Abstract — Analysis of linear time-varying (LTV) systems is closely related to time-varying spectral analysis of nonstationary processes. In this paper, we link together the classical theory of linear time-varying systems and well-known concepts from time-varying spectral analysis. Formulation of the generalized Weyl correspondence provides a unified time-frequency (TF) framework for linear system analysis and synthesis. The conceptual basis of this approach is linear TF analysis of LTV systems by quadratic formulation of the generalized Weyl correspondence provides a unified time-frequency (TF) framework for linear system analysis and synthesis. Applications discussed in the paper include a new model for WSSUS channels and non-redundant TF filter design.

1 INTRODUCTION

We consider a linear time-varying (LTV) system $H$ with input–output relation (all integrals go from $-\infty$ to $\infty$)

$$ (Hx)(t) = \int h(t, t')x(t')dt', $$

where $h(t, t')$ is the impulse response of the system $H$.

The Fourier transform is a powerful tool for the analysis and synthesis of LTI systems. The LTI system's transfer function characterizes the system's behavior as frequency-selective (complex) weighting. It is intuitively appealing to generalize the transfer–function concept from LTI to LTV systems in the sense of a time-varying transfer function, thus a joint function of time and frequency.

Zadeh's time–varying transfer function. In a classical paper, Zadeh proposed the following mapping of the LTV system's impulse response $h(t, t')$ onto a function of time and frequency [1]

$$ \text{ZH}(t, f) \overset{\text{def}}{=} \int h(t, t - \tau)e^{-j2\pi f\tau}d\tau, $$

that is commonly referred to as the time–varying transfer function of an LTV system. Since $\text{ZH}(t, f)$ is a joint function of time and frequency, one may expect an interpretation in terms of a time–frequency (TF) signal representation of the input signal. It has been shown recently that such an interpretation indeed exists [2].

TF reinterpretation of Zadeh's function. The estimation of eigenvalues is fundamental for system identification and spectral estimation. For an LTV system $H$ and an unit-energy input signal $x(t)$, an optimal eigenvalue estimate $c_{\text{opt}}$ can be obtained by energy minimization of the error signal $\epsilon(t)$ in the output signal's split-up $(Hx)(t) = cx(t) + \epsilon(t)$. This yields the eigenvalue estimate

$$ c_{\text{opt}} = (Hx, x) = \int \int h(t, t')x(t')x(t)dt'dt, $$

namely, the Rayleigh quotient of $H$ and $x(t)$. In the case of an eigensignal of $H$ the error signal $\epsilon(t)$ vanishes and the Rayleigh quotient $c_{\text{opt}}$ yields the corresponding eigenvalue. The Rayleigh quotient theory is often referred to as orthogonality principle, since (3) is equivalent to $(x, c_{\text{opt}}) = 0$ in the split-up $(Hx)(t) = c_{\text{opt}}x(t) + \epsilon(t)$.

The fundamental TF property of Zadeh's function $\text{ZH}(t, f)$ is a TF formulation of (3) in terms of the input signal's Rihaczek distribution [3] $\text{Rx}(t, f) = \int x(t)x^*(t - \tau)e^{-j2\pi f\tau}d\tau$:

$$ c_{\text{opt}} = (\text{ZH}, \text{Rx}) = \int \int \text{ZH}(t, f)\text{Rx}(t, f)dt'df. $$

For an LTI system, the Rayleigh quotient $c_{\text{opt}}$ can be written as a frequency domain inner product

$$ c_{\text{opt}} = (H, |X|^2) = \int \int H(f)|X(f)|^2df, $$

where $X(f)$ is the spectrum of the input signal $x(t)$. Eq. (5) is indeed a special case of (4), since i) Zadeh's function is consistent with the LTI system's transfer function $\text{ZH}(t, f) = H(f)$, and ii) the Rihaczek distribution satisfies the marginal property, $\int \text{Rx}(t, f)dt = |X(f)|^2$.

Weyl symbol. The Rihaczek distribution $\text{Rx}(t, f)$ as time–varying generalization of the energy density spectrum $|X(f)|^2$ uniquely determines Zadeh's function $\text{ZH}(t, f)$ as time–varying generalization of the LTI system's transfer function. The alternative choice of the Wigner distribution $\text{Wx}(t, f) = \int x(t + \frac{\tau}{2})x^*(t - \frac{\tau}{2})e^{-j2\pi f\tau}d\tau$ yields another definition of a time–varying transfer function, namely the Weyl symbol [4, 2]:

$$ \text{LH}(t, f) \overset{\text{def}}{=} \int h(t + \frac{\tau}{2}, t - \frac{\tau}{2})e^{-j2\pi f\tau}d\tau, $$

The mapping of the LTV system (linear operator) $H$ onto the TF–function $\text{LH}(t, f)$ is called Weyl correspondence [4]. The fundamental property of the Weyl symbol,

$$ c_{\text{opt}} = \text{LH}, \text{Wx} = \int \int \text{LH}(t, f)\text{Wx}(t, f)dt'df, $$

establishes the Wigner–Weyl framework, which has received recent interest [5, 2, 6, 7, 8] due to its versatile applicability. We now formulate a unified approach that incorporates both the Rihaczek–Zadeh and the Wigner–Weyl framework.
2 GENERALIZED WEYL CORRESPONDENCE

The Wigner distribution \( W_\alpha(t, f) \) and the Rihaczek distribution \( R_\alpha(t, f) \) are both members of the continuously parametrized family of the generalized Wigner distributions (GWDs). The GWD \( W_\alpha(t, f) \) is defined as [6]

\[
W_\alpha(t, f) = \mathcal{A}(\alpha) q_x(t) e^{-j2\pi f t} d\tau,
\]

where \( q_x(t) \) is the signal product kernel and the mapping \( \mathcal{A} \) is defined as \( \alpha \in \mathbb{R} \):

\[
(\mathcal{A}(\alpha) h)(t, \tau) = h(t + \frac{1}{2} - \alpha) \tau - \frac{1}{2}(1 + \alpha) \tau.
\]

The GWD reduces to the Wigner distribution for \( \alpha = 0 \) and to the Rihaczek distribution for \( \alpha = 1/2 \). Any member of the GWD family satisfies a number of properties [3] that are of specific importance for LTV system analysis.

Clearly, the freedom in the definition of a time-varying signal spectrum as manifested in the \( \alpha \)-parametrization of TF system representations of the GWD family satisfies a number of properties [3] that are of specific importance for LTV system analysis.

Following this line leads to a parametrized family of linear TF system representations

\[
L^{(\alpha)}(t, f) = \int_r (\mathcal{A}(\alpha) h)(t, \tau) e^{j2\pi f \tau} d\tau,
\]

that shall be called generalized Weyl symbol (GWS). The GWS \( L^{(\alpha)}(t, f) \) reduces to Zadeh’s function \( Z_H(t, f) \) for \( \alpha = 1/2 \) and to the Weyl symbol \( L_H(t, f) \) for \( \alpha = 0 \). The unitary map of the impulse response \( h(t, t') \) onto the GWS \( L^{(\alpha)}(t, f) \) is then called generalized Weyl correspondence. The inverse generalized Weyl correspondence is given by

\[
h(t, t') = \int_r (\mathcal{A}(\alpha) h)(t, \tau) e^{j2\pi f(t-t')} d\tau.
\]

This relation can be used for a conceptually simple TF filter design [6, 9].

The GWS \( L^{(\alpha)}(t, f) \), as a linear TF system representation, is in abstract one-to-one correspondence to the GWD \( W^{(\alpha)}(t, f) \) as quadratic TF signal representation. This correspondence can be formulated either through a deterministic or a stochastic relation.

Deterministic relation. The fundamental deterministic TF property of the GWS \( L^{(\alpha)}(t, f) \) is a time-frequency formulation of the bilinear form \((Hx, y)\) in terms of the GWD \( W^{(\alpha)}(x, y) \):

\[
(Hx, y) = \left( L^{(\alpha)}_H(t, f), W^{(\alpha)}_{x,y}(t, f) \right).
\]

We emphasize that validity of (12) uniquely determines \( L^{(\alpha)}_H(t, f) \) as linear TF system representation based on the choice of \( W^{(\alpha)}_{x,y}(t, f) \) as time-varying signal spectrum. For an unit-energy signal \( z(t) = y(t) \) the bilinear form reduces to the conventional Rayleigh quotient (cf. (3)).

Stochastic relation. The generalized Weyl correspondence is equivalent to the map of the cross-correlation function \( r_{x,y}(t, t') = E[x(t)y^*(t')] \) of two non-stationary processes \( x(t) \) and \( y(t) \) onto their expected cross-GWD

\[
E \{ W^{(\alpha)}_{x,y}(t, f) \} = \int_r (\mathcal{A}(\alpha) r_{x,y})(t, \tau) e^{-j2\pi f \tau} d\tau.
\]

In other words, the GWS \( L^{(\alpha)}_H(t, f) \) of a system \( H \) with the impulse response \( h(t, t') = r_{x,y}(t', t) \) equals the expected cross-GWD \( E \{ W^{(\alpha)}_{x,y}(t, f) \} \) of the processes \( x(t) \) and \( y(t) \).

If the processes \( x(t) \) and \( y(t) \) are wide-sense stationary, then the expected cross-GWD reduces to the conventional cross-power spectrum \( S_{x,y}(f) \).

Eq. (13) provides another way to interpret the GWS as a consistent generalization of the LTI system’s transfer function. Application of white noise \( n(t) \) (with correlation function \( r_n(t, t') = \delta(t-t') \)) as input signal yields the GWS \( L^{(\alpha)}_H(t, f) \) as expected cross-GWD of the output process \( (Hn)(t) \) and the input process \( n(t) \)

\[
L^{(\alpha)}_H(t, f) = E \{ W^{(\alpha)}_{H_{x,y}}(t, f) \}.
\]

For LTI systems, (14) reduces to the well-known Wiener filter relation,

\[
H(f) = S_{x,y}(f),
\]

where \( S_{x,y}(f) \) is the cross-power spectral density of the (white-noise) input process \( n(t) \) and the output process \( (Hn)(t) \).

Properties. The GWS \( L^{(\alpha)}_H(t, f) \) satisfies the following properties:

- Linearity. The GWS of a weighted, parallel combination \( aG + bH \) (with \( a, b \in \mathbb{C} \)) of two LTV systems \( G \) and \( H \) is given by the weighted sum of the GWSs of the two systems

\[
L^{(\alpha)}_{aG+bH}(t, f) = aL^{(\alpha)}_G(t, f) + bL^{(\alpha)}_H(t, f).
\]

- Unitarity. The GWS preserves the inner product of the kernels \( g(t, t') \) and \( h(t, t') \) of two systems \( G \) and \( H \)

\[
(g, h) = L^{(\alpha)}_G(L^{(\alpha)}_H).
\]

- LTI system. For an LTI system with impulse response \( h(t, t') = h(t-t') \) the GWS is consistent with the usual transfer function:

\[
L^{(\alpha)}_H(t, f) = H(f).
\]

- LFI system. In the dual case of a linear, frequency-invariant system with impulse response \( h(t, t') = \delta(t-t')m(t) \) the GWS yields the time-domain weighting characteristic:

\[
L^{(\alpha)}_H(t, f) = m(t).
\]

- Perfect Reconstruction. The GWS of the identity system is one. Consequently, parallel connection of two LTV systems \( G \) and \( H \) with complementary symbols yields perfect reconstruction,

\[
L^{(\alpha)}_G(t, f) + L^{(\alpha)}_H(t, f) = 1 \iff Gx + Hr = x.
\]

- Energetic TF spectral decomposition. The GWS of a system with \( h(t, t') \in L^2(\mathbb{R}^2) \) can be expanded into a singular-value weighted sum of cross-GWD’s

\[
L^{(\alpha)}_H(t, f) = \sum_{k=0}^{\infty} \sigma_k W^{(\alpha)}_{\sigma_k,\nu_k}(t, f),
\]

where the weights \( \sigma_k \geq 0 \) are the singular values, and the orthonormal bases \( \{ \sigma_k \} \) and \( \{ \nu_k \} \) are the left and right singular signals of the system.
function. As a unique system representation, the GWS allows to predict both weighting and displacement effects into a weighted sum of ideal TF shift systems behavior of an LTV system. It reduces to the conventional yields an infinitesimal decomposition of the LTV system provides a transparent specification of the TF shifting because of the 

The underlying TF shift operator is defined as

\[ \left( S_{(a),r,\nu} \right)(t, t') \triangleq \delta(t - t' - \tau)e^{j2\pi\nu(t - t')/a}. \]

Here, \( a \) reflects the freedom in the definition of a TF shift operator (different values of \( a \) can be obtained by appropriately splitting up and interchanging the time and frequency shifts). The GSF \( S_H^{(a)}(r, \nu) \) is in two–dimensional (symplectic–Fourier correspondence to the GWS \( L_H^{(a)}(t, f) \))

\[ S_H^{(a)}(r, \nu) = \int_c \int_d L_H^{(a)}(t, f)e^{-j2\pi(r(C-\tau-f))}df. \]

This duality of the GWS \( L_H^{(a)}(t, f) \) and the GSF \( S_H^{(a)}(r, \nu) \) is a manifestation of Heisenberg’s uncertainty principle in LTV system theory: Restriction of potential TF shifts (i.e. the GSF \( S_H^{(a)}(r, \nu) \) is zero outside of a small region around the \((r, \nu)\)–plane origin) requires that the GWS \( L_H^{(a)}(t, f) \) is essentially a two–dimensional lowpass function.

The Fourier transform duality (30) carries over to a duality in the properties of the GWS and the GSF.

### 5 SPECIFIC SYSTEMS

**STFT based system representation.** A well–known TF model for LTV systems is based on the short–time Fourier transform (STFT) [3]. The system model consists of (i) STFT–analysis of the input signal \( x(t) \) with analysis window \( g(t) \), (ii) multiplying the STFT by a TF weight function \( M(t, f) \), and (iii) synthesis of the output signal from the weighted STFT with the synthesis window \( \gamma(t) \), \( \tilde{X}(t, f) = \int_r x(t') \gamma(t' - t) e^{-j2\pi f't} dt' \),

\[ M(t, f) = \int_r \int_c S_{H(t, f)}^{(a)}(r, \nu) \mathcal{F}(x(t)) \mathcal{F}^{-1}(M(t, f)) d\nu dt. \]

The overall system \( M \) is significantly influenced by the choice of the analysis window \( \gamma(t) \) and the synthesis window \( \gamma(t) \). The influence of the windows becomes apparent in the system’s GWS, which can be shown to be the TF weight function \( M(t, f) \) smoothed by the GWD of the windows,

\[ L_M^{(a)}(t, f) = M(t, f) \mathcal{F}(x(t)) \mathcal{F}^{-1}(M(t, f)) \].
The GSF expression for an STFT-system,
\[ S_{st}^{(\alpha)}(\tau, \nu) = F_2(t,f)_{(\nu,\tau)} \{ M(t,f) \} d_{\nu,\tau}(\tau, \nu), \tag{35} \]
shows that, for usual analysis/synthesis windows and bounded TF weight function \( M(t,f) \), STFT-systems do not introduce considerable TF displacement.

**WSSUS-Systems.** The concept of wide-sense stationary uncorrelated scattering (WSSUS) is a statistical model for time-varying communication channels [10]. The WSSUS model is based on the assumption that Zadeh’s function \( Z_H(t, f) \) is a zero-mean, wide-sense stationary process,
\[ E \{ Z_H(t, f) \} = 0, \tag{36} \]
\[ E \{ Z_H(t, f)Z_H(t', f') \} = R_H(t - t', f - f'). \tag{37} \]
It can be shown that this definition of WSSUS is \( \sigma \)-invariant in the sense of the GWS-family:
\[ E \{ L_{st}^{(\sigma)}(t, f) \} = 0, \tag{38} \]
\[ E \{ L_{st}^{(\sigma)}(t, f)L_{st}^{(\sigma)*}(t', f') \} = R_H(t - t', f - f'). \tag{39} \]
The samples \( L_{st}^{(\sigma)}(t, f) \), however, are generally \( \sigma \)-dependent. The TF correlation function \( R_H(t, f) \) is in Fourier correspondence to the WSSUS system’s scattering function \( C_H(\tau, \nu) \), \( R_H(t, f) = F_2(\nu,\tau)(-\tau,-f)C_H(\tau, \nu). \)

Systems with constrained spreading. Practically important LTV systems satisfy a spreading constraint that we describe by a \( \sigma \)-valued indicator function \( I(\tau, \nu) \),
\[ S_{st}^{(\sigma)}(\tau, \nu) = S_H^{(\sigma)}(\tau, \nu)I(\tau, \nu). \tag{40} \]
Such a spreading constraint appears in different situations:
(i) As an a priori knowledge, e.g., in WSSUS system identification, where \( I(\tau, \nu) \) is determined by the support of the channel’s scattering function, \( C_H(\tau, \nu) = C_H(\tau, \nu)I(\tau, \nu) \).
(ii) As an a priori requirement in TF filter design, where \( I(\tau, \nu) \) is an inherent demand to avoid TF displacement of the input signal’s components.

It must be stressed that in any case the indicator function \( I(\tau, \nu) \) will be \( \sigma \)-invariant, since the GSF’s magnitude is \( \sigma \)-invariant. Eq. (40) carries over to the TF weighting domain,
\[ L_{st}^{(\sigma)}(t, f) = L_{st}^{(\sigma)}(t, f) \ast \ast P(t, f) \tag{41} \]
where \( P(t, f) \) is the 2D Fourier transform of the indicator function, \( P(t, f) = F_2(\nu,\tau)(-\tau,-f)I(\tau, \nu). \) In view of condition (40), the GWS \( L_{st}^{(\sigma)}(t, f) \) is highly redundant. This redundancy can be removed by TF sampling of the GWS. The choice of a rectangular sampling grid yields the following representation of the GWS (the summations go from \(-\infty \) to \( \infty \))
\[ L_{st}^{(\sigma)}(t, f) = \sum \Pi_{n,l} \delta_{nT,f}(nT, f)P(t - nT, f - IF), \tag{42} \]
with \( P(t, f) = TFP(t, f) \). Eq. (42) is valid under the condition that \( I(\tau, \nu) \) is zero outside of a rectangle with dimensions \( 1/F \times 1/T \). Note that the constants \( T \) and \( F \) of the critical sampling grid are \( \sigma \)-invariant. TF spectral decomposition of the reconstruction kernel \( P(t, f) \) (according to (21)) yields the following expression for the GWS:
\[ L_{st}^{(\sigma)}(t, f) = \sum \Pi_{n,l} \delta_{nT,f}(nT, f)\Pi_{u,k}^{(\sigma)}(t - nT, f - IF). \tag{43} \]

Comparison of (43) with (34) shows that the system can be realized as a singular-value weighted parallel combination of STFT filters with the identical TF multiplicator function
\[ M(t, f) = \sum \Pi_{n,l} \delta_{nT,f}(nT, f)\delta(t - nT)\delta(f - IF), \tag{44} \]
and with \( u_k(t)/v_k(t) \) as synthesis/analysis windows. The conceptual advantage of this model depends on the convergence of the spectral decomposition of the TF reconstruction kernel \( P(t, f) \). If the singular values \( \alpha_k \) converge rapidly to zero, a small number of STFT-systems provides a sufficient approximation.

**5 CONCLUSION**

It has been shown that the freedom in the definition of a time-varying signal spectrum carries over to a corresponding freedom in the definition of a time-varying transfer function of an LTV system. The orthogonality principle either in its deterministic variant, Rayleigh quotient theory, or in its stochastic variant, Wiener filter theory, constitutes the formal link between linear system representations and quadratic signal representations. TF formulation of the orthogonality principle in terms of the generalized Wigner distribution led us to the formulation of the generalized Weyl correspondence. The generalized Weyl symbol (GWS) represents an LTV system in the sense of TF weighting characteristic, while the generalized spreading function (GSF) specifies the TF shifting behavior of an LTV system. For the practically important class of LTV systems with constrained TF displacement, the Fourier duality between the GSF and GWS suggests a system theoretic version of Gabor’s classical principle of TF discretization.

**REFERENCES**