $G \times \widehat{G}$. Furthermore, we emphasize the relevanve of the existence of a trace and of the C^*-algebra norm on a finite-dimensional matrix algebra structure. The main aim of this section is to make our exposition self-contained and accesible to engineers and graduate students interested in Gabor analysis. The main result of Section 2 is the Theorem 2.9, where we state the spreading representation of a linear operator on $\mathbb{C}^N$. In Section 3 we treat (multi-window) Gabor systems generated with the help of a subgroup $\Lambda \lhd G \times \widehat{G}$ of a finite time-frequency plane. Our main tool is the spreading characterization of Λ-invariant operators and a generalization of Janssen's representation for such operators. Finally we discuss the Wexler-Raz biorthogonality relations and the duality principle of Ron-Shen for finite-dimensional Gabor frames. In the last section, Sectopm 4, we state some results about the optimality of the canonical dual and tight Gabor window with respect to different measures. We are especially interested in the relation between the tight canonical Gabor window and the Löwdin orthogonalization, [29].

2. Basics on matrix algebras

Let A be a linear mapping from $\mathbb{C}^N$ to $\mathbb{C}^M$ with orthonormal bases $\{e_j : j = 1, ..., N\}$ and $\{f_j : j = 1, ..., M\}$. Then A can be identified with a $M \times N$ matrix $A = (a_{i,j})$ with entries $a_{i,j} = \langle Ae_j, f_i\rangle$ for $i = 1, ..., M$ and $j = 1, ..., N$. We identify the set of all linear mappings from $\mathbb{C}^N$ to $\mathbb{C}^M$ with the $M \times N$-matrices $\mathcal{M}_{M\times N}(\mathbb{C})$. Furthermore $\mathcal{M}_{M\times N}(\mathbb{C})$ possess an *involution* $\star$ which for $A = (a_{i,j})$ is defined by $A^\star = (\overline{a_{j,i}})$. The map $A \to A^\star$ is an anti-isomorphism, which means that $(\lambda A)^\star = \overline{\lambda} A^\star$ for $\lambda \in \mathbb{C}$ and $(AB)^\star = B^\star A^\star$. Recall that for $A \in \mathcal{M}_{M\times N}(\mathbb{C})$ the matrices $AA^\star$ and $A^\star A$ are hermitian, which have the same non-zero eigenvalues $\lambda_1, ..., \lambda_r$, counting multiplicities. The square roots of the eigenvalues $A^\star A$ are called the *singular values* $s_1 \geq s_2 \geq \cdots \geq s_r > 0$ where r is the rank of A. A main result on rectangular matrices A in $\mathcal{M}_{M\times N}(\mathbb{C})$ is the *singular value decomposition*. It tells us that for every $A \in \mathcal{M}_{M\times N}(\mathbb{C})$ there exist unitary matrices $U \in \mathcal{M}_{M\times M}(\mathbb{C})$ and $V \in \mathcal{M}_{N\times N}(\mathbb{C})$ and a diagonal matrix D with non-negative diagonal entries $\mathrm{diag}(s_1, ..., s_r, 0, ..., 0)$ such that $A = UDV^\star$. Despite the possible lack of uniqueness of the factorization, the sequence of singular values is uniquely determined, because they are just the eigenvalues of $A^\star A$. Hence they can be used to define a family of norms $\{\|.\|_{S^p}\}$ on

$\mathcal{M}_{\mathrm{M}\times\mathrm{N}}(\mathbb{C})$, for $\mathrm{p} \in [1,\infty]$, which are indeed the finite-dimensional versions of the so-called Schatten-von Neumann classes of compact operators on a Hilbert space. They are given by

$$\|\mathrm{A}\|_{\mathrm{S}^\mathrm{p}} = \Big(\sum_{\mathrm{i}=1}^{\mathrm{r}} \mathrm{s_i}^\mathrm{p}\Big)^{1/\mathrm{p}}.$$

The case $\|.\|_{\mathrm{S}^2}$ arises in a variety of applications and is called *Frobenius, Hilbert-Schmidt* or *Schur norm.* In the following we will denote $\|\mathrm{A}\|_{\mathrm{S}^2}$ by $\|\mathrm{A}\|_{\mathrm{Fro}}$ and refer to it as the Frobenius norm of $\mathrm{A} \in \mathcal{M}_{\mathrm{M}\times\mathrm{N}}(\mathbb{C})$. The other important Schatten-von Neumann norm $\|.\|_{\mathrm{S}^\mathrm{p}}$ arises for $\mathrm{p} = \infty$ which equals s_1 since

$$\|\mathrm{A}\|_{\mathrm{S}^\infty} = \lim_{\mathrm{p}\to\infty} \|\mathrm{A}\|_{\mathrm{S}^\mathrm{p}} = \mathrm{s}_1.$$

By definition s_1 is the largest eigenvalue of A which is the *operator norm* of A and we denote it by $\|\mathrm{A}\|_{\mathrm{op}}$ for $\mathrm{A} \in \mathcal{M}_{\mathrm{M}\times\mathrm{N}}(\mathbb{C})$.

The norms $\|\mathrm{A}\|_{\mathrm{Fro}}$ and $\|\mathrm{A}\|_{\mathrm{op}}$ endow $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ with a Euclidean structure and a C^*-algebraic structure respectively. Later we will explore these two structures further which will allow us to draw some elementary conclusions.

Recall that an *algebra* $\mathcal{A}$ is a vector space over $\mathbb{C}$ with some multiplication compatible with the linear structure. If $\mathcal{A}$ has a unit element, then $\mathcal{A}$ is called *unital.* Let $\mathcal{S}$ be a subset of $\mathcal{A}$. Then the *subalgebra generated* by the set $\mathcal{S}$ is denoted by $\mathrm{alg}(\mathcal{S})$ and is defined to be the smallest subalgebra of $\mathcal{A}$ that contains $\mathcal{S}$, i.e. $\mathrm{alg}(\mathcal{S}) = \mathrm{span}\{\mathrm{s}_1\cdots\mathrm{s}_\mathrm{n} : \mathrm{s}_1,...,\mathrm{s}_\mathrm{n} \in \mathcal{S}\}$, see [12] for more information on finite-dimensional algebras.

Let $(\mathrm{G},\cdot)$ be a finite group and let V be a finite-dimensional vector space. Then a *projective group representation* $\rho : \mathrm{G} \to \mathrm{V}$ of G is a family of unitary mappings $\{\rho(\mathrm{g}) : \mathrm{g} \in \mathrm{G}\}$ such that $\rho(\mathrm{g}_1\cdot\mathrm{g}_2) = \mathrm{c_G}(\mathrm{g}_1,\mathrm{g}_2)\rho(\mathrm{g}_1)\rho(\mathrm{g}_2)$ for unimodular numbers $\mathrm{c_G}(\mathrm{g}_1,\mathrm{g}_2)$. The projective group representation $\rho : \mathrm{G} \to \mathrm{V}$ of G is called *irreducible* if $\{0\}$ and V are the only ρ-invariant subspaces of V.

Recall that the *commutant* $\mathcal{S}'$ of a set $\mathcal{S} \subseteq \mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ is

$$\mathcal{S}' := \{\mathrm{A} \in \mathcal{S} : \mathrm{AB} = \mathrm{BA} \text{ for all } \mathrm{B} \in \mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})\}.$$

Observe that $\mathcal{S}'$ is always a subalgebra of $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ containing all scalar multiples of the unit in $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$. Moreover the commutant of $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ is called the *center* of $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$.

Let $\mathrm{A} \in \mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$. Then the *trace* of $\mathrm{A} = (\mathrm{a_{i,j}})$ is defined as the sum of its diagonal elements, i.e.

$$\mathrm{tr}(\mathrm{A}) = \mathrm{a}_{1,1} + \cdots + \mathrm{a}_{\mathrm{N},\mathrm{N}}.$$

The trace tr on $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ has the following properties:

(1) tr is a linear functional on $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$.
(2) $\mathrm{tr}(\mathrm{A}^\star) = \overline{\mathrm{tr}(\mathrm{A})}$.
(3) $\mathrm{tr}(\mathrm{AB}) = \mathrm{tr}(\mathrm{BA})$ for all $\mathrm{A},\mathrm{B} \in \mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ *(tracial property).*

From the definition of the trace and the product of matrices $\mathrm{A}, \mathrm{B} \in \mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ we obtain that

$$\mathrm{tr}(\mathrm{A}^\star \mathrm{B}) = \sum_{\mathrm{i,j}=1}^{\mathrm{N}} \overline{\mathrm{a_{i,j}}} \mathrm{b_{i,j}} = \overline{\mathrm{tr}(\mathrm{B}^\star \mathrm{A})}.$$

The identification of $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ with $\mathbb{C}^{\mathrm{N}^2}$ gives that $(\mathrm{A}, \mathrm{B}) \mapsto \mathrm{tr}(\mathrm{AB}^\star)$ is an inner product on $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$. In particular tr is *non-degenerate*, i.e.

$$\mathrm{tr}(\mathrm{AB}) = 0 \text{ for all } \mathrm{B} \in \mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C}) \Rightarrow \mathrm{A} = 0.$$

Therefore $(\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C}), \mathrm{tr})$ is a N^2-dimensional Hilbert space. As a consequence we know that every linear functional $\phi : \mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C}) \to \mathbb{C}$ is of the form $\phi(\mathrm{A}) = \mathrm{tr}(\mathrm{AB})$ for some $\mathrm{B} \in \mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$. The tracial property and Schur's lemma finally imply uniqueness of the trace.

Lemma 2.1. *If ϕ is a linear functional on $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ with the tracial property $\phi(\mathrm{AB}) = \phi(\mathrm{BA})$, then there exists a constant $\mathrm{c} \in \mathbb{C}$ such that $\phi(\mathrm{A}) = \mathrm{c}\,\mathrm{tr}(\mathrm{A})$ for all $\mathrm{A} \in \mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$.*

Proof. Let B in $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ such that $\phi(\mathrm{A}) = \mathrm{tr}(\mathrm{AB}^\star)$. Then $\mathrm{tr}((\mathrm{BC} - \mathrm{CB})\mathrm{A}) = \phi(\mathrm{AC}^\star) - \phi(\mathrm{CA}^\star) = 0$ for all $\mathrm{A}, \mathrm{C} \in \mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$. Since trace is non-degenerate $\mathrm{BC} = \mathrm{CB}$ for all $\mathrm{C} \in \mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$, which means that B is in the center of $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$. Consequently $\mathrm{B} = \mathrm{c}\mathbb{I}_{\mathrm{N}}$ for some $\mathrm{c} \in \mathbb{C}$. □

Therefore $\langle \mathrm{A}, \mathrm{B}\rangle_{\mathrm{Fro}} := \mathrm{tr}(\mathrm{AB}^\star)$ is the unique inner product on $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ and

$$\|\mathrm{A}\|_{\mathrm{Fro}}^2 := \mathrm{tr}(\mathrm{AA}^\star) = \mathrm{tr}(\mathrm{AA}^\star) = \sum_{\mathrm{i,j}=1}^{\mathrm{N}} |\mathrm{a_{i,j}}|^2$$

defines a norm which endows $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ with a *unique* Euclidean structure.

At this point we invoke the C^*-algebraic structure of $\mathcal{A}(\mathbb{Z}_{\mathrm{N}} \times \mathbb{Z}_{\mathrm{N}})$, i.e. of $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ with $\|.\|_{\mathrm{op}}$. Recall that an involutive normed algebra $(\mathcal{A}, \|.\|_{\mathcal{A}})$ is a C^*-algebra if the norm $\|.\|_{\mathcal{A}}$ fulfills

$$\|\mathrm{A}^\star \mathrm{A}\|_{\mathcal{A}} = \|\mathrm{A}\|_{\mathcal{A}}^2 \text{ for all } \mathrm{A} \in \mathcal{A}.$$

The study of the state space for a finite-dimensional C^*-algebra provides the link to Grassmannian frames and finite geometry. First of all a *finite-dimensional C^*-algebra $\mathcal{A}$* is isomorphic to a full matrix algebra $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ with involution given by the adjoint of a matrix A, i.e. the finite-dimensionality forces $\mathcal{A}$ to be unital. We know that $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ with the operator norm provides a natural example of a C^*-algebra. The next lemma states that this exhausts all possible finite-dimensional C^*-algebras.

Lemma 2.2. *Let $\mathcal{A}$ be a finite-dimensional algebra with norm $\|.\|_{\mathcal{A}}$. Then $\|.\|_{\mathcal{A}}$ is a C^*-norm if and only if it is the operator norm $\|.\|_{\mathrm{op}}$.*

Proof. By the preceding observation $\mathcal{A}$ is isomorphic to $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ for some N. The argument relies on the relation between the Schatten-von Neumann classes and the operator norm $\|.\|_{\mathrm{op}}$. Let A be in $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ and $\mathrm{D} = \mathrm{AA}^\star$. Then we have to show that $\|\mathrm{D}\|_{\mathcal{A}} = \|\mathrm{D}\|_{\mathrm{op}}$. Let $\mathrm{d}_1 \geq \mathrm{d}_2 \geq \cdots \mathrm{d}_{\mathrm{r}} > 0$ be the singular values of A, i.e. the eigenvalues of $AA^\star$. Then by the equivalence of all norms on a finite-dimensional vector space we get that

$$d_1 = \inf_{R\geq 0}\Big(\lim_{n\to\infty}\frac{\|D^{2^m}\|_{\mathcal{A}}}{R^{2^m}}\Big) = \inf_{R\geq 0}\Big(\lim_{n\to\infty}\frac{\|D\|_{\mathcal{A}}^{2^m}}{R^{2^m}}\Big).$$

In the last equality we have used the C^*-algebra property of $\|.\|_{\mathcal{A}}$. Consequently $d_1 = \|D\|_{\mathcal{A}} = \|A^\star A\|_{op} = \|D\|_{op}$, the desired assertion. □

This elementary fact will be very useful in our approach to Ron-Shen duality for Gabor frames in $\mathbb{C}^N$.

3. Spreading representation

In the last years frames for finite-dimensional Hilbert spaces have been constructed from representations of finite groups since these frames inherit symmetries from the groups. We study frames associated with representations of the Heisenberg group, which is a non-commutative group, a so-called two-step nilpotent group, see [18, 38] for a detailed treatment of Heisenberg groups.

We denote the circle group by $\mathbb{T} = \{\tau \in \mathbb{C} : |\tau| = 1\}$, and the *(Weyl)-Heisenberg group* by

$$\mathbb{H}(\mathbb{Z}_N) = \{(\tau, k, s) : k, s \in \mathbb{Z}_N, \tau \in \mathbb{T}\}$$

with multiplication given by

$$(\tau_1, k_1, r_1)(\tau_2, k_2, r_2) = (\tau_1\tau_2 \cdot e^{2\pi i k_2 r_1}, k_1 + k_2, r_1 + r_2). \tag{1}$$

Next we have to fix some notations concerning Gabor frames in the finite-dimensional Hilbert space $\mathbb{C}^N$. Its elements are considered as discrete time signals of period N, i.e. $f(k) = f(k + mN)$ for all $m \in \mathbb{Z}$ and if k exceeds the period N then k is taken modulo N. In the sequel we represent a signal $\mathbf{f} = (f(k)) = (f(0), ..., f(N-1))^T$ as a *column vector* of $\mathbb{C}^N$. The Euclidian structure of $\mathbb{C}^N$ induces an inner-product on discrete N-periodic signals by $\langle \mathbf{f}, \mathbf{g}\rangle_{\mathbb{C}^N} = \sum_{k=0}^{N-1} f(k)\overline{g(k)}$ for $\mathbf{f}, \mathbf{g} \in \mathbb{C}^N$.

The key players of our investigation are *time-frequency shifts* of discrete N-periodic signals. For an integer k the *translation operator* T_k is defined by

$$T_k\mathbf{f} = (f(k), f(k+1), ..., f(k-1)),\ \ \mathbf{f} = (f(j)) \in \mathbb{C}^N,$$

the *modulation operator* M_r is given by

$$M_r\mathbf{f} = (f(0), e^{2\pi i r/N}f(1), e^{2\pi i 2r/N}f(2), ..., e^{2\pi i r(N-1)/N}f(N-1)),\ \ \mathbf{f} = (f(j)) \in \mathbb{C}^N,$$

and the *time-frequency shift* $\pi(k, r)$ of $\mathbf{f}$ by

$$\pi(k, r)\mathbf{f} = M_rT_k\mathbf{f},\ \text{ for } \mathbf{f} = (f(j)) \in \mathbb{C}^N.$$

Next we turn our attention to the key players of our investigation: the *time-frequency shifts*, which are made up from *translations* and *modulations* of a signal.

We identify $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ with $\mathbb{C}^{\mathrm{N}^2}$ and denote the standard basis of $\mathbb{C}^{\mathrm{N}^2}$ by $\{\delta_{\mathrm{i,j}} : \mathrm{i,j} = 1, ..., \mathrm{N}\}$. Then the matrix representations of translation T_1 and of the modulation M_1 are given by

$$\mathrm{T}_1 = \begin{pmatrix} 0 & 1 & 0\cdots & 0 \\ \vdots & \ddots & 1 & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 0 & 0\cdots & 0 \end{pmatrix} \text{ and } \mathrm{M}_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{2\pi i/\mathrm{N}} & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{2\pi i(\mathrm{N}-1)/\mathrm{N}} \end{pmatrix}.$$

An elementary computation yields ${\mathrm{T}_1}^{\mathrm{N}} = {\mathrm{M}_1}^{\mathrm{N}} = \mathbb{I}_{\mathrm{N}}$ and

$$\mathrm{M}_1\mathrm{T}_1 = e^{2\pi i/\mathrm{N}}\mathrm{T}_1\mathrm{M}_1.$$

The preceding equation is a finite-dimensional analogue of the non-commutativity of translation and modulation. In his work on the foundations of quantum mechanics Weyl treated finite-dimensional analogues of the commutation relations which appeared in the works of Born, Dirac, Heisenberg, Jordan and Schrödinger on quantum mechanics, [45]. In these investigations Weyl was heading towards the fundamental theorem that all solutions of Born's commutation relation are unitarian equivalent to each other. He was only able to prove this result for the finite-dimensional case. Later Stone and von Neumann independently turned Weyl's formal arguments into a rigorous proof which is the famous Stone-von Neumann theorem. This theorem is one of the most important facts in non-commutative harmonic analysis and lies at the heart of time-frequency analysis.

Theorem 3.1 (Weyl)**.** *Let* U *and* V *be unitary operators on* $\mathbb{C}^{\mathrm{N}}$ *such that the algebra* *alg*(U,V) *generated by* U *and* V *is* $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ *and* U,V *satisfy the commutation relation*

(2) $$\mathrm{VU} = \mathrm{e}^{2\pi \mathrm{ik}/\mathrm{N}}\mathrm{UV} \textit{ for } \gcd(\mathrm{k},\mathrm{N}) = 1.$$

Then U *and* V *are unitarily equivalent to* T_1 *and* M_1*, i.e. there exists a unitary operator* Z *such that* $\mathrm{Z}^\star\mathrm{UZ} = \mathrm{T}_1$ *and* $\mathrm{Z}^\star\mathrm{VZ} = \mathrm{M}_1$*.*

Weyl had described the result in terms of projective representations of $\mathbb{Z}_{\mathrm{N}} \times \mathbb{Z}_{\mathrm{N}}$.

Proposition 3.2 (Weyl)**.** *Let* $\{\rho(\mathrm{k},\mathrm{s}) : \mathrm{k},\mathrm{s} \in \mathbb{Z}_{\mathrm{N}}\}$ *be an irreducible projective representation of* $\mathbb{Z}_{\mathrm{N}} \times \mathbb{Z}_{\mathrm{N}}$*. Then* ρ *is unitarily equivalent to the projective representation of* $\mathbb{Z}_{\mathrm{N}} \times \mathbb{Z}_{\mathrm{N}}$ *by time-frequency shifts* $\{\pi(\mathrm{k},\mathrm{r}) = \mathrm{M}_{\mathrm{r}}\mathrm{T}_{\mathrm{k}} : \mathrm{k},\mathrm{r} \in \mathbb{Z}_{\mathrm{N}}\}$*.*

It is a fundamental fact of great importance for us that the mapping from (k,r) to the time-frequency shift operators $\{\pi(\mathrm{k},\mathrm{r}) : \mathrm{k},\mathrm{r} \in \mathbb{Z}_{\mathrm{N}}\}$ defines an *irreducible projective representation* of $\mathbb{Z}_{\mathrm{N}} \times \mathbb{Z}_{\mathrm{N}}$, i.e.

(3) $$\pi(\mathrm{k},\mathrm{r})\pi(\mathrm{l},\mathrm{s}) = e^{2\pi i(\mathrm{l}\cdot\mathrm{r}-\mathrm{k}\cdot\mathrm{s})}\pi(\mathrm{l},\mathrm{s})\pi(\mathrm{k},\mathrm{r}).$$

One advantage of the projective representation instead of unitary representation for the Heisenberg group of $\mathbb{Z}_{\mathrm{N}}$ is that the matrix-coefficients $\langle \mathbf{f}, \pi(\mathrm{k},\mathrm{r})\mathbf{g}\rangle$ for $\mathbf{f},\mathbf{g} \in \mathbb{C}^{\mathrm{N}}$ of $\{\pi(\mathrm{k},\mathrm{r}) : \mathrm{l},\mathrm{r} \in \mathbb{Z}_{\mathrm{N}}\}$ have a concrete practical meaning and we do not have to take care of additional phase factors. Namely as the *Short-time Fourier transform* of $\mathbf{f} \in \mathbb{C}^{\mathrm{N}}$ with respect to a fixed *window* $\mathbf{g} \in \mathbb{C}^{\mathrm{N}}$

$$\mathrm{V}_{\mathbf{g}}\mathbf{f}(\mathrm{k},\mathrm{r}) = \sum_{\mathrm{l}\in\mathbb{Z}_{\mathrm{N}}} \mathrm{f}(\mathrm{l})\overline{\mathrm{g}}(\mathrm{k}-\mathrm{l})\mathrm{e}^{-2\pi \mathrm{irl}/\mathrm{N}} = \langle \mathbf{f}, \pi(\mathrm{k},\mathrm{r})\mathbf{g}\rangle_{\mathbb{C}^{\mathrm{N}}}.$$

If we regard the elements of $\mathbb{Z}_N \times \mathbb{Z}_N$ as linearly independent vectors, most conveniently as $\delta_{k,0} \cdot \delta_{r,0}$ for $k, r \in \mathbb{Z}_N$, then the span of the elements $\delta_{k,0} \cdot \delta_{r,0}$ is the vector space

$$\mathcal{A}(\mathbb{Z}_N \times \mathbb{Z}_N) = \Big\{ \sum_{k,r \in \mathbb{Z}_N} a(k,r)\delta_{k,0} \cdot \delta_{r,0} \Big\}.$$

The vector space $\mathcal{A}(\mathbb{Z}_N \times \mathbb{Z}_N)$ has a product which arises from the group product by a twist of a uni-modular number. Therefore the arising group algebra is referred to as the *twisted group algebra* of $\mathbb{Z}_N \times \mathbb{Z}_N$. More precisely, the multiplication of $\delta_{k,0}\delta_{r,0}$ and $\delta_{l,0}\delta_{s,0}$ is the *(left) twisted translation* $\delta_{k,0}\delta_{r,0}\delta_{l,0}\delta_{s,0} = e^{2\pi i(l-k)\cdot r/N}\delta_{k+l,0}\delta_{r+s,0}$. For the elements of $\mathcal{A}(\mathbb{Z}_N \times \mathbb{Z}_N)$ this yields

$$\begin{aligned} & \Big(\sum_{k,r} a(k,r)\delta_{k,0} \cdot \delta_{r,0}\Big)\Big(\sum_{l,s} b(l,s)\delta_{l,0} \cdot \delta_{s,0}\Big) \\ = & \sum_{k,r}\sum_{l,s} a(k,r)b(l,s)e^{2\pi i r \cdot l/rmN}\delta_{k+l,0} \cdot \delta_{r+s,0} \\ = & \sum_{k,r}\Big(\sum_{l,s} a(l,s)b(l-k,s-r)e^{2\pi i(l-k)\cdot r/N}\delta_{k,0} \cdot \delta_{r,0}\Big). \end{aligned}$$

This calculation motivates the definition of a "twisted" product for $\mathcal{A}(\mathbb{Z}_N \times \mathbb{Z}_N)$:

Definition 3.3. *The complex vector space $\mathcal{A}(\mathbb{Z}_N \times \mathbb{Z}_N)$ of functions on $\mathbb{Z}_N \times \mathbb{Z}_N$ is called the twisted group algebra when given the twisted convolution $\natural$ as a product. For $\mathbf{a} = (a(k,r))$ and $\mathbf{b} = (b(l,s))$ in $\mathcal{A}(\mathbb{Z}_N \times \mathbb{Z}_N)$ we define the the* **twisted convolution** *of* $\mathbf{a}$ *and* $\mathbf{b}$ *by*

$$(\mathbf{a}\natural\mathbf{b})(k,r) = \sum_{l,s} a(l,s)b(l-k,s-r)e^{2\pi i(l-k)\cdot s}. \tag{4}$$

A projective representation of $\mathbb{Z}_N \times \mathbb{Z}_N$ induces a representation of the twisted group algebra $\mathcal{A}(\mathbb{Z}_N \times \mathbb{Z}_N)$. We define a representation $\pi_{\mathcal{A}}$ for $\mathcal{A}(\mathbb{Z}_N \times \mathbb{Z}_N)$ by

$$\pi_{\mathcal{A}}(\mathbf{a}) := \sum_{k,r \in \mathbb{Z}_N} a(k,r)\pi(k,r) \quad \text{for } \mathbf{a} \in \mathcal{A}(\mathbb{Z}_N \times \mathbb{Z}_N).$$

For non-discrete groups the above representation of the twisted group algebra is the so-called *integrated representation*. By definition the representation $\pi_{\mathcal{A}}$ of $\mathcal{A}(\mathbb{Z}_N \times \mathbb{Z}_N)$ is intimately related with the representation of the group $\mathbb{Z}_N \times \mathbb{Z}_N$. Therefore we collect some of the basic properties of time-frequency shifts which are elementary consequences of the commutation relation.

Lemma 3.4. *Let* (k,r) *and* (l,s) *be in* $\mathbb{Z}_N \times \mathbb{Z}_N$*. Then*

(i) $\pi(k,r)^* = T_{-k}M_{-r} = e^{2\pi i k\cdot r/N}\pi(-k,-r)$,
(ii) $\pi(k,r)\pi(l,s) = e^{2\pi i l\cdot r/N}\pi(k+l,r+s)$,
(iii) $\pi(k,r)\pi(l,s) = e^{2\pi i(l\cdot r-k\cdot s)/N}\pi(l,s)\pi(k,r)$.

Lemma 3.4 (i) suggests the following natural *twisted involution*: on $\mathcal{A}(\mathbb{Z}_N \times \mathbb{Z}_N)$.

$$\mathbf{a}^\star(k,r) = e^{2\pi i k\cdot r/N}\,\overline{a(-k,-r)}. \tag{5}$$

Altogether we have the following properties of $\pi_{\mathcal{A}}$:

Proposition 3.5. *The mapping* $\mathbf{a} \mapsto \pi_{\mathcal{A}}(\mathbf{a})$*, defined on* $\mathcal{A}(\mathbb{Z}_N \times \mathbb{Z}_N)$*, given by*

$$\pi_{\mathcal{A}}(\mathbf{a}) = \sum_{k,r\in\mathbb{Z}_N} a(k,r)\pi(k,r) \tag{6}$$

defines an involutive representation of $\mathcal{A}(\mathbb{Z}_N \times \mathbb{Z}_N)$*, i.e. one has:*

(a) $\pi_{\mathcal{A}}(\mathbf{a}) + \pi_{\mathcal{A}}(\mathbf{b}) = \pi_{\mathcal{A}}(\mathbf{a}+\mathbf{b})$,
(b) $\pi_{\mathcal{A}}(\mathbf{a})\,\pi_{\mathcal{A}}(\mathbf{b}) = \pi_{\mathcal{A}}(\mathbf{a}\natural\mathbf{b})$,
(c) $\pi_{\mathcal{A}}(\mathbf{a}^\star) = \pi_{\mathcal{A}}(\mathbf{a})^\star$,
(d) $\pi_{\mathcal{A}}(\delta_{0,0}) = 1_N$.

Conversely, if an involutive representation $\pi_{\mathcal{A}}$ *of* $\mathcal{A}(\mathbb{Z}_N \times \mathbb{Z}_N)$ *satisfies* (a) − (d)*, then there exists a projective representation* π *of* $\mathbb{Z}_N \times \mathbb{Z}_N$ *satisfying* (6)*, i.e., all such algebra representations arise in this way.*

We still have to show the converse, for which we need the definition of a projective representation π for $\mathbb{Z}_N \times \mathbb{Z}_N$ by (6).

$$\pi(k,r) := \pi_{\mathcal{A}}(\delta_{k,0}\cdot\delta_{r,0}). \tag{7}$$

The situation just described is a special case of a more general result, according to which there is a one-to-one correspondence between projective representations of a finite group G and involutive representations of its twisted group algebra $\mathcal{A}(G\times G)$. Another close relation between a group and its group algebra is the fact that $\pi_{\mathcal{A}}$ is an irreducible involutive representation of $\mathcal{A}(G\times G)$ if and only if π is an irreducible projective representation of $G \times G$. Schur's lemma formulates this more precisely.

Proposition 3.6 (Schur's Lemma)**.** *Let* $\pi_{\mathcal{A}}$ *be an irreducible involutive representation of* $\mathcal{A}(\mathbb{Z}_N \times \mathbb{Z}_N)$ *on* $\mathbb{C}^N$*. If a linear mapping* A *of* $\mathbb{C}^N$ *satisfies*

$$\pi_{\mathcal{A}}(\mathbf{a})\,A = A\,\pi_{\mathcal{A}}(\mathbf{a}) \quad \text{for all } \mathbf{a} \in \mathcal{A}(\mathbb{Z}_N \times \mathbb{Z}_N)$$

or equivalently $\pi(k,r)A = A\pi(k,r)$ *for all* $k,r \in \mathbb{Z}_N$*. Then* $A = c\mathbb{I}_N$ *for some* $c \in \mathbb{C}$.

For the proof we refer the reader to the excellent book [43] by A. Terras, which provides further information about representations of finite groups.

By construction the twisted group algebra $\mathcal{A}(\mathbb{Z}_N \times \mathbb{Z}_N)$ coincides with the full matrix algebra $\mathcal{M}_{N\times N}(\mathbb{C})$. From this point of view Schur's lemma states that the center of $\mathcal{M}_{N\times N}(\mathbb{C})$ consists of all scalar multiples of the identity $\{c\mathbb{I}_N : c \in \mathbb{C}\}$.

In our discussion of the spreading function we make use of the existence of a *trace* on $\mathcal{M}_{N\times N}(\mathbb{C})$.

The Hilbert space $\mathcal{M}_{N\times N}(\mathbb{C})$ with $\langle A,B\rangle_{Fro} = \mathrm{tr}(AB^\star)$ has a natural orthonormal basis which leads to the spreading representation of linear operators on $\mathbb{C}^N$.

Lemma 3.7. *Let* $\{\pi(k,r) : k,r \in \mathbb{Z}_N\}$ *be the family of all time-frequency shift operators in* $\mathcal{M}_{N\times N}(\mathbb{C})$*. Then* $\{N^{-1/2}\pi(k,r) : k,r \in \mathbb{Z}_N\}$ *is an orthonormal basis for* $(\mathcal{M}_{N\times N}(\mathbb{C}), \|.\|_{Fro})$*.*

Proof. The observation that the cardinality of $\{\pi(k,r) : k,r \in \mathbb{Z}_N\}$ is equal to the dimension of $\mathcal{M}_{N\times N}(\mathbb{C})$ and the following calculation gives the desired assertion. Let

$\pi(\mathrm{k},\mathrm{r})$ and $\pi(\mathrm{l},\mathrm{s})$ be two time-frequency shifts for $\mathrm{k},\mathrm{r}\in\mathbb{Z}_\mathrm{N}$ and $(\mathrm{l},\mathrm{s})\in\mathbb{Z}_\mathrm{N}$. Then

$$\begin{aligned}\langle\pi(\mathrm{k},\mathrm{r}),\pi(\mathrm{l},\mathrm{s})\rangle &= \mathrm{tr}(\pi(\mathrm{k},\mathrm{r})\pi(\mathrm{l},\mathrm{s})^*)\\ &= \mathrm{tr}(\pi(\mathrm{k}-\mathrm{l},\mathrm{r}-\mathrm{s})) = \mathrm{N}\delta_{k-l,s-r}.\end{aligned}$$

□

As a consequence we are able to expand every operator $\mathrm{A}\in\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ with respect to *all* time-frequency shifts from the discrete time-frequency plane $\mathbb{Z}_\mathrm{N}\times\widehat{\mathbb{Z}}_\mathrm{N}$.

Theorem 3.8 (Spreading representation)**.** *For* $\mathrm{A}\in\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ *we have*

$$\begin{aligned}\mathrm{A} &= \sum_{\mathrm{k},\mathrm{r}\in\mathbb{Z}_\mathrm{N}}\langle\mathrm{A},\pi(\mathrm{k},\mathrm{r})\rangle_{\mathrm{Fro}}\pi(\mathrm{k},\mathrm{r})\\ &= \sum_{\mathrm{k},\mathrm{r}\in\mathbb{Z}_\mathrm{N}}\eta_\mathrm{A}(\mathrm{k},\mathrm{r})\pi(\mathrm{k},\mathrm{r}),\end{aligned}$$

where $\eta_\mathrm{A}=(\eta_\mathrm{A}(\mathrm{k},\mathrm{r}))_{\mathrm{k},\mathrm{r}\in\mathbb{Z}_\mathrm{N}}$ *is called the* ***spreading function*** *of* A*. Furthermore*

$$\eta_\mathrm{A}(\mathrm{k},\mathrm{r})=\mathrm{N}^{-1}\sum_{\mathrm{l}\in\mathbb{Z}_\mathrm{N}}\mathrm{a}(\mathrm{l},\mathrm{l}-\mathrm{k})e^{-2\pi i\mathrm{lr}/\mathrm{N}}\quad \textit{for}\quad \mathrm{k},\mathrm{r}\in\mathbb{Z}_\mathrm{N}. \tag{8}$$

The expression for the spreading function is a straightforward consequence from the definitions. Traditionally a linear operator A on $\mathbb{C}^\mathrm{N}$ is formulated via a kernel by

$$\mathrm{Af}(\mathrm{j})=\sum_{\mathrm{i}\in\mathbb{Z}_\mathrm{N}}\mathrm{k}_\mathrm{A}(\mathrm{i},\mathrm{j})\mathrm{f}(\mathrm{i}),\ \text{for } \mathbf{f}=(\mathrm{f}(\mathrm{j}))\in\mathbb{C}^\mathrm{N}.$$

Then the relation between the spreading representation and the kernel of A is given by

$$\eta_\mathrm{A}(\mathrm{k},\mathrm{r})=\sum_{\mathrm{i}\in\mathbb{Z}_\mathrm{N}}\mathrm{k}_\mathrm{A}(\mathrm{i},\mathrm{i}-\mathrm{k})e^{-2\pi i\mathrm{r}\cdot\mathrm{i}/\mathrm{N}}$$

and the inversion formula is

$$\mathrm{k}_\mathrm{A}(\mathrm{i},\mathrm{j})=\sum_{\mathrm{r}\in\mathbb{Z}_\mathrm{N}}\eta_\mathrm{A}(\mathrm{i}-\mathrm{j},\mathrm{r})e^{2\pi i\mathrm{r}\cdot\mathrm{i}/\mathrm{N}}.$$

The correspondence between $\mathrm{A}\in\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ and its spreading coefficients $(\eta_\mathrm{A}(\mathrm{k},\mathrm{r}))$ may be considered as non-commutative analogue of the finite Fourier transform. First we have as a non-commutative analogue of Parseval's Theorem which follows from the orthogonality of $\{\pi(\mathrm{k},\mathrm{r}):\mathrm{k},\mathrm{r}\in\mathbb{Z}_\mathrm{N}\}$ and $\mathrm{tr}(A)=\mathrm{N}\eta_\mathrm{A}(0,0)$:

$$\langle\mathrm{A},\mathrm{A}\rangle_{\mathrm{Fro}}=\mathrm{tr}(\mathrm{A}^\star\mathrm{A})=\mathrm{N}\sum_{\mathrm{k},\mathrm{r}\in\mathbb{Z}_\mathrm{N}}|\eta_\mathrm{A}(\mathrm{k},\mathrm{r})|^2\ \text{for } \mathrm{A}\in\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C}).$$

Recall that the Fourier coefficients provide the best least square approximation under all trigonometric polynomials. An analogous argument yields the same for general orthonormal bases in abstract Hilbert space. For the spreading function this implies:

Theorem 3.9. *Let* A *be in* $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$*. Then for every subset* $\mathrm{F}\subseteq\mathbb{Z}_\mathrm{N}\times\mathbb{Z}_\mathrm{N}$ *the best approximation to* A *among all finite linear combinations of time-frequency shifts from F - in the Frobenius norm* $\|.\|_{\mathrm{Fro}}$ *- is given by* $\sum_{\mathrm{F}\subseteq\mathbb{Z}_\mathrm{N}}\mathrm{a}(\mathrm{k},\mathrm{r})\pi(\mathrm{k},\mathrm{r})$.

Proof. The orthogonality of the time-frequency shifts $\{\pi(\mathrm{k},\mathrm{r}) : \mathrm{k},\mathrm{r} \in \mathbb{Z}_\mathrm{N}\}$ implies that it is sufficient to prove the following: Let U be a unitary matrix in $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$. Then A has to be approximated by a scalar multiple of the unitary matrix U.

$$\|\mathrm{A}-\mathrm{cU}\|_{\mathrm{Fro}}^2 = \mathrm{tr}((\mathrm{A}-\mathrm{cU})^*(\mathrm{A}-\mathrm{cU})) = \|\mathrm{A}\|_{\mathrm{Fro}}^2 - 2\,\mathrm{Re}\,\overline{\mathrm{c}}\,\mathrm{tr}(\mathrm{AU}^\star) + \mathrm{N}|\mathrm{c}|^2$$

is minimized for $\mathrm{c} = \mathrm{N}^{-1}\,\mathrm{tr}(\mathrm{AU}^\star)$. The choice $\mathrm{U} = \pi(\mathrm{k},\mathrm{r})$ for some $\mathrm{k},\mathrm{r} \in \mathbb{Z}_\mathrm{N}$ gives the desired assertion since $\eta_\mathrm{A}(\mathrm{k},\mathrm{r}) = \mathrm{N}^{-1}\,\mathrm{tr}(\mathrm{A}\pi(\mathrm{k},\mathrm{r})^\star)$. □

The spreading representation was introduced by Kailath in the electrical engineering context of time-variant systems [26] completely independent from mathematical physics and representation theory. The parallelism with representation theory of the Heisenberg group becomes evident through the work of Feichtinger and Kozek in [14]. A real world realization of time-variant filters requires a finite-dimensional model which can be implemented on a computer. The spreading representation allows a symbolic calculus for time-variant filters which in mathematics are called pseudo-differential operators.

Let A be our finite-dimensional model of a time-variant system. Then the spreading function η_A can be considered as a symbol which contains all information about the time-frequency concentration of A. Given the spreading representations of $A = \sum_{\mathrm{k},\mathrm{r}\in\mathbb{Z}_\mathrm{N}} \eta_\mathrm{A}(\mathrm{k},\mathrm{r})\pi(\mathrm{k},\mathrm{r})$ and $B = \sum_{\mathrm{k},\mathrm{r}\in\mathbb{Z}_\mathrm{N}} \eta_\mathrm{B}(\mathrm{k},\mathrm{r})\pi(\mathrm{k},\mathrm{r})$, then after an easy direct computation of the spreading representation AB reveals that

$$\mathrm{AB} = \sum_{\mathrm{k},\mathrm{r}\in\mathbb{Z}_\mathrm{N}} (\eta_\mathrm{A}\natural\eta_\mathrm{B})(\mathrm{k},\mathrm{r})\pi(\mathrm{k},\mathrm{r}). \tag{9}$$

These observations allow us to complement recent work of Wildberger on a symbolic calculus for finite abelian groups. In [46] the Weyl quantization was considered on finite abelian groups and one of the main results states that there is no good symbolic calculus for groups of even order. Our results do not rely on the order of $\mathbb{Z}_\mathrm{N}$ which is possible because we use a different kind of quantization, namely the Kohn-Nirenberg quantization. In [21] the Kohn-Nirenberg quantization is discussed in full generality for locally compact abelian groups, and its relevance for time-frequency analysis is shown in [40]. The presentation given here makes use of the finite dimensionality of the space of time-variant filters, i.e. linear operators on $\mathbb{C}^\mathrm{N}$.

The singular value decomposition of A of rank r in $\mathcal{M}_{\mathrm{N}\times\mathrm{N}}(\mathbb{C})$ may be considered as the decomposition of A as the sum of r rank-one operators $\mathbf{g}_\mathrm{i}\otimes\overline{\mathbf{h}_\mathrm{i}}$ for $\mathbf{g}_\mathrm{i},\mathbf{h}_\mathrm{i}\in\mathbb{C}^\mathrm{N}$ for $\mathrm{i}=1,...,\mathrm{r}$ where $\mathbf{g}\otimes\overline{\mathbf{h}}$ denotes the rank-one operator $\mathrm{P}_{\mathbf{g},\mathbf{h}}$

$$\mathrm{P}_{\mathbf{g},\mathbf{h}}\mathbf{f} := \big(\mathbf{g}\otimes\overline{\mathbf{h}}\big)\mathbf{f} = \langle\mathbf{f},\mathbf{h}\rangle\mathbf{g} \text{ for } \mathbf{h}\in\mathbb{C}^\mathrm{N}. \tag{10}$$

More concretely, if A has the singular value decomposition UDV^* with singular values $\mathrm{d}_1 \geq \mathrm{d}_2 \geq \cdots \geq \mathrm{d}_\mathrm{r} > 0$, then $\mathbf{g}_\mathrm{i}$ and $\overline{\mathbf{h}_\mathrm{i}}$ are the i-th column $\mathbf{u}_\mathrm{i}$ of U and i-th row $\mathbf{v}_\mathrm{i}$ of V, i.e.

$$\mathrm{A} = \mathrm{d}_1\mathbf{u}_1\otimes\overline{\mathbf{v}_1} + \cdots + \mathrm{d}_\mathrm{r}\mathbf{u}_\mathrm{r}\otimes\overline{\mathbf{v}_\mathrm{r}}.$$

It thus turns out that the spreading representation of A is given by

$$\mathrm{A} = \sum_{\mathrm{i}=1}^{\mathrm{r}}\sum_{\mathrm{l},\mathrm{s}\in\mathbb{Z}_\mathrm{N}} \mathrm{d}_\mathrm{i}\langle\mathrm{P}_{\mathbf{u}_\mathrm{i},\mathbf{v}_\mathrm{i}},\pi(\mathrm{l},\mathrm{s})\rangle_{\mathrm{Fro}}\pi(\mathrm{l},\mathrm{s}),$$

i.e. the problem is reduced to the spreading representation of a rank-one operator $\mathrm{P}_{\mathbf{g},\mathbf{h}}$ for $\mathbf{g}$ and $\mathbf{h}$ as mentioned above. By definition we get

$$\mathrm{P}_{\mathbf{g},\mathbf{h}} = \mathrm{N}^{-1} \sum_{\mathrm{k},\mathrm{r}\in\mathbb{Z}_\mathrm{N}} \langle \mathbf{h}, \pi(\mathrm{k},\mathrm{r})\mathbf{g}\rangle_{\mathrm{Fro}}\, \pi(\mathrm{k},\mathrm{r}). \tag{11}$$

The symbolic calculus for linear operators allows us to transfer properties of the operators with relations for their spreading functions. As an example we look at the product of two rank-one operators and their spreading representation. Let $\mathbf{g}_1, \mathbf{g}_2, \mathbf{h}_1, \mathbf{h}_2$ be in $\mathbb{C}^\mathrm{N}$. Then an elementary computation gives that

$$\mathrm{P}_{\mathbf{g}_1,\mathbf{h}_1}\mathrm{P}_{\mathbf{g}_2,\mathbf{h}_2} = \langle \mathbf{g}_2, \mathbf{h}_1\rangle \mathrm{P}_{\mathbf{g}_1,\mathbf{h}_2}. \tag{12}$$

The spreading representation of a rank-one operator and the symbolic calculus for linear operators yields the so-called *reproducing property*

$$\langle \mathbf{g}_1, \pi(k,r)\mathbf{h}_1\rangle \natural \langle \mathbf{g}_2, \pi(k,r)\mathbf{h}_2\rangle = \langle \mathbf{g}_2, \mathbf{h}_1\rangle\langle \mathbf{g}_1, \pi(k,r)\mathbf{h}_2\rangle.$$

In our discussion of Gabor frames we will return to the coefficients of the spreading representation of a rank-one operator and place it into the setting of representation theory for a pair of twisted group algebras. At the moment we want to emphasize the role of projection operators and their associated one-dimensional subspaces. Our discussion indicates the connection between our presentation of the spreading representation and the work of Calderbank in [22] and Grassmannian frames [41].

3.1. **Discrete time-frequency plane** $\mathrm{G}\times\widehat{\mathrm{G}}$**.** So far we have restricted ourselves to cyclic groups $\mathbb{Z}_\mathrm{N}$. This is no real restriction, since the structure theorem for finite abelian groups G allows us to move on to the more general setting of a finite abelian group G.

Lemma 3.10 (Decomposition of Finite Abelian Groups)**.** *Let* G *be a finite Abelian group of order* N*. Then* G *is isomorphic to a direct product of groups* $\mathbb{Z}_{\mathrm{p}_\mathrm{i}^{\mathrm{N}_\mathrm{i}}}$ *where* $\mathrm{N} = \mathrm{p}_1^{\mathrm{N}_1}\cdots\mathrm{p}_\mathrm{k}^{\mathrm{N}_\mathrm{k}}$ *is the prime number decomposition of N, i.e.* $\mathrm{G}\cong\mathbb{Z}_{\mathrm{p}_1^{\mathrm{N}_1}}\times\cdots\times\mathbb{Z}_{\mathrm{p}_\mathrm{k}^{\mathrm{N}_\mathrm{k}}}$.

A character $\mathrm{c_G}$ of a finite abelian group is a mapping from $\mathrm{G}\to\mathbb{T}$, such that $\mathrm{c_G}(\mathrm{x}+\mathrm{y},\omega) = \mathrm{c_G}(\mathrm{x},\omega)\mathrm{c_G}(\mathrm{y},\omega)$ for all $\mathrm{x},\mathrm{y}\in\mathrm{G}$. The set of all characters of G form a group with respect to multiplication which we denote by $\widehat{\mathrm{G}}$. A basic result about finite abelian groups asserts that $\widehat{\mathrm{G}}$ is naturally isomorphic to G. In addition we have an explicit knowledge of $\mathrm{c_G}$ in terms of the building blocks of G, i.e.

$$\mathrm{c_G}(\mathrm{x},\omega) = \mathrm{c}_{\mathbb{Z}_1}(\mathrm{l}_1,\mathrm{s}_1)\cdots\mathrm{c}_{\mathbb{Z}_\mathrm{k}}(\mathrm{l}_\mathrm{k},\mathrm{s}_\mathrm{k})\ \mathrm{x} = (\mathrm{l}_1,\cdots,\mathrm{l}_\mathrm{k})\in\mathrm{G}, \omega = (\mathrm{s}_1,\cdots,\mathrm{s}_\mathrm{k})\in\widehat{\mathrm{G}},$$

where we abbreviated $\mathbb{Z}_{\mathrm{p}_\mathrm{i}^{\mathrm{N}_\mathrm{i}}}$ by $\mathbb{Z}_\mathrm{i}$.

We refer to $\mathrm{G}\times\widehat{\mathrm{G}}$ as the discrete time-frequency plane. We are therefore faced with the discussion of twisted group algebras $\mathcal{A}(\mathrm{G}_1\times\mathrm{G}_2)$ for G_1 for abelian groups G_1 and G_2 which turns out to be the tensor product $A(\mathrm{G}_1\times\mathrm{G}_1)\otimes\mathcal{A}(\mathrm{G}_2\times\mathrm{G}_2)$. Consequently the twisted group algebra $\mathcal{A}(\mathrm{G}\times\widehat{\mathrm{G}}) = \mathcal{A}(\mathbb{Z}_{\mathrm{p}_1^{\mathrm{N}_1}}\times\mathbb{Z}_{\mathrm{p}_1^{\mathrm{N}_1}})\otimes\cdots\otimes\mathcal{A}(\mathbb{Z}_{\mathrm{p}_\mathrm{k}^{\mathrm{N}_\mathrm{k}}}\times\mathbb{Z}_{\mathrm{p}_\mathrm{k}^{\mathrm{N}_\mathrm{k}}})$ is isomorphic to a product of full matrix algebras $\mathcal{M}_{|\mathrm{G}\times\widehat{\mathrm{G}}|} = \mathcal{M}_{\mathrm{p}_1^{\mathrm{N}_1}\times\mathrm{p}_1^{\mathrm{N}_1}}\otimes\cdots\otimes\mathcal{M}_{\mathrm{p}_\mathrm{k}^{\mathrm{N}_\mathrm{k}}\times\mathrm{p}_\mathrm{k}^{\mathrm{N}_\mathrm{k}}}$. In the following we denote an element of $\mathrm{G}\times\widehat{\mathrm{G}}$ by (k,r) to emphasize that k,r are elements of $\mathrm{G}\cong\mathbb{Z}_{\mathrm{p}_1^{\mathrm{N}_1}}\times\cdots\times\mathbb{Z}_{\mathrm{p}_\mathrm{k}^{\mathrm{N}_\mathrm{k}}}$.

4. Gabor frames

In this section we discuss Gabor frames from a matrix algebra point of view. If $\mathcal{G}(\mathbf{g},\Lambda)$ is a Gabor frame for $\mathbb{C}^{|\mathrm{G}|}$, then the Gabor system $\mathcal{G}(g,\Lambda^\circ)$ over the adjoint subgroup Λ° is naturally related to the original Gabor system. We will consider the twisted group algebras of Λ and Λ° and the investigation of their structure allows a unified treatment of Gabor frames. The understanding of the twisted group algebras for the subgroups Λ and Λ° requires some harmonic analysis over the time-frequency plane. We therefore develop this prerequisites in the first part of this section. In the second part we introduce multi-window Gabor frames and as a motivation we treat the set of all time-frequency shifts as the Gabor system $\mathcal{G}(\mathbf{g},\mathrm{G}\times\widehat{\mathrm{G}})$ and state the *Resolution of Identity* as a "continuous" analogue for reconstruction formulas for a general Gabor system.

4.1. **Harmonic Analysis over** $\mathrm{G}\times\widehat{\mathrm{G}}$**.** Harmonic analysis over the time-frequency plane $\mathrm{G}\times\widehat{\mathrm{G}}$ is the study of the Fourier transform of $\mathrm{G}\times\widehat{\mathrm{G}}$ and its properties which differ from the Euclidean Fourier transform of $\mathrm{G}\times\mathrm{G}$. The difference between $\mathrm{G}\times\mathrm{G}$ and $\mathrm{G}\times\widehat{\mathrm{G}}$ arises from the symplectic structure of the time-frequency plane. We motivate our investigation of $\mathrm{G}\times\widehat{\mathrm{G}}$ with the commutation relations for time-frequency shifts:

$$\pi(\mathrm{l},\mathrm{s})[(\mathrm{k},\mathrm{r})]=\mathrm{c_G}\big((\mathrm{k},\mathrm{r}),(\mathrm{l},\mathrm{s})\big)\bar{\mathrm{c}}_{\mathrm{G}}\big((\mathrm{l},\mathrm{s}),(\mathrm{k},\mathrm{r})\big)\pi(\mathrm{l},\mathrm{s})=e^{2\pi i\Omega((\mathrm{k},\mathrm{r}),(\mathrm{l},\mathrm{s}))/|\mathrm{G}|}\pi(\mathrm{l},\mathrm{s}),$$

where $\Omega:\mathrm{G}\times\widehat{\mathrm{G}}\to\mathrm{G}\times\widehat{\mathrm{G}}$ denotes the *standard symplectic form*

$$\Omega((\mathrm{k},\mathrm{r}),(\mathrm{l},\mathrm{s})):=\mathrm{l}\cdot\mathrm{r}-\mathrm{k}\cdot\mathrm{s}.$$

Therefore the commutation relations for time-frequency shifts endow the discrete time-frequency plane $\mathrm{G}\times\widehat{\mathrm{G}}$ with an intrinsic *symplectic structure*.

From the algebraic point of view $\mathrm{G}\times\widehat{\mathrm{G}}$ is isomorphic to $\mathrm{G}\times\mathrm{G}$ but from a harmonic analysis point of view these two groups are different objects. Namely, the characters of $\mathrm{G}\times\widehat{\mathrm{G}}$ are not of the form $\mathrm{c}_{\mathrm{G}\times\mathrm{G}}=\mathrm{c_G}\cdot\mathrm{c_G}$, i.e. the character group of $\mathrm{G}\times\widehat{\mathrm{G}}$ is not $\widehat{\mathrm{G}}\times\widehat{\mathrm{G}}$. Observe that in the time-frequency plane the character group $\widehat{\mathrm{G}}$ and G are orthogonal to each other, i.e. a rotation by $\pi/2$ moves G onto $\widehat{\mathrm{G}}$. These facts should convince you that the "correct" characters of $\mathrm{G}\times\widehat{\mathrm{G}}$ are Euclidean characters rotated by $\pi/2$. More precisely, we define a *symplectic character* $\mathrm{c}^{\mathbf{s}}_{|\mathrm{G}\times\widehat{\mathrm{G}}|}\big((\mathrm{k},\mathrm{r}),(\mathrm{l},\mathrm{s})\big)$ for a fixed (l,s) by

$$\mathrm{c}^{\mathbf{s}}_{|\mathrm{G}\times\widehat{\mathrm{G}}|}\big((\mathrm{k},\mathrm{r}),(\mathrm{l},\mathrm{s})\big)=\mathrm{c_G}\big((\mathrm{k},\mathrm{r}),(\mathrm{l},\mathrm{s})\big)\bar{\mathrm{c}}_{\mathrm{G}}\big((\mathrm{l},\mathrm{s}),(\mathrm{k},\mathrm{r})\big).$$

Lemma 4.1. *The character group of* $\mathrm{G}\times\widehat{\mathrm{G}}$ *is* $\{\mathrm{c}^{\mathbf{s}}_{|\mathrm{G}\times\widehat{\mathrm{G}}|}\big((\mathrm{k},\mathrm{r}),(\mathrm{l},\mathrm{s})\big):(\mathrm{k},\mathrm{r})\in\mathrm{G}\times\widehat{\mathrm{G}}\}$, *i.e. it is isomorphic to* $\widehat{\mathrm{G}}\times\mathrm{G}$.

This observation motivates the following "symplectic" analogues of translation and modulation operators which are actually the translation and modulation operators for $\mathrm{G}\times\widehat{\mathrm{G}}$. Let $\mathbf{F}$ be in $\mathbb{C}^{|\mathrm{G}\times\widehat{\mathrm{G}}|}$. Then we define the *symplectic translation* operator by

$$\mathrm{T}^{\mathbf{s}}_{(\mathrm{k}',\mathrm{r}')}\mathbf{F}(\mathrm{k},\mathrm{r})=\mathbf{F}((\mathrm{k}-\mathrm{k}',\mathrm{r}-\mathrm{r}'))\text{ for }(\mathrm{k}',\mathrm{r}')$$

and the *symplectic modulation* operator by

$$\mathrm{M}^{\mathbf{s}}_{(\mathrm{l},\mathrm{s})}\mathbf{F}((\mathrm{k},\mathrm{r})) = e^{2\pi i\Omega((\mathrm{k},\mathrm{r}),(\mathrm{l},\mathrm{s}))/|\mathrm{G}|}\mathbf{F}\big((\mathrm{k},\mathrm{r})\big)$$

for $(\mathrm{k},\mathrm{r}),(\mathrm{l},\mathrm{s}) \in \mathrm{G}\times\widehat{\mathrm{G}}$. As for translation and modulation operators we have the non-commutativity of symplectic translations and modulations:

$$\mathrm{T}^{\mathbf{s}}_{(\mathrm{k},\mathrm{r})}\mathrm{M}^{\mathbf{s}}_{(\mathrm{l},\mathrm{s})} = \mathrm{c}^{\mathbf{s}}_{|\mathrm{G}\times\widehat{\mathrm{G}}|}\big((\mathrm{k},\mathrm{r}),(\mathrm{l},\mathrm{s})\big)\mathrm{M}^{\mathbf{s}}_{(\mathrm{l},\mathrm{s})}\mathrm{T}^{\mathbf{s}}_{(\mathrm{k},\mathrm{r})}$$

In other words $\{\mathrm{T}^{\mathbf{s}}_{(\mathrm{k},\mathrm{r})}\mathrm{M}^{\mathbf{s}}_{(\mathrm{l},\mathrm{s})} : (\mathrm{k},\mathrm{r})(\mathrm{l},\mathrm{s}) \in \mathrm{G}\times\widehat{\mathrm{G}}\}$ is a projective representation of $\mathrm{G}\times\widehat{\mathrm{G}}\times\mathrm{G}\times\widehat{\mathrm{G}}$. Therefore we may define the symplectic analogue of the short-time Fourier transform and analyze objects on $\mathrm{G}\times\widehat{\mathrm{G}}$ with respect to this symplectic short-time Fourier transform. Gröchenig uses this object implicitly in his discussion of time-frequency localization operators and pseudo-differential operators, [5].

The structure of the character group of $\mathrm{G}\times\widehat{\mathrm{G}}$ indicates that the "correct" Fourier transform in time-frequency analysis is not the standard Euclidean Fourier transform but the *symplectic Fourier transform* of $\mathbf{F}\in\mathbb{C}^{|\mathrm{G}\times\widehat{\mathrm{G}}|}$:

$$\mathcal{F}_{\mathbf{s}}(\mathrm{k},\mathrm{r}) = |\mathrm{G}\times\widehat{\mathrm{G}}|^{-1/2}\sum_{(\mathrm{l},\mathrm{s})}\mathrm{c}^{\mathbf{s}}_{|\mathrm{G}\times\widehat{\mathrm{G}}|}\big((\mathrm{k},\mathrm{r}),(\mathrm{l},\mathrm{s})\big)\mathbf{F}\big((\mathrm{l},\mathrm{s})\big).$$

An elementary computation establishes the following property of the symplectic Fourier transform.

Lemma 4.2. *The symplectic Fourier transform $\mathcal{F}_{\mathbf{s}}$ is a self-inverse mapping of order two on $\mathbb{C}^{|\mathrm{G}\times\widehat{\mathrm{G}}|}$, i.e.*

$$\mathcal{F}_{\mathbf{s}}^{-1} = \mathcal{F}_{\mathbf{s}}, \qquad \mathcal{F}_{\mathbf{s}}^{2} = \mathrm{I}_{\mathbb{C}^{|\mathrm{G}\times\widehat{\mathrm{G}}|}}.$$

Since the symplectic characters $\mathrm{c}^{\mathbf{s}}_{|\mathrm{G}\times\widehat{\mathrm{G}}|}$ are obtained from the characters $\mathrm{c}_{|\mathrm{G}\times\mathrm{G}|}$ by a rotation of $\pi/2$ the Euclidean Fourier transform $\mathcal{F}$ and the symplectic Fourier transform $\mathcal{F}_{\mathbf{s}}$ are related by

$$\mathcal{F}_{\mathbf{s}} = \mathcal{F}\circ\mathrm{J} \text{ for } \mathrm{J} = \begin{pmatrix} 0 & \mathrm{I}_{|\mathrm{G}|} \\ -\mathrm{I}_{|\mathrm{G}|} & 0 \end{pmatrix},$$

where J describes the rotation by $\pi/2$ of the time-frequency plane $\mathrm{G}\times\widehat{\mathrm{G}}$. We want to have a Poisson summation formula for the symplectic Fourier transform but what is the symplectic analogue of the dual subgroup $\Lambda^{\perp}$ of a subgroup Λ? After our discussion of symplectic characters it should be clear that the correct choice is $\mathrm{J}\Lambda^{\perp}$ because this set consists of all points (k,l) such that $\mathrm{c}_{|G|}(\lambda,(\mathrm{k},\mathrm{r}))\overline{\mathrm{c}_{|G|}}((\mathrm{k},\mathrm{r}),\lambda) = 1$ for all $\lambda\in\Lambda$. First this implies that the set of all points $(\mathrm{k},\mathrm{r})\in\mathrm{G}\times\widehat{\mathrm{G}}$ which satisfies this condition is another subgroup Λ° of $\mathrm{G}\times\widehat{\mathrm{G}}$. In Gabor analysis Λ° is called the *adjoint subgroup* of Λ and its relevance was first discovered by Feichtinger and Kozek in [14]. Their approach provided an explanation of Janssen's representation of the frame operator for separable lattices in $\mathbb{R}^{\mathrm{d}}\times\widehat{\mathbb{R}}^{\mathrm{d}}$ in [6, 23] and for certain non-separable lattices due to Ron and Shen in [36, 37]. For different reasons Rieffel was led to consider Λ°, which he calls the orthogonal subgroup of Λ [35], and recently Digernes and Varadarajan came across this object Λ° and they refer to it as the polar of Λ, [7].

Theorem 4.3 (Poisson summation formula). *For any subgroup $\Lambda \lhd \mathrm{G}\times\widehat{\mathrm{G}}$ one has:*

$$\sum_{\lambda\in\Lambda}\mathbf{F}(\lambda)=|\Lambda|^{-1}\sum_{\lambda^\circ\in\Lambda^\circ}\mathbf{F}(\lambda^\circ)\quad \textit{for all}\quad \mathbf{F}\in\mathbb{C}^{|\mathrm{G}\times\widehat{\mathrm{G}}|}. \tag{13}$$

An important consequence of the Poisson summation formula for the symplectic Fourier transform is the *Fundamental Identity of Gabor Analysis*(FIGA). The following lemma about the symplectic Fourier transform of two STFT's appears at different places in the engineering and mathematical literature, see [11] in various degrees of generality. Following [42] we call it the Sussman identity. Therefore the next result is a finite analogue of Sussman's identity.

Proposition 4.4. *Let $\mathbf{f}_1,\mathbf{f}_2,\mathbf{g}_1,\mathbf{g}_2$ be in $\mathbb{C}^{|\mathrm{G}|}$. Then*

$$\mathcal{F}^s\big[\mathrm{V}_{\mathbf{g}_1}\mathbf{f}_1\overline{\mathrm{V}_{\mathbf{g}_2}\mathbf{f}_2}\big]((\mathrm{l},\mathrm{s}))=\mathrm{V}_{\mathbf{g}_1}\mathbf{g}_2\overline{\mathrm{V}_{\mathbf{f}_1}\mathbf{f}_2}((\mathrm{l},\mathrm{s})).$$

Proof.

$$\begin{aligned}\mathcal{F}^s\big[\mathrm{V}_{\mathbf{g}_1}\mathbf{f}_1\overline{\mathrm{V}_{\mathbf{g}_2}\mathbf{f}_2}\big]((\mathrm{l},\mathrm{s})) &= \sum_{\mathrm{k},\mathrm{r}}\mathrm{V}_{\mathbf{g}_1}\mathbf{f}_1(\mathrm{k},\mathrm{r})\overline{\mathrm{V}_{\mathbf{g}_2}\mathbf{f}_2(\mathrm{k},\mathrm{r})}e^{2\pi i\Omega((\mathrm{l},\mathrm{s}),(\mathrm{k},\mathrm{r}))}\\ &= \sum_{\mathrm{k},\mathrm{r}}\langle\pi(\mathrm{k},\mathrm{r})\mathbf{f}_1,\pi(\mathrm{l},\mathrm{s})\pi(\mathrm{k},\mathrm{r})\mathbf{g}_1\rangle\overline{\langle\mathbf{f}_2,\pi(\mathrm{k},\mathrm{r})\mathbf{g}_2\rangle}e^{2\pi i\Omega((\mathrm{l},\mathrm{s}),(\mathrm{k},\mathrm{r}))}\\ &= \sum_{\mathrm{k},\mathrm{r}}\langle\pi(\mathrm{l},\mathrm{s})\mathbf{f}_1,\pi(\mathrm{k},\mathrm{r})\pi(\mathrm{l},\mathrm{s})\mathbf{g}_1\rangle\overline{\langle\mathbf{f}_2,\pi(\mathrm{k},\mathrm{r})\mathbf{g}_2\rangle}\\ &= \overline{\langle\mathbf{f}_2,\pi(\mathrm{l},\mathrm{s})\mathbf{f}_1\rangle}\langle\mathbf{g}_2,\pi(\mathrm{l},\mathrm{s})\mathbf{g}_1\rangle.\end{aligned}$$

□

The next theorem is just an application of the symplectic Poisson summation formula to Sussman's Identity, which is the above mentioned *Fundamental Identity of Gabor analysis.*

Theorem 4.5 (FIGA). *Let Λ be a subgroup of $\mathrm{G}\times\widehat{\mathrm{G}}$ and $\mathbf{f}_1,\mathbf{f}_2,\mathbf{g}_1,\mathbf{g}_2\in\mathbb{C}^{|\mathrm{G}|}$. Then*

$$\sum_{\lambda\in\Lambda}\mathrm{V}_{\mathbf{g}_1}\mathbf{f}_1(\lambda)\overline{\mathrm{V}_{\mathbf{g}_2}\mathbf{f}_2}(\lambda)=|\Lambda|^{-1}\sum_{\lambda^\circ\in\Lambda^\circ}\mathrm{V}_{\mathbf{g}_1}\mathbf{g}_2(\lambda^\circ)\overline{\mathrm{V}_{\mathbf{f}_1}\mathbf{f}_2}(\lambda^\circ).$$

In the continuous case Janssen was the first to call this identity the FIGA because important results such as the Wexler-Raz biorthogonality conditions and the Ron-Shen duality principle are easily derived from it, [23].

The relevance of the symplectic structure of the time-frequency plane $\mathrm{G}\times\widehat{\mathrm{G}}$ was first pointed out by Feichtinger and Kozek in the context of quantization of operators and Gabor frames for elementary locally compact abelian groups [14]. Their approach invokes the full power of abstract harmonic analysis and is therefore very technical and not accessible to the majority of workers in Gabor analysis. On the other hand these groups are applying Gabor analysis to real world problems and one reason for this article is to transfer the main results of Feichtinger and Kozek into a form which requires only a modest background in harmonic analysis.

4.2. **Multi-window Gabor frames.** In this section we explore multi-window Gabor frames with the help of the spreading representation. This allows for a unified discussion of non-separable Gabor systems which generalize the known results for separable Gabor systems due to Wexler-Raz, Tolimieri, Qiu and Strohmer, [44, 34, 39]. First of all we treat just Gabor frames and obtain their main properties and at the end of this section we indicate how these results are extended to multi-window Gabor frames, i.e. a finite sum of Gabor frames.

Definition 4.6. *Let* Λ *be a subgroup of* $\mathrm{G}\times\widehat{\mathrm{G}}$ *and* $\mathbf{g}$ *a Gabor atom in* $\mathbb{C}^{|\mathrm{G}|}$*. Then* $\mathcal{G}(\mathbf{g},\Lambda)=\{\pi(\lambda)\mathbf{g}:\lambda\in\Lambda\}$ *is called a Gabor system. If the frame operator*

$$\mathrm{S}_{\mathbf{g},\Lambda}\mathbf{f}=\sum_{\lambda\in\Lambda}\langle\mathbf{f},\pi(\lambda)\mathbf{g}\rangle\pi(\lambda)\mathbf{g}$$

is invertible, then $\mathcal{G}(\mathbf{g},\Lambda)$ *is called a Gabor frame.*

The signal $\tilde{\mathbf{g}}:=\mathrm{S}_{\mathbf{g},\Lambda}^{-1}\mathbf{g}$ is the *canonical dual window* and $\mathbf{h}_0:=\mathrm{S}_{\mathbf{g},\Lambda}^{-1/2}\mathbf{g}$ is the *canonical tight window* of the Gabor frame $\mathcal{G}(\mathbf{g},\Lambda)$.

We consider the elements of the Gabor system $\{\pi(\lambda)\mathbf{g}:\lambda\in\Lambda\}$ as columns of a $|G|\times|\Lambda|$-matrix $\mathrm{D}_{\mathbf{g},\Lambda}=[\pi(\lambda_1)\mathbf{g},...,\pi(\lambda_{|\Lambda|})\mathbf{g}]$ for some ordering of the elements of Λ. Then the Gabor frame operator $\mathrm{S}_{\mathbf{g},\Lambda}$ may be written as $\mathrm{S}_{\mathbf{g},\Lambda}=\mathrm{D}_{\mathbf{g},\Lambda}\circ\mathrm{D}^{\star}_{\mathbf{g},\Lambda}$, i.e. $\mathrm{S}_{\mathbf{g},\Lambda}$ acts on vectors in $\mathbb{C}^{|\mathrm{G}|}$ and implements the coefficient operator for $\mathcal{G}(\mathbf{g},\Lambda)$. Observe that the operator $\mathrm{D}^{\star}_{\mathbf{g},\Lambda}\circ\mathrm{D}_{\mathbf{g},\Lambda}$ with respect to the canonical basis $\{\delta_{\lambda,0}:\lambda\in\Lambda\}$ is represented by the **Gram matrix** $\mathrm{G}_{\mathbf{g},\Lambda}$ of $\mathcal{G}(\mathbf{g},\Lambda)$ with entries $\big(\mathrm{G}_{\lambda,\mu}=\langle\mathbf{g}_\mu,\mathbf{g}_\lambda\rangle\big)_{\lambda,\mu\in\Lambda}$. Consequently the Gram matrix $\mathrm{G}_{\mathbf{g},\Lambda}$ acts on vectors in $\mathbb{C}^{|\Lambda|}$.

As a motivation of a general Gabor system $\mathcal{G}(\mathbf{g},\Lambda)$ we first treat the Gabor frame $\mathcal{G}(\mathbf{g},0\})$, i.e. $\{\pi(\mathrm{k},\mathrm{r}):(\mathrm{k},\mathrm{r})\in\mathrm{G}\times\widehat{\mathrm{G}}\}$. Then the coefficient operator and synthesis operator are given by

$$\mathrm{C}_{\mathcal{G}}\mathbf{f}=\Big(\langle\mathbf{f},\pi(\mathrm{k},\mathrm{r})\mathbf{g}\rangle\Big)_{(\mathrm{k},\mathrm{r})}\text{ and }\mathrm{D}_{\mathcal{G}}\mathbf{c}=\sum_{(\mathrm{k},\mathrm{r})\in\mathrm{G}\times\widehat{\mathrm{G}}}c(\mathrm{k},\mathrm{r})\pi(\mathrm{k},\mathrm{r})\mathbf{g}\text{ for }\mathbf{c}\in\mathbb{C}^{|\mathrm{G}\times\widehat{\mathrm{G}}|}.$$

Proposition 4.7. *For* $\mathbf{g}\in\mathbb{C}^{|\mathrm{G}|}$ *we have that* $\mathcal{G}=\{\pi(\mathrm{k},\mathrm{r})\mathbf{g}:(\mathrm{k},\mathrm{r})\in\mathrm{G}\times\widehat{\mathrm{G}}\}$ *is a tight frame for* $\mathbb{C}^{|\mathrm{G}|}$ *with frame constants* $\mathrm{A},\mathrm{B}=\|\mathbf{g}\|^2_{\mathbb{C}^{|\mathrm{G}|}}$*, i.e.*

$$\|\mathbf{g}\|^2_{\mathbb{C}^{|\mathrm{G}|}}\|\mathbf{f}\|^2_{\mathbb{C}^{|\mathrm{G}|}}=\frac{1}{|\mathrm{G}\times\widehat{\mathrm{G}}|}\sum_{\mathrm{k},\mathrm{r}\in\mathrm{G}}|\langle\mathbf{f},\pi(\mathrm{k},\mathrm{r})\mathbf{g}\rangle|^2\ \textit{for all}\ \mathbf{f}\in\mathbb{C}^{|\mathrm{G}|}.$$

Corollary 4.8 (Resolution of Identity)**.** *Let* $\mathbf{g}\in\mathbb{C}^{|\mathrm{G}|}$ *with* $\|\mathbf{g}\|_{\mathbb{C}^{|\mathrm{G}|}}=1$*. Then for every* $\mathbf{h}\in\mathbb{C}^{|\mathrm{G}|}$ *with* $\langle\mathbf{g},\mathbf{h}\rangle\neq0$ *one has*

$$\mathbf{f}=\frac{1}{|\mathrm{G}|\cdot\langle\mathbf{g},\mathbf{h}\rangle}\sum_{(\mathrm{k},\mathrm{r})\in\mathrm{G}\times\widehat{\mathrm{G}}}\langle\mathbf{f},\pi(\mathrm{k},\mathrm{r})\mathbf{g}\rangle\pi(\mathrm{k},\mathrm{r})\mathbf{h}.$$

The corollary follows from our discussion of frames in the introduction with $\mathbf{h}_0:=\mathrm{S}_{\mathcal{G}}^{-1}\mathbf{g}$. Our proof that $\{\pi(\mathrm{k},\mathrm{r})\mathbf{g}:(\mathrm{k},\mathrm{r})\in\mathrm{G}\times\widehat{\mathrm{G}}\}$ is a tight frame for $\mathbb{C}^{|\mathrm{G}|}$ relies on the commutation relations for time-frequency shifts and Schur's lemma. The frame operator $\mathrm{S}_{\mathcal{G}}$ of $\{\pi(\mathrm{k},\mathrm{r})\mathbf{g}:(\mathrm{k},\mathrm{r})\in\mathrm{G}\times\widehat{\mathrm{G}}\}$ has the following property:

Lemma 4.9. *For every* (l,s) *in* $G\times\widehat{G}$ *we have*

$$S_{\mathcal{G}} = \pi(l,s)\circ S_{\mathcal{G}}\circ\pi(l,s)^{\star}.$$

Proof. The commutation relations $\pi(l,s)^{\star}\pi(k,r) = e^{-2\pi i(k-l)\cdot s/|G|}\pi(k-l,r-s)$ yields

$$\begin{aligned}\pi(l,s)\circ S_{\mathcal{G}}\circ\pi(l,s)^{\star}\mathbf{f} &= \sum_{k,r\in G}\langle\pi(l,s)^{\star}\mathbf{f},\pi(k,r)\mathbf{g}\rangle\pi(k,r)\mathbf{g}\\ &= \sum_{k,r\in G}\langle\mathbf{f},\pi(k-l,r-s)\mathbf{g}\rangle\pi(k-l,r-s)\mathbf{g} = S_{\mathcal{G}}\mathbf{f}.\end{aligned}$$

□

Recall that $\{\pi(k,r) : (k,r) : G\times\widehat{G}\}$ is an irreducible representation of $G\times\widehat{G}$ and that it generates $\mathcal{M}_{|G\times G|}(\mathbb{C})$, hence the preceding observation yields that $S_{\mathcal{G}}$ is in the commutant of $\mathcal{M}_{|G\times G|}(\mathbb{C})$. Hence by Schur's lemma $S_{\mathcal{G}}$ is a multiple of the identity operator,

$$S_{\mathcal{G}} = C\cdot\mathbb{I}_{|G|}\quad\text{for some } C\in\mathbb{C}.$$

The determination of the constant C requires the calculation of the inner-product of two STFT's which we state in a slightly general form. In the continuous setting this identity is the so-called Moyal Identity.

Proposition 4.10 (Moyal's Formula)**.** *Let* $\mathbf{f}_1,\mathbf{g}_1$ *and* $\mathbf{f}_2,\mathbf{g}_2$ *in* $\mathbb{C}^{|G|}$*. Then*

$$\langle V_{\mathbf{g}_1}\mathbf{f}_1, V_{\mathbf{g}_2}\mathbf{f}_2\rangle_{\mathbb{C}^{|G\times\widehat{G}|}} = \langle\mathbf{f}_1,\mathbf{f}_2\rangle_{\mathbb{C}^{|G|}}\overline{\langle\mathbf{g}_1,\mathbf{g}_2\rangle_{\mathbb{C}^{|G|}}}.$$

The proof is straightforward and relies on the definition of the STFT and a change of summation. Technically speaking Moyal's identity expresses the orthogonality of two matrix coefficients of the irreducible projective representation $\{\pi(k,r) : (k,r)\in G\times\widehat{G}\}$ of $G\times\widehat{G}$. An application of Moyal's Identity with $\mathbf{f} = \mathbf{f}_1 = \mathbf{f}_1$ and $\mathbf{g} = \mathbf{g}_1 = \mathbf{g}_2$ yields that $C = \|\mathbf{g}\|^2_{\mathbb{C}^{|G|}}$, which implies the assertion that $\{\pi(k,r) : (k,r)\in G\times\widehat{G}\}$ is a tight frame for $\mathbb{C}^{|G|}$.

Before we move on to general Gabor frames we note some important properties of the spreading representation, which allows us to justify the name "Resolution of Identity". First of all we remark that the projective representation of $G\times\widehat{G}$ gives rise to a unitary representation on $\mathcal{M}_{|G\times G|}(\mathbb{C})$. In other words

$$(k,r)\mapsto A[(k,r)] = \pi(k,r)A\pi(k,r)^{*},\ A\in\mathcal{M}_{|G\times G|}(\mathbb{C})$$

is an involutive automorphism of $\mathcal{M}_{|G\times G|}(\mathbb{C})$.

Proposition 4.11. *The mapping* $(k,r)\mapsto A[(k,r)]$ *defines a unitary representation of* $G\times\widehat{G}$ *on the Hilbert space* $\mathcal{M}_{|G\times G|}(\mathbb{C})$ *with the Frobenius norm, i.e.*

(1) $A[(k,r)]\circ A[(l,s)] = A[k+l,r+s]$,
(2) $\langle A[(k,r)],B[(k,r)]\rangle_{Fro} = \langle A,B\rangle_{Fro}$.

We are interested in the relation between the spreading function of an operator $A\in\mathcal{M}_{|G\times G|}(\mathbb{C})$ and of $A[(k,r)]$ for $(k,r)\in G\times\widehat{G}$. Since the spreading representation of A is an expansion with respect to the basis $\{\pi(k,r) : (k,r)\in G\times\widehat{G}\}$ the problem reduces to the understanding of conjugation by $\pi(l,s)$ for time-frequency shifts.

Lemma 4.12. *For* $\mathrm{A} \in \mathcal{M}_{|\mathrm{G}\times\mathrm{G}|}(\mathbb{C})$ *and* $(\mathrm{k},\mathrm{r}) \in \mathrm{G} \times \widehat{\mathrm{G}}$ *one has* $\eta_{\mathrm{A}[(\mathrm{k},\mathrm{r})]} = \mathrm{M}^{\mathbf{s}}_{(\mathrm{k},\mathrm{r})}\eta_{\mathrm{A}}$.

Proof.

$$\begin{aligned} \pi(\mathrm{k},\mathrm{r}) \circ \mathrm{A} \circ \pi(\mathrm{k},\mathrm{r})^\star &= \sum_{\mathrm{l},\mathrm{s}} \eta_{\mathrm{A}}(\mathrm{l},\mathrm{s})\pi(\mathrm{k},\mathrm{r})\pi(\mathrm{l},\mathrm{s})\pi(\mathrm{k},\mathrm{r})^\star \\ &= \sum_{\mathrm{l},\mathrm{s}} \chi_{\mathrm{G}}\big((\mathrm{k},\mathrm{r}),(\mathrm{l},\mathrm{s})\big)\overline{\chi}_{\mathrm{G}}\big((\mathrm{l},\mathrm{s}),(\mathrm{k},\mathrm{r})\big)\eta_{\mathrm{A}}(\mathrm{l},\mathrm{s})\pi(\mathrm{l},\mathrm{s}) \\ &= \sum_{\mathrm{l},\mathrm{s}} \mathrm{M}^{\mathbf{s}}_{(\mathrm{l},\mathrm{s})}\eta_{\mathrm{A}}\pi(\mathrm{l},\mathrm{s}). \end{aligned}$$

□

This behaviour of the spreading coefficients under conjugation of the operator will be crucial in our discussion of the Janssen representation of Gabor frame operators. At the moment we want to point out that the preceding properties of the spreading representation gives the following form of the Resolution of Identity (4.8). Namely, recall the rank-one operator $\mathrm{P}_{\mathbf{g},\mathbf{h}}\mathbf{f} = \langle \mathbf{f}, \mathbf{g}\rangle\mathbf{h}$ for $\mathbf{f}, \mathbf{g} \in \mathbb{C}^{|\mathrm{G}|}$. Then an elementary calculation gives that

$$\big[\pi(\mathrm{k},\mathrm{r}) \circ \mathrm{P}_{\mathbf{g},\mathbf{h}} \circ \pi(\mathrm{k},\mathrm{r})^\star\big]\mathbf{f} = \langle \mathbf{f}, \pi(\mathrm{k},\mathrm{r})\mathbf{g}\rangle\pi(\mathrm{k},\mathrm{r})\mathbf{h}.$$

Therefore conjugation of a rank-one operator $\mathrm{P}_{\mathbf{g},\mathbf{h}}$ by a time-frequency shift moves it to the point (k,r) in the time-frequency plane $\mathbb{C}^{|\mathrm{G}\times\widehat{\mathrm{G}}|}$. We denote conjugation by $\pi(\mathrm{k},\mathrm{r})$ of a linear operator A on $\mathbb{C}^{|\mathrm{G}|}$ by

$$\mathrm{A}[(\mathrm{k},\mathrm{r})] := \pi(\mathrm{k},\mathrm{r}) \circ \mathrm{A} \circ \pi(\mathrm{k},\mathrm{r})^\star.$$

Consequently the Resolution of Identity can be expressed in the following way,

$$\mathbb{I}_{|\mathrm{G}|} = \frac{1}{\langle \mathbf{g}, \mathbf{h}\rangle} \sum_{\mathrm{k},\mathrm{r}} \mathrm{P}_{\mathbf{g},\mathbf{h}}[(\mathrm{k},\mathrm{r})],$$

for $\langle \mathbf{f}, \mathbf{h}\rangle \neq 0$. If $\mathbf{g} = \mathbf{h}$ then the identity operator is a linear combination of orthogonal projections onto the one-dimensional spaces generated by the family $\{\pi(\mathrm{k},\mathrm{r})\mathbf{g}\}$. Recall that the pure states of $\mathcal{M}_{|\mathrm{G}\times\mathrm{G}|}(\mathbb{C})$ are the rank one operators. Therefore the resolution of identity may be understood as shifting a pure state of $\mathcal{M}_{|\mathrm{G}\times\mathrm{G}|}(\mathbb{C})$ over the discrete time-frequency plane $\mathbb{C}^{|\mathrm{G}\times\widehat{\mathrm{G}}|}$.

After these preparations we want to explore the structure of Gabor frames $\mathcal{G}(\mathbf{g}, \Lambda)$ for $\Lambda \lhd \mathrm{G} \times \widehat{\mathrm{G}}$, i.e. discrete analogues of the Resolution of Identity:

$$\mathbf{f} = \sum_{\lambda\in\Lambda} \langle \mathbf{f}, \pi(\lambda)\mathbf{h}\rangle\, \pi(\lambda)\mathbf{g} \quad \text{for a suitable } \mathbf{h} \in \mathbb{C}^{|\mathrm{G}|}.$$

The last equation indicates that operators of the type

$$\mathrm{S}_{\mathbf{g},\mathbf{h},\Lambda}\mathbf{f} = \sum_{\lambda\in\Lambda} \langle \mathbf{f}, \pi(\lambda)\mathbf{h}\rangle\pi(\lambda)\mathbf{g}$$

are closely related with the frame operator of a Gabor system $\mathcal{G}(\mathbf{g}, \Lambda)$. Due to this fact they are called *Gabor frame-type operators*. The following property of Gabor frame-type operators is crucial for an understanding of the structure of Gabor systems and is the very reason for all duality principles in Gabor analysis.

Lemma 4.13. *Let* Λ *be a subgroup of* $\mathrm{G} \times \widehat{\mathrm{G}}$ *and* $\mathbf{g}$ *a Gabor atom in* $\mathbb{C}^{|\mathrm{G}|}$. *Then the Gabor frame-type operator* $\mathrm{S}_{\mathbf{g},\mathbf{h},\Lambda}$ *commutes with* $\pi(\lambda)$ *for all* $\lambda \in \Lambda$, *i.e.*

$$\pi(\lambda) \circ \mathrm{S}_{\mathbf{g},\mathbf{h},\Lambda} \circ \pi(\lambda)^\star = \mathrm{S}_{\mathbf{g},\mathbf{h},\Lambda}. \tag{14}$$

Proof. The commutation relation for time-frequency shifts yields for $\mu \in \Lambda$.:

$$\begin{aligned}
[\pi(\mu)\mathrm{S}_{\mathbf{g},\mathbf{h},\Lambda}\pi(\mu)^\star]\mathbf{f} &= \sum_{\lambda\in\Lambda}\langle \pi(\mu)^\star\mathbf{f}, \pi(\lambda)\mathbf{h}\rangle \pi(\mu)\pi(\lambda)\mathbf{g} \\
&= \sum_{\lambda\in\Lambda}\langle \mathbf{f}, \pi(\mu)\pi(\lambda)\mathbf{h}\rangle \pi(\mu)\pi(\lambda)\mathbf{g} \\
&= \sum_{\lambda\in\Lambda}\langle \mathbf{f}, \pi(\mu+\lambda)\mathbf{h}\rangle \pi(\mu+\lambda)\mathbf{g} = \mathrm{S}_{\mathbf{g},\mathbf{h},\Lambda}\mathbf{f}.
\end{aligned}$$

□

Since (14) is a crucial property of $\mathrm{S}_{\mathbf{g},\mathbf{h},\Lambda}$ we want to explore the structure of the set $\mathcal{B}(\Lambda)$ of all linear operators $\mathrm{A} \in \mathcal{M}_{|\mathrm{G}\times\mathrm{G}|}$ which commute with all time-frequency shifts $\pi(\lambda)$ from a given subgroup Λ of $\mathrm{G} \times \widehat{\mathrm{G}}$. We call these operators Λ-*invariant*.

Proposition 4.14. *For every subgroup* $\Lambda \lhd G \times \widehat{G}$ *the set* $\mathcal{B}(\Lambda)$ *is an involutive subalgebra of* $M_{|\mathrm{G}\times\mathrm{G}|}(\mathbb{C})$.

Proof. If A, B are Λ-invariant operators, then we have that:

(1) $\mathrm{A} + \mathrm{B} \in \mathcal{B}(\Lambda)$ and $c\mathrm{A} \in \mathcal{B}(\Lambda)$ for all $c \in \mathbb{C}$.
(2) $\mathrm{AB} \in \mathcal{B}(\Lambda)$ since $\pi(\lambda)\mathrm{A}\pi(\lambda)^* = \mathrm{A}$ and $\pi(\lambda)\mathrm{B}\pi(\lambda)^* = \mathrm{B}$ implies that

$$\mathrm{AB} = \pi(\lambda)\mathrm{A}\pi(\lambda)^*\pi(\lambda)\mathrm{B}\pi(\lambda)^* = \pi(\lambda)\mathrm{AB}\pi(\lambda)^*.$$

(3) $\mathrm{A} \in \mathcal{B}(\Lambda)$ implies $\pi(\lambda)\mathrm{A}^\star\pi(\lambda)^* = \mathrm{A}^\star$.

□

The algebra $\mathcal{B}(\Lambda)$ of Λ-invariant operator A is the commutant of the twisted group algebra $\mathcal{A}(\Lambda)$ within $\mathcal{M}_{|\mathrm{G}|}$ is equal to the twisted group algebra $\mathcal{A}(\Lambda^\circ)$. In detail:

Proposition 4.15. *Let* Λ *be a subgroup of* $\mathrm{G} \times \widehat{\mathrm{G}}$. *Then we have*

(1) $\mathcal{B}(\Lambda) = \mathcal{A}(\Lambda^\circ)$;
(2) *The commutant of* $\mathcal{B}(\Lambda)$ *is* $\mathcal{A}(\Lambda)$ *and the commutant of* $\mathcal{A}(\Lambda^\circ)$ *is* $\mathcal{A}(\Lambda)$;
(3) *The center of* $\mathcal{B}(\Lambda) = \mathcal{A}(\Lambda \cap \Lambda^\circ)$;
(4) $\mathcal{B}(\Lambda)$ *is commutative if and only if* $\Lambda^\circ \subseteq \Lambda$.

The last assertions indicate that the structure of Λ is essential for the properties of $\mathcal{B}(\Lambda)$. In analogy to symplectic vector spaces we call the subgroup Λ of $\mathrm{G} \times \widehat{\mathrm{G}}$ *isotropic* if the symplectic form Ω vanishes identically on Λ. The largest subgroup of $\mathrm{G} \times \widehat{\mathrm{G}}$ with this property is naturally called *maximal isotropic*. A moment of reflection shows that a Λ is isotropic if and only if $\Lambda^\circ \subseteq \Lambda$. Therefore $\mathcal{B}(\Lambda)$ is commutative if and only if Λ is isotropic. Now the maximal commutative subalgebra of $\mathcal{M}_{|\mathrm{G}\times\mathrm{G}|}(\mathbb{C})$ is the algebra of diagonal matrices which implies that Λ is maximal isotropic if and only if Λ is a product lattice $\Lambda \times \Lambda^\perp$ for $\Lambda \lhd \mathrm{G}$ and $\Lambda^\perp \lhd \widehat{\mathrm{G}}$. In other words, the Gabor frame operator $\mathrm{S}_{\mathcal{G},\Lambda\times\Lambda^\perp}$ for a product lattice $\Lambda \times \Lambda^\perp$ is unitarily equivalent to a diagonal operator. The Zak transform is the interwinding operator which diagonalizes the Gabor frame operator $\mathrm{S}_{\mathcal{G},\Lambda\times\Lambda^\perp}$, see [19].

Note that $\{\pi(\lambda) : \lambda \in \Lambda\}$ defines a *reducible* projective representation of Λ if Λ is a proper subgroup, because then its commutant $\mathcal{A}(\Lambda^\circ)$ is non-trivial.

Proposition 4.16. *Let Λ be a subgroup of $\mathrm{G} \times \widehat{\mathrm{G}}$. Then $\{\pi(\lambda) : \lambda \in \Lambda\}$ defines a reducible projective representation of Λ.*

The previous statement is valid for any subgroup Λ of $\mathrm{G} \times \widehat{\mathrm{G}}$, especially for Λ°. The twisted group algebras $\mathcal{A}(\Lambda)$ and $\mathcal{A}(\Lambda^\circ)$ are the matrix algebras underlying Gabor analysis over finite abelian groups. Therefore we investigate their structure in detail. We use the notion of *matrix coefficients for a twisted group algebra*. Let $\pi_{\mathcal{A}(\Lambda)}$ be a representation of the twisted group algebra $\mathcal{A}$. Then we define a *matrix coefficient* of $\mathcal{A}(\Lambda)$ as

$$\langle \pi_{\mathcal{A}(\Lambda)}(\mathbf{a})\mathbf{g}, \mathbf{h}\rangle_{\mathbb{C}^{|\Lambda|}} = \sum_{\lambda\in\Lambda} a(\lambda)\langle \pi(\lambda)\mathbf{g}, \mathbf{h}\rangle_{\mathbb{C}^{|\Lambda|}} \text{ for } \mathbf{a} = (a(\lambda)), \mathbf{g}, \mathbf{h} \in \mathbb{C}^{|\Lambda|}.$$

There is a close relation between the matrix coefficients of $\mathcal{A}(\Lambda)$ and $\mathcal{A}(\Lambda^\circ)$.

Theorem 4.17. *Let Λ be a subgroup of $\mathrm{G} \times \widehat{\mathrm{G}}$. Then we have*

$$\sum_{\lambda\in\Lambda}\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{d_i}\langle \mathbf{g}_\mathrm{i}, \pi(\lambda)\mathbf{h}_\mathrm{i}\rangle\langle \pi(\lambda)\mathbf{g}, \mathbf{h}\rangle = \frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ}\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{d_i}\langle \mathbf{g}_\mathrm{i}, \pi(\lambda^\circ)\mathbf{g}\rangle\langle \pi(\lambda^\circ)\mathbf{h}, \mathbf{h}_\mathrm{i}\rangle.$$

Proof. Recall that every $\mathrm{A} \in \mathcal{A}(\Lambda)$ may be written as

$$\pi_{\mathcal{A}(\Lambda)}(\eta_\mathrm{A}) = \sum_{\lambda\in\Lambda}\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{d_i}\langle \mathbf{g}_\mathrm{i}, \pi(\lambda)\mathbf{h}_\mathrm{i}\rangle\pi(\lambda)$$

and this yields to

$$\begin{aligned}\langle \pi_{\mathcal{A}(\Lambda)}(\eta_\mathrm{A})\mathbf{g}, \mathbf{h}\rangle_{\mathbb{C}^{|\Lambda|}} &= \sum_{\lambda\in\Lambda}\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{d_i}\langle \mathbf{g}_\mathrm{i}, \pi(\lambda)\mathbf{h}_\mathrm{i}\rangle\langle \pi(\lambda)\mathbf{g}, \mathbf{h}\rangle \\ &= \frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ}\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{d_i}\langle \mathbf{g}_\mathrm{i}, \pi(\lambda^\circ)\mathbf{g}\rangle\langle \pi(\lambda^\circ)\mathbf{h}, \mathbf{h}_\mathrm{i}\rangle,\end{aligned}$$

where we applied in the last equation the FIGA. The right side of the last equation may be understood as the matrix coefficient for a certain element of $\pi_{\mathcal{A}(\Lambda^\circ)}$. □

If we consider the special case $\Lambda = \mathrm{G} \times \widehat{\mathrm{G}}$, then the statement of the last theorem specializes to Moyal's Formula, i.e. the Schur orthogonality relations for STFT's. In this sense we consider the last theorem as a generalization of Schur's orthogonality relation to reducible group representations.

The preceding observations yield the following representation of a Λ-invariant operator, which includes the representation of Gabor frame operators as given by Tolimieri-Orr, Qiu, Strohmer, and Wexler-Raz for product lattices $\Lambda_1 \times \Lambda_2$.

Theorem 4.18 (Janssen representation)**.** *Let* Λ *be a subgroup of* $\mathrm{G} \times \widehat{\mathrm{G}}$*. Then for a* Λ*-invariant operator* A *we have a prototype matrix* P *such that*

$$\mathrm{A} = \sum_{\lambda \in \Lambda} \mathrm{P}[\lambda] = \sum_{\lambda \in \Lambda} \pi(\lambda)^\star \circ \mathrm{P} \circ \pi(\lambda)$$

is the Λ*-periodization of* P *or*

$$\mathrm{A} = \sum_{\lambda^\circ \in \Lambda^\circ} \langle \mathrm{P}, \pi(\lambda^\circ) \rangle_{\mathrm{Fro}} \pi(\lambda^\circ),$$

i.e. the spreading coefficients of A *are the sampled spreading coefficients of the prototype operator* P *to the adjoint lattice* Λ°*.*

Proof. An application of the commutation relations for time-frequency shifts in the following form

$$\pi(\mu)\pi(\lambda)\pi(\mu)^* = \mathrm{c}^{\mathbf{s}}_{|\mathrm{G}\times\widehat{\mathrm{G}}|}(\lambda, \mu)\pi(\lambda - \mu) \quad \text{for all } \lambda, \mu \in \Lambda$$

implies that $\pi(\mu) \circ \pi_{\mathcal{A}(\Lambda)}(\mathbf{a}) \circ \pi(\mu)$ is another element of $\mathcal{A}(\Lambda)$ for a suitable translated and shifted version of $\mathbf{a}$. In terms of the spreading representation, the imposition of Λ-invariance on a linear operator A implies that the spreading coefficients are periodic,i.e. $\mathrm{A} \in \mathcal{B}(\Lambda)$ is equivalent to $\mathrm{A}[\lambda] = \mathrm{A}$, i.e. that for all $\lambda \in \Lambda$ one has:

$$\eta_{\mathrm{A}[\lambda]}(\mu) = \mathrm{c}_{|G|}(\lambda, \mu)\overline{\mathrm{c}_{|G|}}(\mu, \lambda)\eta_{\mathrm{A}}(\mu) \text{ for all } \mu \in \Lambda.$$

In other words every Λ-invariant operator has a representation of the form

$$\mathrm{A} = \sum_{\lambda^\circ \in \Lambda^\circ} \mathrm{a}(\lambda^\circ)\pi(\lambda^\circ)$$

for some vector $\mathbf{a} = \big(a(\lambda^\circ)\big) \in \mathbb{C}^{|\Lambda^\circ|}$.

The prototype operator P is the sum of projection operators arising from the singular value decomposition of A, i.e. $\mathrm{P} = \sum_{\mathrm{i}=1}^{\mathrm{r}} \mathrm{d_i} \mathbf{u}_\mathrm{i} \otimes \overline{\mathbf{v}_\mathrm{i}}$. Let A be the Gabor frame operator of $\mathcal{G}(\mathbf{g}, \Lambda)$. Then

$$\mathrm{S}_{\mathbf{g},\Lambda} = |\Lambda|^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} \langle \mathbf{g}, \pi(\lambda^\circ)\mathbf{g} \rangle \pi(\lambda^\circ).$$

□

We interpret the Janssen representation of a Λ-invariant operator in terms of multi-window Gabor frames. At the moment we draw some consequences of the Janssen representation which generalize results of Wexler-Raz on Gabor frame operators for separable lattices in cyclic groups. In their work on the inversion of Gabor frame operators over cyclic groups Wexler and Raz gave the impetus for the duality theory of Gabor frames which was developed independently by several groups of researchers. Our general results on the structure of Λ-invariant operators lead naturally to the study of a finite number of Gabor frames. Let $\mathbf{g}_1, ..., \mathbf{g}_\mathrm{r} \in \mathbb{C}^{|\mathrm{G}|}$ be Gabor atoms of the Gabor systems $\mathcal{G}(\mathbf{g}_1, \Lambda), ..., \mathcal{G}(\mathbf{g}_\mathrm{r}, \Lambda)$, then we speak of a *multi-window Gabor system.* If the associated frame operator

$$\mathrm{S}_{\mathcal{G}}\mathbf{f} = \sum_{\mathrm{j}=1}^{\mathrm{r}} \mathrm{S}_{\mathbf{g}_\mathrm{j},\Lambda}\mathrm{f} = \sum_{\mathrm{j}=1}^{\mathrm{r}} \sum_{\lambda \in \Lambda} \langle \mathbf{f}, \pi(\lambda)\mathbf{g}_\mathrm{j} \rangle \pi(\lambda)\mathbf{g}_\mathrm{j}$$

is invertible, then the system $\mathcal{G} = \bigcup_{j=1}^{r} \mathcal{G}(\mathbf{g}_j, \Lambda)$ is called a *multi-window Gabor frame*. Therefore our discussion of the Janssen representation of a Λ-invariant operator A states that A is a multi-window Gabor frame operator with $\mathbf{g}_\mathrm{i} = \mathbf{u}_\mathrm{i}$ for $\mathrm{i} = 1, ..., \mathrm{r}$. We refer the reader to [47, 48, 8] for further information on multi-window Gabor frames.

The multi-window Gabor frame operator is actually a finite sum of rank-one operators

$$\mathrm{S}_{\mathcal{G}} = \sum_{\lambda \in \Lambda} (\mathbf{g}_1 \otimes \overline{\mathbf{g}_1}[\lambda] + \cdots + \mathbf{g}_\mathrm{r} \otimes \overline{\mathbf{g}_\mathrm{r}}[\lambda])$$

and is a Λ-invariant operator and therefore has a Janssen representation.

The main problem in Gabor analysis over finite groups is an understanding of all dual pairs $(\mathbf{g}, \mathbf{h})$ for a given Gabor system $\mathcal{G}(\mathbf{g}, \Lambda)$. In other words, we look for all $(\mathbf{g}, \mathbf{h})$ such that $\mathrm{S}_{\mathbf{g},\mathbf{h},\Lambda} = \mathbb{I}_\mathrm{G}$. We attack this problem by an application of the Janssen representation of Λ-invariant operators.

Corollary 4.19. *Let Λ be a subgroup of $\mathrm{G} \times \widehat{\mathrm{G}}$ and $\mathcal{G}(\mathbf{g}, \Lambda)$ a Gabor system for $\mathbf{g} \in \mathbb{C}^{|\mathrm{G}|}$. Then the following are equivalent:*

(1) $\mathbf{f} = \sum_{\lambda \in \Lambda} \langle \mathbf{f}, \pi(\lambda)\mathbf{h} \rangle \pi(\lambda)\mathbf{g}$ *for all* $\mathbf{f} \in \mathbb{C}^{|\mathrm{G}|}$.
(2) $\langle \mathbf{g}, \pi(\lambda^\circ)\mathbf{h} \rangle = |\Lambda| \cdot \delta_{\lambda^\circ, 0}$ *for all* $\lambda^\circ \in \Lambda^\circ$.

Proof. The equivalence of (1) and (2) follows immediately from the uniqueness of the spreading representation, the fact that $\eta_{\mathbb{I}_{|\mathrm{G}|}} = \delta_{\lambda^\circ, 0}$, and the identity

$$S_{\mathbf{g},\mathbf{h},\Lambda}\mathbf{f} = |\Lambda|^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} \langle \mathbf{g}, \pi(\lambda^\circ)\mathbf{h} \rangle \pi(\lambda^\circ)\mathbf{f}, \ \text{ for all } \mathbf{f} \in \mathbb{C}^{|\mathrm{G}|}.$$

□

The biorthogonality relations of Wexler-Raz are a relation between traces of $\mathcal{A}(\Lambda)$ and $\mathcal{A}(\Lambda^\circ)$. Namely, the commutation relations for time-frequency shifts and the tracial property yields that

$$\mathrm{tr}_{\mathcal{A}(\Lambda)}(\mathrm{A}) = \eta_\mathrm{A}(0) = \mathrm{a}_{0,0} \text{ and } \mathrm{tr}_{\mathcal{A}(\Lambda^\circ)}(\mathrm{A}) = \mathrm{c}\eta_\mathrm{A}(0) \text{ for some } \mathrm{c} \in \mathbb{C}.$$

The Poisson summation formula for the symplectic Fourier transform allows us to identity c with $1/|\Lambda|$, i.e. the traces of $\mathcal{A}(\Lambda)$ and $\mathcal{A}(\Lambda^\circ)$ are multiples of each other:

$$\mathrm{tr}_{\mathcal{A}(\Lambda)} = \frac{1}{|\Lambda|} \, \mathrm{tr}_{\mathcal{A}(\Lambda^\circ)} \, .$$

Let A be a Gabor frame-type operator. Then the preceding equation yields the Wexler-Raz biorthogonality relations. More generally, we get a "weighted" Wexler-Raz biorthogonality relation for multi-window Gabor frame-type operators.

The Janssen representation of a Gabor frame operator provides an alternative route to the inversion of the Gabor frame operator: Namely, a Gabor frame operator may be considered as a twisted convolution operator on Λ°. More generally,

Lemma 4.20. *Let* Λ *be a subgroup of* $\mathrm{G} \times \widehat{\mathrm{G}}$ *and* $\mathrm{A}, \mathrm{B} \in \mathcal{B}(\Lambda)$*. Then the spreading coefficients of* AB *are given by the twisted convolution of those for* A *and* B*:*

$$\eta_{\mathrm{AB}}(\lambda^\circ) = \sum_{\mu^\circ \in \Lambda^\circ} \eta_{\mathrm{A}}(\lambda^\circ - \mu^\circ)\eta_{\mathrm{B}}(\mu^\circ)\mathrm{c}_{\mathrm{G}}(\lambda^\circ - \mu^\circ, \mu^\circ).$$

Therefore we have a symbolic calculus for Λ-invariant operators. We have that $\mathrm{S}_{\mathbf{g},\Lambda} \times \mathbb{I}_\Lambda = \mathrm{S}_{\mathbf{g},\Lambda}$, i.e.

$$\eta_{\mathrm{S}_{\mathbf{g},\Lambda}}(\lambda^\circ) = \eta_{\mathrm{S}_{\mathbf{g},\Lambda}} \natural_{\Lambda^\circ} \eta_{\mathbb{I}_\Lambda}(\lambda^\circ) = |\Lambda|^{-1} \sum_{\mu^\circ} \langle \mathbf{g}, \pi(\lambda^\circ - \mu^\circ)\mathbf{g}\rangle \mathrm{c}_{\mathrm{G}}(\lambda^\circ - \mu^\circ, \mu^\circ).$$

On the level of representation coefficients the discussion of Gabor frames corresponds to the study of the mapping

$$\mathbf{c} = (\mathrm{c}_{\lambda^\circ}) \mapsto \mathrm{G}\mathbf{c}$$

with $\mathrm{G} = \big(\mathrm{G}(\lambda^\circ, \mu^\circ \in \Lambda^\circ)\big) = \big(\langle \mathbf{g}, \pi(\lambda^\circ - \mu^\circ)\mathbf{g}\rangle \mathrm{c}_{\mathrm{G}}(\lambda^\circ - \mu^\circ, \mu^\circ)\big)$. By the definition of the Gram matrix and the commutation relation, G is the Gram matrix of $\{\pi(\lambda^\circ)\mathbf{g} : \lambda^\circ \in \Lambda^\circ\}$. Now the question of invertibility of the Gabor frame operator translates into the invertibility of the twisted convolution of the Gram matrix of $\mathcal{G}(\mathbf{g}, \Lambda^\circ)$. More precisely, $\mathcal{G}(\mathbf{g}, \Lambda)$ is a Gabor frame for $\mathbb{C}^{|\mathrm{G}|}$ if and only if G is invertible. Let $\mathbf{c}$ be a vector in $\mathbb{C}^{|\Lambda^\circ|}$. Then $\mathbf{c}$ represents the canonical dual window $\mathbf{h}_0 = \mathrm{S}_{\mathbf{g},\Lambda}^{-1}\mathbf{g}$, if it is the unique solution of the system of equations

$$\mathrm{G}\mathbf{c} = |\Lambda|\delta_{\lambda^\circ,0}.$$

Consider the algebra of all Λ-invariant operators $\mathcal{B}(\Lambda)$ as a C^*-algebra of matrices, i.e. we equip $\mathcal{M}_{|\mathrm{G}|\times|\Lambda|}$ with the operator norm. Then we may express an element of $\mathcal{B}(\Lambda)$ in terms of $\{\pi(\lambda) : \lambda \in \Lambda\}$ or $\{\pi(\lambda^\circ) : \lambda^\circ \in \Lambda^\circ\}$. Now by the spreading representation we obtain a relation between the operator norm of $\|\mathrm{S}_{\mathbf{g},\Lambda}\|_{\mathrm{op}}$ and $\|\mathrm{G}\|_{\mathrm{op}}$. By the uniqueness of the C^*-algebra norm we get the existence of a constant k such that $\|\mathrm{S}_{\mathbf{g},\Lambda}\|_{\mathrm{op}} = \mathrm{k}\|\mathrm{G}\|_{\mathrm{op}}$ which for $\|\mathrm{S}_{\mathbf{g},\Lambda}\|_{\mathrm{op}}$ equal to the identity yields $\mathrm{k} = |\Lambda|^{-1}$. As a summary of the previous statements we get the other duality result for Gabor frames, the *Ron-Shen duality principle.*

Theorem 4.21 (Ron-Shen duality)**.** *Let* Λ *be a subgroup of* $G \times \widehat{G}$*. Then* $\mathcal{G}(\mathbf{g}, \Lambda)$ *is a frame for* $\mathbb{C}^{|\mathrm{G}|}$ *if and only if* $\mathcal{G}(\mathbf{g}, \Lambda^\circ)$ *is a (Riesz) basis for* $\mathbb{C}^{|\Lambda^\circ|}$*. Let* A_Λ, B_Λ *be the frame bounds of* $\mathcal{G}(\mathbf{g}, \Lambda)$ *and* $A_{\Lambda^\circ}, B_{\Lambda^\circ}$ *the (Riesz) basis bounds of* $\mathcal{G}(\mathbf{g}, \Lambda^\circ)$*. Then*

$$A_{\Lambda^\circ} = |\Lambda| A_\Lambda \text{ and } B_{\Lambda^\circ} = |\Lambda| B_\Lambda.$$

5. Dual windows, canonical tight windows and Löwdin orthogonalization

In this section we want to stress the importance of the canonical dual and canonical tight Gabor window, especially the relation between the structure of the canonical tight Gabor frame and the Löwdin orthogonalization of the original Gabor frame.

Let $\mathcal{G}(\mathbf{g}, \Lambda)$ be a Gabor frame for $\mathbb{C}^{|\mathrm{G}|}$. Then we denote by $\Gamma_{\mathbf{g},\Lambda}$ the set of all dual pairs $(\mathbf{g}, \mathbf{h})$, i.e.

$$\Gamma_{\mathbf{g},\Lambda} = \{\mathbf{h} : \mathrm{C}^\star_{\mathbf{g},\Lambda}\mathrm{C}_{\mathbf{g},\Lambda} = \mathbb{I}_{|\mathrm{G}|}\}.$$

The exploration of this set is one of the core problems in Gabor analysis. Observe that the Λ-invariance has as important consequence, the Λ-invariance of arbitrary powers of the Gabor frame operator:

$$\pi(\lambda) \circ \mathrm{S}^{\alpha}_{\mathbf{g},\Lambda} \circ \pi(\lambda)^{\star} = \mathrm{S}^{\alpha}_{\mathbf{g},\Lambda} \text{ for all } \alpha \in \mathbb{R}. \tag{15}$$

Therefore

$$\mathbf{f} = \mathrm{S}^{-1}_{\mathbf{g},\Lambda}\mathrm{S}_{\mathbf{g},\Lambda}\mathbf{f} = \sum_{\lambda\in\Lambda} \langle \mathbf{f}, \pi(\lambda)\mathbf{g}\rangle \pi(\lambda)(\mathrm{S}^{-1}_{\mathbf{g},\Lambda}\mathbf{g})$$

and $\mathcal{G}(\tilde{\mathbf{g}},\Lambda)$ is called the *canonical dual Gabor frame* given by $\tilde{\mathbf{g}} = \mathrm{S}^{-1}_{\mathbf{g},\Lambda}\mathbf{g}$. The duality theory of Gabor frames implies that

$$\mathrm{S}_{\tilde{\mathrm{g}},\Lambda} = \frac{1}{|\Lambda|} \sum_{\lambda^\circ\in\Lambda^\circ} \langle \tilde{\mathrm{g}}, \pi(\lambda^\circ)\tilde{\mathrm{g}}\rangle \pi(\lambda^\circ),$$

i.e. $\mathrm{S}_{\tilde{\mathrm{g}},\Lambda}$ is an element of the linear span of $\mathcal{G}(\mathbf{g},\Lambda^\circ) = \{\pi(\lambda^\circ)\mathbf{g} : \lambda^\circ \in \Lambda^\circ\}$. By the Wexler-Raz biorthogonality relation we can identify $\Gamma_{\mathbf{g},\Lambda}$.

Theorem 5.1 (Wexler-Raz)**.** *Let $\mathcal{G}(\mathbf{g},\Lambda)$ be a Gabor frame for $\mathbb{C}^{|\mathrm{G}|}$. Then $\mathbf{h} \in \Gamma_{\mathbf{g},\Lambda}$ if and only if*

$$\mathbf{h} \in \tilde{\mathbf{g}} + \mathcal{G}(\mathbf{g},\Lambda^\circ)^{\perp}.$$

Proof. Our argument closely follows [20]. Every $\mathbf{h} \in \Gamma_{\mathbf{g},\Lambda}$ satisfies the Wexler-Raz biorthogonality conditions, therefore we have $\langle \mathbf{h} - \tilde{\mathbf{g}}, \pi(\lambda^\circ)\mathbf{g}\rangle = 0$ for all $\lambda^\circ \in \Lambda^\circ$, i.e. $\mathbf{h} - \tilde{\mathbf{g}} \in \mathcal{G}(\mathbf{g},\Lambda^\circ)^{\perp}$. Conversely, letting $\mathbf{h} \in \tilde{\mathbf{g}} + \mathcal{G}(\mathbf{g},\Lambda^\circ)^{\perp}$, then $\mathbf{h}$ fulfills the Wexler-Raz biorthogonality relation, i.e. $\mathrm{S}_{\mathbf{g},\mathbf{h},\Lambda} = \mathbb{I}_{|\mathrm{G}|}$. □

The canonical dual Gabor atom $\tilde{\mathbf{g}} = \mathrm{S}^{-1}_{\mathbf{g},\Lambda}\mathbf{g}$ is a distinguished element of $\Gamma_{\mathbf{g},\Lambda}$ since it is the "optimal" dual window in various ways.

Theorem 5.2. *Let $\mathcal{G}(\mathbf{g},\Lambda)$ be a Gabor frame for $\mathbb{C}^{|\mathrm{G}|}$. Then for all $\mathbf{h} \in \Gamma_{\mathbf{g},\Lambda}$ the following (equivalent, characteristic properties) hold true:*

$$\begin{aligned}
\|\tilde{\mathbf{g}}\|_{\mathbb{C}^{|\mathrm{G}|}} &\leq \|\mathbf{h}\|_{\mathbb{C}^{|\mathrm{G}|}} \quad (\textit{minimal norm}),\\
\|\mathrm{C}_{\tilde{\mathbf{g}},\Lambda}\mathbf{f}\|_{\mathbb{C}^{|\mathrm{G}|}} &\leq \|\mathrm{C}_{\tilde{\mathbf{h}},\Lambda}\mathbf{f}\|_{\mathbb{C}^{|\mathrm{G}|}} \quad (\textit{minimal norm coefficients}),\\
\|\mathbf{g} - \tilde{\mathbf{g}}\|_{\mathbb{C}^{|\mathrm{G}|}} &\leq \|\mathbf{g} - \mathbf{h}\|_{\mathbb{C}^{|\mathrm{G}|}} \quad (\textit{closest to Gabor atom}),\\
\|\mathbf{g}/\|\mathbf{g}\|_{\mathbb{C}^{|\mathrm{G}|}} - \tilde{\mathbf{g}}/\|\tilde{\mathbf{g}}\|_{\mathbb{C}^{|\mathrm{G}|}}\|_{\mathbb{C}^{|\mathrm{G}|}} &\leq \|\mathbf{g}/\|\mathbf{g}\|_{\mathbb{C}^{|\mathrm{G}|}} - \mathbf{h}/\|\mathbf{h}\|_{\mathbb{C}^{|\mathrm{G}|}}\|_{\mathbb{C}^{|\mathrm{G}|}} \quad (\textit{most likely}).
\end{aligned}$$

We refer the interested reader to [32] for the proof of the preceding theorem and for a detailed discussion of the structure of $\Gamma_{\mathbf{g},\Lambda}$ and that it actually characterizes the dual Gabor atom. We only want to focus on the minimal norm property of the coefficient mapping for the canonical dual Gabor frame. The following computation shows the relevance of the Moore-Penrose inverse for Gabor frames.

$$\begin{aligned}
\mathcal{C}_{\tilde{\mathbf{g}},\Lambda}\mathbf{f} &= (\langle \mathbf{f}, \pi(\lambda)\mathrm{S}^{-1}_{\mathbf{g},\Lambda}\mathbf{g}\rangle) = (\langle \mathbf{f}, \mathrm{S}^{-1}_{\mathbf{g},\Lambda}\pi(\lambda)\mathbf{g}\rangle)\\
&= (\langle \mathrm{S}^{-1}_{\mathbf{g},\Lambda}\mathbf{f}, \pi(\lambda)\mathbf{g}\rangle) = \mathcal{C}_{\tilde{\mathbf{g}},\Lambda}(\mathcal{C}^{\star}_{\tilde{\mathbf{g}},\Lambda}\mathcal{C}_{\tilde{\mathbf{g}},\Lambda})^{-1}\mathbf{f} = \mathcal{C}^{+}_{\mathbf{g},\Lambda}\mathbf{f},
\end{aligned}$$

where $\mathcal{C}^{+}_{\mathbf{g},\Lambda}$ denotes the Moore-Penrose inverse of the coefficient operator $\mathcal{C}_{\mathbf{g},\Lambda}$. Then it is well-known that $\mathcal{C}^{+}_{\mathbf{g},\Lambda}$ is the minimal solution of the system $\mathcal{C}_{\mathbf{g},\Lambda}\mathbf{f} = \mathbf{a}$ for a given coefficient vector $\mathbf{a}$ and such that $\mathbf{f}$ may be expressed in terms of $(\langle \mathbf{f}, \pi(\lambda)\mathbf{g}\rangle)$, i.e. $\mathbf{f} = \mathrm{B}\mathbf{a}$ under the constraint that

$$\|\mathcal{C}_{\mathbf{g},\Lambda}\mathbf{f} - \mathbf{a}\|^2_{\mathbb{C}^{|G|}} = \|(\mathcal{C}_{\mathbf{g},\Lambda}\mathrm{B} - \mathbb{I}_{|G|})\mathbf{a}\|^2_{\mathbb{C}^{|G|}} = \text{minimal.}$$

In other words we have to minimize $\|\mathcal{C}_{\mathbf{g},\Lambda}\mathrm{B} - \mathbb{I}_{|G|}\|^2_{\mathrm{Fro}}$ over all B which is solved by the Moore-Penrose inverse $B = \mathrm{C}^{+}_{\mathbf{g},\Lambda}$. In [17] it has been shown that the Moore-Penrose inverse minimizes

$$\|\mathrm{C}_{\mathbf{g},\Lambda}\mathrm{B} - I\|_{\mathrm{S_p}}$$

for all Schatten-von Neumann norms, $\|.\|_{\mathrm{S_p}}$ for all $p \in [1, \infty]$. Consequently, the coefficients of the canonical dual Gabor frame are minimal for all ℓ^{p}-norms.

There exists a canonical way to associate a tight Gabor frame with $\mathcal{G}(\mathbf{g}, \Lambda)$. By the Λ-invariance of $\mathrm{S}_{\mathbf{g},\Lambda}$ we have

$$\mathrm{S}^{-1/2}_{\mathbf{g},\Lambda}\mathrm{S}_{\mathbf{g},\Lambda}\mathrm{S}^{1/2}_{\mathbf{g},\Lambda} = \sum_{\lambda\in\Lambda}\langle \mathbf{f}, \pi(\lambda)\mathrm{S}^{-1/2}_{\mathbf{g},\Lambda}\mathbf{g}\rangle \pi(\lambda)\mathrm{S}^{-1/2}_{\mathbf{g},\Lambda}\mathbf{g}$$

Observe that $\mathcal{G}(\mathbf{h}^\circ, \Lambda)$ is a tight frame generated by the canonical tight Gabor atom $\mathbf{h}^\circ = \mathrm{S}^{-1/2}_{\mathbf{g},\Lambda}\mathbf{g}$. More generally, we investigate the set of all tight Gabor frames $\mathcal{G}(\mathbf{h}, \Lambda)$ for a given subgroup Λ in $\mathbb{C}^{|G\times\widehat{G}|}$, i.e. $\Gamma_{\mathrm{tight}}(\mathbf{g}, \Lambda) = \{\mathbf{h} \in \mathbb{C}^{|G|} : \mathrm{S}_{\mathbf{h},\Lambda} = \mathbb{I}_{|G|}\}$.

Theorem 5.3. *Assume that* $\mathbf{h} \in \Gamma_{tight}(\mathbf{g}, \Lambda)$. *Then there exists a unitary operator* U *such that* $\mathrm{U}\mathrm{C}_{\mathbf{h},\mathbf{\Lambda}} = \mathrm{C}_{\mathbf{h}^\circ,\mathbf{\Lambda}}$.

Proof. The statement of the theorem may be expressed as follows. If $\mathrm{C}^\star_{\mathbf{h}^\circ,\mathbf{\Lambda}}\mathrm{C}_{\mathbf{h}^\circ,\mathbf{\Lambda}} = \mathrm{C}^\star_{\mathbf{h},\mathbf{\Lambda}}\mathrm{C}_{\mathbf{h},\mathbf{\Lambda}}$ holds, then there exists a unitary U such that $\mathrm{C}_{\mathbf{h}^\circ,\mathbf{\Lambda}} = \mathrm{U}\mathrm{C}_{\mathbf{h}^\circ,\mathbf{\Lambda}}$. We define U on the range $\mathrm{ran}(\mathrm{C}_{\mathbf{h}^\circ,\mathbf{\Lambda}})$ by $\mathrm{U}\mathrm{C}_{\mathbf{h}^\circ,\mathbf{\Lambda}}\mathbf{f} = \mathrm{C}_{\mathbf{h},\mathbf{\Lambda}}\mathbf{f}$ and by zero on the orthogonal complement. Then $\mathrm{C}^\star_{\mathbf{h}^\circ,\mathbf{\Lambda}}\mathrm{C}_{\mathbf{h}^\circ,\mathbf{\Lambda}} = \mathrm{C}^\star_{\mathbf{h},\mathbf{\Lambda}}\mathrm{C}_{\mathbf{h},\mathbf{\Lambda}}$ implies $\ker(\mathrm{C}_{\mathbf{h}^\circ,\mathbf{\Lambda}}) \subseteq \ker(\mathrm{C}_{\mathbf{h},\mathbf{\Lambda}})$. Consequently U is an isometry satisfying $\mathrm{U}\mathrm{C}_{\mathbf{h},\mathbf{\Lambda}} = \mathrm{C}_{\mathbf{h}^\circ,\mathbf{\Lambda}}$. □

The previous theorem corresponds to the fact that there is some freedom in taking square-roots of a positive operator. The canonical tight window $\mathbf{h}^\circ$ is the tight frame which minimizes $\|\mathbf{g} - \mathbf{h}\|_{\mathbb{C}^{|G|}}$ among all tight frames. By the Ron-Shen duality principle this is equivalent to the minimization over all orthonormal basis $\mathcal{G}(\mathbf{h}, \Lambda)$ for the linear span of $\mathcal{G}(\mathbf{g}, \Lambda)$. The problem for the orthonormal basis was first solved by Löwdin in [29] and corresponds to the choice $\mathrm{A} = \mathrm{G}^{-1/2} = (\mathrm{C}^\star_{\mathbf{g},\mathbf{\Lambda}}\mathrm{C}_{\mathbf{g},\mathbf{\Lambda}})^{-1/2}$. We formulate Löwdin's orthogonalization for a general set of linearly independent vectors $\mathcal{G} = \{\mathbf{g}_1, ..., \mathbf{g}_{\mathrm{N}}\}$ of $\mathbb{C}^{\mathrm{M}}$. Let C be the $\mathrm{M} \times \mathrm{N}$ matrix with $\{\mathbf{g}_1, ..., \mathbf{g}_{\mathrm{N}}\}$ as columns.

Theorem 5.4 (Löwdin)**.** *Let* $\mathcal{G} = \{\mathbf{g}_1, ..., \mathbf{g}_{\mathrm{N}}\}$ *be a collection of linearly independent vectors in* $\mathbb{C}^{\mathrm{M}}$ *and singular value decomposition* $\mathrm{C} = \mathrm{U}\mathrm{D}\mathrm{V}^\star$ *of* C. *Then* $\|\mathrm{C} - X\|_{\mathrm{Fro}}$ *is minimized for unitary* X *by* $\mathrm{L} = \mathrm{C}(\mathrm{C}^\star\mathrm{C})^{-1/2} = \mathrm{U}\mathrm{V}^\star$.

Proof. First we show that L is unitary and secondly we show that it is the unique minimizer. Since $\mathrm{C} = \mathrm{U}\mathrm{D}\mathrm{V}^\star$ we have that

$$\begin{aligned} \mathrm{L} &= \mathrm{UDV}^\star(\mathrm{VDU}^\star\mathrm{UDV}^\star)^{-1/2} = \mathrm{UDV}^\star(\mathrm{VD}^2\mathrm{V}^\star)^{-1/2} \\ &= \mathrm{UDV}^\star(\mathrm{VDV}^\star)^{-1} = \mathrm{UV}^\star. \end{aligned}$$

Since U, V are unitary, we have that L is unitary.

$$\|\mathrm{C} - \mathrm{X}\|_{\mathrm{Fro}} = \|\mathrm{UDV}^\star - \mathrm{X}\|_{\mathrm{Fro}} = \|\mathrm{D} - \mathrm{U}^\star\mathrm{XV}\|_{\mathrm{Fro}} = \min,$$

if $\mathrm{U}^\star\mathrm{XV} = \mathbb{I}$, i.e. $\mathrm{X} = \mathrm{UV}^\star$. □

In [16] it is demonstrated that L is the minimal orthogonalization for any unitarily invariant norm, especially for all Schatten-von Neumann classes $\|.\|_{\mathcal{S}^p}$ and $p \in [0, \infty]$. Many authors call the Löwdin orthogonalization the *symmetric orthogonalization* because it is invariant under a permutation of the linearly independent vectors of G. More precisely, let P be a $\mathrm{N} \times \mathrm{M}$ permutation matrix. Then LP is the best orthogonalization of CP. The proof is by contradiction.

We close with a few words on the case of Gabor frames. Since the Gram matrix of a Gabor frame has a certain structure, the absolute value of entries are circulant, and the Frobenius norm becomes the norm of the Gabor atom, see Janssen and Strohmer for a more detailed discussion [24].

6. Conclusion

The main goal of this note is to survey Gabor frames for finite-dimensional Hilbert spaces and its duality theory, such as the Wexler-Raz biorthogonality relations, Janssen's representation of the Gabor frame operator and the Ron-Shen's duality principle. The presentation relies on elementary properties of twisted group algebras, which are matrix algebras in the given finite-dimensional setting. Our focus on twisted group algebras allows an elementary and self-contained approach. The interplay of these twisted group algebras underlies the well-known results derived by Wexler-Raz, Qiu, Strohmer and others on Gabor frames for product lattices. Therefore we are able to generalize their results to arbitrary subgroups of the time-frequency plane using algebraic methods instead of complicated multi-index calculations. The understanding of these twisted group algebras is also intimately related to the symplectic structure of the time-frequency plane. Consequently we have devoted a part of this survey to describe harmonic analysis on the time-frequency plane. As our note is mostly addressed to applied mathematicians as well as engineers we did not go so far as to explain in which sense the Morita equivalence of the twisted group algebras of the subgroup and its adjoint subgroup. However, from a deeper view-point it can be seen as the very reason for the validity of duality theory of Gabor frames (see [30] for details).

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