FAST LOCAL RECONSTRUCTION METHODS FOR NONUNIFORM SAMPLING IN SHIFT INVARIANT SPACES

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Abstract. We present a new method for the fast reconstruction of a function f from its samples $f(x_j)$ under the assumption that f belongs to a shift invariant space $V(\varphi)$. If the generator φ has compact support, then the reconstruction is local quite in contrast to methods based on band-limited functions. Using frame theoretic arguments, we show that the matrix of the corresponding linear system of equations is a positive-definite banded matrix. This special structure makes possible the fast local reconstruction algorithm in $O(S^2J)$ operations, where J is the number of samples and S is the support length of the generator φ. Further optimization can be achieved by means of data segmentation. Ample numerical simulation is provided.

Key words. Shift-invariant space, nonuniform sampling, banded matrix, localization, data segmentation, denoising.

AMS subject classifications. 41A15, 42C15, 46A35, 46E15, 46N99, 47B37

1. Introduction. Shift-invariant spaces serve as a universal model for uniform and nonuniform sampling of functions. The objective of the so-called sampling problem is either to recover a signal (function) f from its samples $\{f(x_j) : j \in \mathbb{Z}\}$ or to approximate a data set (x_j, y_j) by a suitable function f satisfying $f(x_j) \approx y_j$. Obviously this problem is ill-posed, and so a successful reconstruction requires some a priori information about the signal. Usually it is assumed that f is contained in the span of integer translates of a given generator φ. In technical terms, the original function f has the form $f(x) = \sum_{k\in\mathbb{Z}} c_k\varphi(x - k)$ and belongs to the shift-invariant space $V(\varphi)$.

Until recently the only choice for φ was the cardinal sine function $\varphi(x) = \frac{\sin \pi\alpha x}{\pi\alpha x}$, since in this case $V(\varphi)$ coincides with the band-limited functions of band-width 2α. Despite the existence of fast numerical methods [9], this model has some drawbacks because it is non-local, and the behavior of f at a point x also depends on samples far away from x. For this reason, one works with truncated versions of the cardinal sine. This idea leads naturally to work in shift-invariant spaces with a generator φ of compact support.

The concept of shift-invariant spaces first arose in approximation theory and wavelet theory [5, 6,15]. Its potential for the systematic treatment of sampling problems was recognized much later. We refer to [1] for a detailed survey of the state-of-art and an extensive list of references.

Our goal is the investigation of specific numerical issues of nonuniform sampling in shift-invariant spaces. Usually the reconstruction algorithms are based on general frame methods or they follow simple iterative schemes. Of course, these can be applied successfully in the context of shift-invariant spaces as well, see Sections 6 and 7 of [1]. We adopt the philosophy that the optimal numerical algorithms always have to use the special structure of the problem. Hence the general purpose algorithms have to be fine-tuned to achieve their optimal performance. Here we use the peculiar structure of shift-invariant spaces with compactly supported generator to solve the sampling problem.

We want to reconstruct a function $f \in V(\varphi)$ from a *finite* number of samples $f(x_j)$ taken from an interval $x_j \in [M_0, M_1]$. In our derivation of the special structure we use a frame theoretic argument and combine it with the fact that the generator φ has compact support. The resulting algorithm is local in the sense that the complete reconstruction of a function in $V(\varphi)$ on the interval $[M_0, M_1]$ only requires samples from $[M_0, M_1]$ (quite in contrast to bandlimited functions).

*Both authors acknowledge partial support by the Austrian Science Fund project FWF P-14485.

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By comparing reconstructions in different spline spaces we find that the algorithm can be used as an efficient denoising procedure for noisy samples.

We will assume that the reader is familiar with the short section on frames in [8] or in [5] and will not define explicitly these standard concepts.

The paper is organized as follows: In Section 2 we present the precise technical details of shift-invariant spaces, state the well-known equivalence of sampling problems with the construction of certain frames, and discuss some general reconstruction techniques associated to frames. In Section 3 we exploit the special structure of shift-invariant spaces to derive a local reconstruction algorithm of order $O(J)$ and discuss the numerical issues involved. Section 4 explains the results of the numerical simulations and provides the pseudocode of the main algorithm.

Acknowledgement: We would like to thank Thomas Strohmer, Univ. of California at Davis, for his valuable comments and several references.

2. Shift-Invariant Spaces and Sampling: General Theory.

2.1. Shift-Invariant Spaces, Frames, and Sampling. Let φ be a continuous function with compact support of size S so that

$$\operatorname{supp} \varphi \subseteq [-S, S]\,. \tag{2.1}$$

For convenience we assume that S is a positive integer and thus $\varphi(\pm S) = 0$. Then the shift-invariant space $V(\varphi)$ is defined as

$$V(\varphi) = \{f \in L^2(\mathbb{R}) : f(x) = \sum_{k\in\mathbb{Z}} c_k \varphi(x-k) \text{ for } (c_k) \in \ell^2(\mathbb{Z})\}\,. \tag{2.2}$$

To guarantee the stability of these representations, we assume that the generator φ is stable, which means that there exists a constant $C > 0$ such that

$$C^{-1}\,\|c\|_{\ell^2} \le \Big\| \sum_{k\in\mathbb{Z}} c_k \varphi(\cdot - k)\Big\|_2 \le C\,\|c\|_{\ell^2} \tag{2.3}$$

for all finite sequences $c = (c_k)_{k\in\mathbb{Z}}$, or equivalently, the translates $\phi(.-k), k \in \mathbb{Z}$, form a Riesz basis for $V(\varphi)$. As a consequence, $V(\varphi)$ is a closed subspace of $L^2(\mathbb{R})$ and inherits the inner product $\langle .\,,.\rangle$ of $L^2(\mathbb{R})$.

The sampling problem in $V(\varphi)$ is the following problem: Given a set of sampling points $X = \{x_j : j \in \mathbb{Z}\}$ arranged in increasing order $x_j < x_{j+1}$ and a sequence of samples $\{f(x_j) : j \in \mathbb{Z}\}$, of a function $f \in V(\varphi)$, we would like to recover the original function f in a stable and numerically efficient way. Here stability means that there exist (possibly unspecified) constants $A, B > 0$ such that

$$A\|f\|_2 \le \Big(\sum_{j\in\mathbb{Z}} |f(x_j)|^2\Big)^{1/2} \le B\|f\|_2 \qquad \forall f \in V(\varphi)\,. \tag{2.4}$$

A sampling set satisfying (2.4) is called a *set of stable sampling.*

Obviously, for (2.4) to be valid, we need point evaluations in $V(\varphi)$ to be well-defined. This is guaranteed by the following lemma [1].

Lemma 2.1. *If φ is continuous and satisfies the condition $\sum_{k\in\mathbb{Z}} \max_{x\in[0,1]} |f(x+k)| < \infty$, in particular, if φ is continuous with compact support, then for all $x \in \mathbb{R}$ there exists a function $K_x \in V(\varphi)$ such that $f(x) = \langle f, K_x\rangle$ for $f \in V(\varphi)$. We say that $V(\varphi)$ is a reproducing kernel Hilbert space.*

Explicit formulas for K_x are known, see [1], but we do not need them here. Note that with the kernels K_x the sampling inequality (2.4) can be formulated as $A\|f\|_2 \le (\sum_{j\in\mathbb{Z}} |\langle f, K_{x_j}\rangle|^2)^{1/2} \le B\|f\|_2$, which is equivalent to saying that the set $\{K_{x_j} : j \in \mathbb{Z}\}$ is a frame for $V(\varphi)$.

Let U be the infinite matrix with entries

$$U_{jk} = \varphi(x_j - k) \qquad j,k \in \mathbb{Z}\,. \tag{2.5}$$

Then the sampling problem in $V(\varphi)$ can be formulated in several distinct ways [2, Prop. 1.3].

LEMMA 2.2. *If φ satisfies the condition of Lemma 2.1, then the following are equivalent:*
(i) $X = \{x_j : j \in \mathbb{Z}\}$ is a set of sampling for $V(\varphi)$.
(ii) There exist $A, B > 0$ such that

$$A\|c\|_{\ell^2} \le \|Uc\|_{\ell^2} \le B\|c\|_{\ell^2} \qquad \forall c \in \ell^2(\mathbb{Z})\,.$$

(iii) The set of reproducing kernels $\{K_{x_j} : j \in \mathbb{Z}\}$ is a frame for $V(\varphi)$.

REMARK. It is difficult to characterize sets of sampling for $V(\varphi)$. If φ is a B-spline of order N, i.e., $\varphi = \chi_{[0,1]} * \cdots * \chi_{[0,1]}$ ($N+1$ convolutions), then the main result of [2] implies that the maximum gap condition $\sup_{j\in\mathbb{Z}}(x_{j+1} - x_j) = \delta < 1$ is sufficient for the conditions of Lemma 2.2 to hold.

2.2. General Purpose Reconstructions. Lemma 2.2 leads to some general reconstruction techniques that are always applicable.

1. **Linear Algebra Solution.** One could simply try to solve the (infinite) system of linear equations

$$\sum_{k\in\mathbb{Z}} c_k \varphi(x_j - k) = f(x_j) \qquad \forall j \in \mathbb{Z}\,, \tag{2.6}$$

for the coefficients (c_k), or in the notation of (2.5) with $f|_X = (f(x_j))_{j\in\mathbb{Z}}$

$$Uc = f|_X\,. \tag{2.7}$$

2. **The Normal Equations.** Frequently it is better to consider the associated system of normal equations [10]

$$U^*U\, c = U^* f|_X\,. \tag{2.8}$$

This approach has the advantage that the matrix $T := U^*U$ is a positive operator on $\ell^2(\mathbb{Z})$. Furthermore, if the input $y = (y_j)_{j\in\mathbb{Z}}$ does not consist of a sequence of exact samples of $f \in V(\varphi)$, then the function $f = \sum_{k\in\mathbb{Z}} c_k \varphi(\cdot - k)$ corresponding to the solution $c = (U^*U)^{-1}U^*y$ solves the least squares problem

$$\sum_{j\in\mathbb{Z}} |y_j - f(x_j)|^2 = \min_{h\in V(\varphi)} \sum_{j\in\mathbb{Z}} |y_j - h(x_j)|^2\,. \tag{2.9}$$

3. **Frame Approach.** Lemma 2.2(iii) suggests to use versions of the frame algorithm to find a reconstruction of f. By (iii) the frame operator which is defined as

$$Sf(x) = \sum_{j\in\mathbb{Z}} \langle f, K_{x_j}\rangle K_{x_j}(x) = \sum_{j\in\mathbb{Z}} f(x_j) K_{x_j}(x) \tag{2.10}$$

is invertible and its inverse defines the dual frame $\widetilde{K_{x_j}} = S^{-1}K_{x_j}, j \in \mathbb{Z}$. Then the reconstruction is given by

$$f(x) = \sum_{j\in J} \langle f, K_{x_j}\rangle \widetilde{K_{x_j}} = \sum_{j\in J} f(x_j) \widetilde{K}_{x_j}(x). \tag{2.11}$$

We observe that the linear algebra solution (2.7) and the frame method are equivalent. By definition the vector of samples is given by $Uc = f|_X$. The sampled energy of $f \in V(\varphi)$ is

$$\sum_{j\in\mathbb{Z}} |f(x_j)|^2 = \langle f|_X, f|_X\rangle_{\ell^2} = \langle Uc, Uc\rangle_{\ell^2} = \langle U^*Uc, c\rangle_{\ell^2} \,. \tag{2.12}$$

Thus X is a set of sampling if and only if U^*U is invertible on $\ell^2(\mathbb{Z})$.

4. **Iterative Frame Methods.** In nonuniform sampling problems it is usually difficult to calculate the entire dual frame, therefore one often resorts to iterative methods. Since the Richardson-Landweber iteration in the original paper of Duffin-Schaeffer [8] is slow and requires good estimates of the frame bounds, we recommend the conjugate gradient acceleration of the frame algorithm for all problems without additional structure [11]. It converges optimally and does not require the estimate of auxiliary parameters.

3. Exploiting the Structure of the Problem. So far we have discussed general purpose methods for the reconstruction of the function. These could be applied in any situation involving frames and do not take into consideration the particular structure of the sampling problem in shift-invariant spaces.

3.1. A Localization Property. We now exploit the special structure of shift-invariant spaces. The following lemma is simple, but crucial. It is a consequence of the assumption that the generator of $V(\varphi)$ has compact support.

LEMMA 3.1. *If supp $\varphi \subseteq [-S, S]$, then $T = U^*U$ is a band matrix of (upper and lower) band-width $2S$.*

Proof. By definition the entries of U^*U are

$$(U^*U)_{kl} = \sum_{j\in\mathbb{Z}} \overline{U_{jk}} U_{jl} = \sum_{j\in\mathbb{Z}} \overline{\varphi(x_j - k)}\ \varphi(x_j - l) \,.$$

Since φ has compact support, the sum is always locally finite and its convergence does not pose any problem. Since $\varphi(x_j - k) = 0$ if $|x_j - k| \geq S$, we find that $(U^*U)_{kl}$ can be non-zero only if both $|x_j - k| < S$ and $|x_j - l| < S$. In other words, $(U^*U)_{kl} \neq 0$ implies that

$$|k - l| \leq |k - x_j| + |x_j - l| < 2S \,.$$

This means that only $4S - 1$ diagonals of U^*U contain non-zero entries. □

REMARKS. 1. Banded matrices and the resulting numerical advantages occur in a number of related problems. For instance, in the interpolation of scattered data by radial functions with compact support, the interpolation matrix is banded, see [3] and the references there. Likewise, the calculation of the optimal smoothing spline on a finite set of arbitrary nodes requires the inversion of a banded matrix [12].

2. Lemma 3.1 combined with a result of Demko, Moss, and Smith [7] or of Jaffard [13] implies that the inverse matrix possesses exponential decay off the diagonal, i.e., there exist $C, A > 0$ such that

$$|(U^*U)^{-1}_{kl}| \leq Ce^{-A|k-l|} \qquad \forall k, l \in \mathbb{Z} \,.$$

To make our treatment more realistic, we take into account that in any real problem only a finite (albeit large) number of samples is given. It turns out that the model of shift-invariant spaces with compactly supported generator possesses excellent localization properties. These are quantified in the next lemma.

LEMMA 3.2. *The restriction of $f \in V(\varphi)$ to the interval $[M_0, M_1]$ is determined completely by the coefficients c_k for $k \in (M_0 - S, M_1 + S) \cap \mathbb{Z}$.*

Proof. Since $\varphi(x-k)=0$ for $|x-k|\geq S$ and $x\in[M_0,M_1]$, we obtain that

$$M_0-S\leq x-S<k<x+S\leq M_1+S\,.$$

Consequently, as $S\in\mathbb{N}$, we have

$$\begin{aligned}f(x)&=\sum_{k\in\mathbb{Z}}c_k\varphi(x-k)=\sum_{|x-k|<S}c_k\varphi(x-k)\\&=\sum_{k=M_0-S+1}^{M_1+S-1}c_k\varphi(x-k)\,.\end{aligned}$$

□

In other words, the exact reconstruction of $f\in V(\varphi)$ on $[M_0,M_1]$ requires only the M_1-M_0+2S-1 unknown coefficients c_k with $k\in(M_0-S,M_1+S)\cap\mathbb{Z}$. By counting dimensions, we find that we need at least M_1-M_0+2S-1 samples in $[M_0,M_1]$ for the coefficients to be determined uniquely. Usually the length M_1-M_0 is large compared to S, therefore the additional $2S-1$ coefficients amount to a negligible oversampling.

Lemma 3.2 demonstrates an important theoretical and practical advantage of shift-invariant spaces with compactly supported generators. *A function $f\in V(\varphi)$ can be reconstructed exactly on an arbitrary interval solely from samples in that interval.* In contrast, the restriction of a band-limited function to an interval is *not* uniquely determined by any finite number of samples in that interval, but can only be approximated by these samples. The localization property expressed in Lemma 3.2 is one of the main reasons to work with shift-invariant spaces with compactly supported generators as a sampling model!

Finally we remark that uniform sampling at critical density is not local and may even be unstable in this model. If $f\in V(\varphi)$ is sampled at $\xi+k, k\in\mathbb{Z}$, for some $\xi\in[0,1)$, then there exists an interpolating function ψ_ξ of exponential decay, such that $f(x)=\sum_{k\in\mathbb{Z}}f(\xi+k)\psi_\xi(x-k)$ [14]. In this case the restriction of f to $[M_0,M_1]$ is not determined exclusively by the M_1-M_0 values $f(\xi+k)$ for $\xi+k\in[M_0,M_1]$. Moreover, if φ is continuous, then there always exists a $\xi\in[0,1)$ such that the reconstruction $\{f(\xi+k)\}\to f$ is unstable. Janssen's results in [14] indicate that a small amount of oversampling is an essential hypothesis to guarantee the locality and the stability of its reconstruction.

3.2. A Local Reconstruction Algorithm. In practice we perform the calculations with a truncated version of the matrices U and T. We now combine Lemmas 3.1 and 3.2 to a first version of an efficient numerical reconstruction algorithm.

ALGORITHM

Input. We assume that finitely many sampling points $x_1,\dots x_J\in[M_0,M_1]$ are given with associated sampling vector $y=(y_1,\dots,y_J)\in\mathbb{R}^J$. Assume that $J\geq M_1-M_0+2S-1$ and that the truncated matrix $\mathscr{T}$ defined below is invertible.

Step 0. First we define and compute the truncated matrices $\mathscr{U}=\mathscr{U}^{M_0,M_1}$ and $\mathscr{T}=\mathscr{T}^{M_0,M_1}=\mathscr{U}^*\mathscr{U}$, given by their entries

$$\begin{aligned}\mathscr{U}_{jk}&=\varphi(x_j-k)\\\mathscr{T}_{kl}&=\sum_{j=1}^{J}\overline{\varphi(x_j-k)}\varphi(x_j-l)\end{aligned}\tag{3.1}$$

for $j=1,\dots,J$ and $k,l=M_0-S+1,\dots,M_1+S-1$.

Step 1. Compute $b = \mathscr{U}^* y$, i.e.,

$$b_k = \sum_{j=1}^{J} \overline{\varphi(x_j - k)}\, y_j \qquad \text{for } k = M_0 - S + 1, \dots, M_1 + S - 1\,. \tag{3.2}$$

Step 2. Solve the system of equations

$$c = \mathscr{T}^{-1} b\,. \tag{3.3}$$

Step 3. Compute the restriction of f to $[M_0, M_1]$ by

$$f(x) = \sum_{k=M_0-S+1}^{M_1+S-1} c_k \varphi(x-k) \qquad \text{for } x \in [M_0, M_1]\,. \tag{3.4}$$

Then f is the (unique) least square approximation of the given data vector y in the sense that

$$\sum_{j=1}^{J} |y_j - f(x_j)|^2 = \min_{h \in V(\varphi)} \sum_{j=1}^{J} |y_j - h(x_j)|^2\,. \tag{3.5}$$

If y arises as the sampled vector of an $f \in V(\varphi)$, i.e., $y_j = f(x_j)$, then this algorithm provides the exact reconstruction of f.

Proof. The least square property (3.5) is clear, since this is exactly the property of the solution of the system of normal equations $\mathscr{U}^*\mathscr{U} c = \mathscr{U}^* y$. See [10] for details. □

In the case of B-splines a sufficient condition on the sampling density can be extracted from the proofs of Thm. 2.1 and 2.2 of [2]. *Assume that* $x_{j+1} - x_j \le \delta$ *and that*

$$\delta \le \frac{M_1 - M_0}{M_1 - M_0 + 2S - 1} < 1\,. \tag{3.6}$$

Then $\mathscr{T}$ *is invertible.* Condition (3.6) guarantees that there are at least $M_1 - M_0 + 2S - 1$ samples in $[M_0, M_1]$. Then the Schoenberg-Whitney theorem [16, p. 167] implies that $\mathscr{T}$ is invertible. See [2] for the detailed arguments.

3.3. Data Segmentation. A further optimization of the reconstruction procedure is possible by data segmentation. Instead of solving the large system of equations

$$\mathscr{T}^{M_0,M_1} = (\mathscr{U}^{M_0,M_1})^* y\,,$$

with a band matrix of dimension $M_1 - M_0 + 2S - 1$, we will solve t systems of smaller size. For this purpose we partition the large interval $[M_0, M_1]$ into t smaller intervals $[m_r, m_{r+1}]$, $r = 0, \dots, t-1$ with $M_0 = m_0$ and $M_1 = m_t$.

Now we apply Algorithm 3.2 to each interval separately. More precisely, given the data (x_j, y_j) where $x_j \in [m_r, m_{r+1}]$, we set up the matrices $\mathscr{U}^{m_r,m_{r+1}}$ and $\mathscr{T}^{m_r,m_{r+1}}$ and solve t equations

$$\mathscr{T}^{m_r,m_{r+1}} c^{(r)} = (\mathscr{U}^{m_r,m_{r+1}})^* y^{(r)} \tag{3.7}$$

where the vector $y^{(r)}$ consists of those data y_j for which $x_j \in [m_r, m_{r+1}]$ and the coefficient vector $c^{(r)} = (c_{m_r-S+1}, \dots, c_{m_{r+1}+S-1})$.

The segmentation technique has a number of practical advantages.

1. The dimension of vectors and matrices can be reduced drastically. Using data segmentation, we solve t small systems of size $(M_1 - M_0)/t + 2S - 1$ instead of the large system of size $M_1 - M_0 + 2S - 1$.

2. Parallel processing can be applied because non-adjacent intervals can be handled simultaneously.

3. The function can be reconstructed on specified subintervals at smaller cost. See Figure 6.

On the other hand, data segmentation also comes with some caveats:

1. The coefficients c_k with indices $k \in [m_r - S + 1, m_{r+1} + S - 1]$ are computed at least twice because of overlap. Heuristically it has proved best to take averages of the multiply computed coefficients.

2. For a successful execution of the segmentation method it is necessary that each of the small matrices $\mathscr{T}^{m_r, m_{r+1}}$ is invertible. Again by dimension counts we find that the number of data in the interval $[m_r, m_{r+1}]$ should exceed the number of variables, i.e.,

$$\#\Big(X \cap [m_r, m_{r+1}]\Big) \geq m_{r+1} - m_r + 2S - 1\,.$$

Obviously this condition imposes an upper bound for the possible number of segmentations.

3.4. Implementation Issues. 1. In Algorithm 3.2 the most expensive step is the calculation of the matrix $\mathscr{U}$ because it requires the point evaluations of φ. However, if the sampling points x_j are given, then $\mathscr{U}$ and $\mathscr{T}$ can be computed in advance and stored. Thus Step 0 can be taken care of before solving the reconstruction problem.

We handle the pointwise evaluation of φ by "quantizing" the generator. This means that for $\delta > 0$ sufficiently small we create of vector ψ consisting of entries $\varphi(\frac{l}{N})$ for $l = -NS, \dots, NS$ such that

$$|\varphi(x) - \varphi(\frac{l}{N})| < \delta \qquad \text{for } |x - \frac{l}{N}| < \frac{1}{2N}\,.$$

Thus to build the matrix $\mathscr{U}_{jk} = \varphi(x_j - k)$ amount to selecting the appropriate entries of ψ. This approximation of $\mathscr{U}$ works remarkably well and fast in the numerical simulations.

2. For the solution of the banded system (3.3) a number of fast algorithms is available. Golub-van Loan [10, Ch. 4.3] offer several efficient algorithms for this task; other options for the inversion of a banded matrix are mentioned in [12]. Since $\mathscr{T}$ is assumed to be positive definite, the band Cholesky algorithm seems to be a good choice that minimizes the operation count for Step 2. MATLAB provides the commands SPARSE and CHOL to deal with this task.

3. Usually f is reconstructed on a grid $G = \{\frac{l}{N} : l = M_0N, \dots, M_1N\}$. Then (3.4) amounts to a discrete convolution, and thus Step 3 can be performed quickly. Again, since φ has compact support, we can use the banded structure of the associated matrix to perform this step.

3.5. Operation Count. We estimate the number of multiplications for Algorithm 3.2. Recall that J is the number of samples, and $D = M_1 - M_0 + 2S - 1$ is the dimension of the problem.

(a) According to (3.2) each of the D entries of the vector b requires $\#\{j : |x_j - k| < S\}$ multiplications. Consequently Step 1 requires

$$\begin{aligned}
\sum_{k=M_0-S+1}^{M_1+S-1} \#\{j : |x_j - k| < S\} &= \sum_{k=M_0-S+1}^{M_1+S-1} \sum_{j=1}^{J} \chi_{(k-S,k+S)}(x_j) \\
&= \sum_{j=1}^{J} \sum_{k=M_0-S+1}^{M_1+S-1} \chi_{(k-S,k+S)}(x_j) \\
&\leq \sum_{j=1}^{J} 2S = 2SJ
\end{aligned}$$

operations, because a point x is in at most $2S$ translates of the open interval $(-S, S)$.

(b) Likewise to calculate an entry of $\mathscr{T}$ requires $\#\big(\{j : |x_j - k| < S\} \cap \{j : |x_j - l| < S\}\big)$ multiplications, see (3.1). As in (a) we estimate the number of operations to set up the matrix $\mathscr{T}$ by

$$\begin{aligned}
&\sum_{k=M_0-S+1}^{M_1+S-1} \sum_{l=M_0-S+1}^{M_1+S-1} \#\Big(\{j : |x_j - k| < S\} \cap \{j : |x_j - l| < S\}\Big) \\
&= \sum_{k=M_0-S+1}^{M_1+S-1} \sum_{l=M_0-S+1}^{M_1+S-1} \sum_{j=1}^{J} \chi_{(k-S,k+S)}(x_j)\chi_{(l-S,l+S)}(x_j) \\
&= \sum_{j=1}^{J} \Big(\sum_{k=M_0-S+1}^{M_1+S-1} \chi_{(k-S,k+S)}(x_j)\Big) \Big(\sum_{l=M_0-S+1}^{M_1+S-1} \chi_{(l-S,l+S)}(x_j)\Big) \\
&\leq J \cdot (2S)^2 .
\end{aligned}$$

(c) For the solution of the banded system $\mathscr{T} c = b$ by means of the band Cholesky algorithm we need at most

$$D\big((2S)^2 + 16S + 1\big) = (M_1 - M_0 + 2S - 1)\big((2S)^2 + 16S + 1\big) \leq J(4S^2 + 16S + 1)$$

operations (and no square roots), see [10, Ch. 4.3.6].

(d) To compute the reconstruction f on a grid $\{\frac{l}{N} : l = M_0N, \dots, M_1N\}$ we need to calculate $(M_1 - M_0)N$ point evaluations of f via (3.4). Since $\#\{k \in \mathbb{Z} : |x - k| < S\} \leq 2S$, each point evaluation requires at most $2S$ multiplications. Thus for the reconstruction on the grid we need at most

$$(M_1 - M_0)N \cdot 2S \leq J \cdot 2SN$$

multiplications.

Combining these estimates, we find that *Algorithm 3.2 requires*

$$\mathscr{O}\big(J(S^2 + SN)\big) \tag{3.8}$$

operations. In other words, the cost of the algorithm is linear in the number of data and quadratic in the size of the generator!

4. Numerical Simulations. In our simulation we have used MATLAB. We used the shift-invariant spline spaces with the B-spline of order 3

$$\varphi = \underbrace{\chi_{[-1/2,1/2]} * \cdots * \chi_{[-1/2,1/2]}}_{\text{4 times}}$$

as the generator of $V(\varphi)$. Thus supp $\varphi \subseteq [-2, 2]$ and $S = 2$. See Figure 1.

Figure 2 is a plot of the operation count as a function of the number of sampling points. We have reconstructed examples of size $114, 226, 444, 667, 887, 1085$ and used the MATLAB function FLOPS to count the number of operations.

The example in Figure 3 uses a signal on the interval $[0, 128]$. Since $S = 2$, we need at least $M_1 - M_0 + 2S - 1 = 131$ samples. The actual sampling set of Figure 3 consists of ca. 200 points and satisfies the maximum gap condition $\max_j (x_{j+1} - x_j) \approx 0.67 < 1$.

To make the example more realistic, we have added white noise to the sampled values of a given function $f \in V(\varphi)$. Instead of using the correct values $f(x_j)$ in the reconstruction algorithm, we use the noisy values $f_{err}(x_j) = f(x_j) + e_j$ so that

$$f_{err}|_X = f|_X + e\,.$$

The relative error between the original signal and the noisy signal is measured by

$$err_{samp} = \frac{\|f_{err}|_X - f|_X\|_2}{\|f|_X\|_2} = \Big(\frac{\sum_{j=1}^J |e_j|^2}{\sum_{j=1}^J |f(x_j)|^2}\Big)^{1/2}.$$

In our example $err_{samp} = 63.8\%$.

Figure 3 shows the plots of the original signal (top left), of the noisy signal (top right), the plot of the noisy samples which looks rather chaotic (bottom left). The last plot (bottom right) displays the reconstruction (continuous line) means of Algorithm 3.2. For comparison we have added the original function as a dotted line. The relative error err_{rec} of the reconstruction measured at the sampling points with respect to the correct samples is now

$$err_{rec} = \frac{\|f_{rec}|_X - f|_X\|_2}{\|f|_X\|_2} = 18.5\%$$

The noise reduction is thus

$$err_{samp} = 63.8\% \rightarrow err_{rec} = 18.5\%$$

In Figure 4 we investigate the dependence of the reconstruction on the generator φ. In each subplot the generator is a B-spline of order N, i.e. $\varphi_N = \chi_{[-1/2,1/2]} * \cdots * \chi_{[-1/2,1/2]}$ ($N+1$-fold convolution). The data set $(x_j, y_j)_{j=1,\dots,J}$ is generated by sampling a function $f \in V(\varphi_5)$ and then we have added noise. The top left picture shows the original signal and the noisy sampled data. Then each subplot depicts the optimal approximation of these data in the spline space $V(\varphi_N), N = 0,\dots,6$, starting with an approximation by a step function $f_{rec} \in V(\varphi_0)$ via an approximation by a piecewise linear function $f_{rec} \in V(\varphi_1)$ and ending with a smooth approximation $f_{rec} \in V(\varphi_6)$.

In each case we have also plotted the orginal function f (dotted line) for comparison. In addition, the relative error err_{samp} is indicated. The dependency of this error of N is a typical L-curve as it occurs in regularization procedures. In all our examples the best approximation is obtained in the correct space in which f was originally generated. This observation is consistent with the extended literature on smoothing splines, e.g., [4, 12, 17]. The main difference between those methods and the algorithm of Section 3.2 is in the underlying function space. The reconstruction algorithm 3.2 finds the best local reconstruction in the shift-invariant spline space $V(\varphi)$, whereas the smoothing spline of [4] is based on the nodes x_j and does not belong to a fixed function space.

Figure 5 displays the associated banded matrix $\mathscr{T}$ of the linear system (3.3). White squares correspond to zero entries, dark squares signify large entries of $\mathscr{T}$, the shading being proportional to the size. The banded structure is clearly visible.

Figure 6 exhibits the power of the method of data segmentation. Instead of reconstructing the entire signal f, we have reconstructed only the restriction to two disjoint intervals. In the absence of noise the reconstruction is exact. Since $supp(\varphi) \subset [-S, S]$, the calculation for the two intervals can be done locally and simultaneously. This property can be used for parallel processing.

Appendix. Pseudo-Code.

```
function f_rec = reconstruction(xp, xs, x_rec, t);
% xp    ... sampling positions
% xs    ... sampling values
% x_rec ... positions, where the function should be reconstructed
% gen   ... generator with support supp(gen)=[-S,S]
% t     ... number of segmentation

step = round((max(x_rec)-min(x_rec))/t);
for k=min(x_rec):step:max(x_rec)
 xp_rel = {xp: k-S < xp < k+step+S} %relevant sampling positions
                                    %for the interval [k,k+step]
 xp_min = min(xp_rel);
 xp_max = max(xp_rel);
 J = length(xp_rel);                %number of sampling points

% calculation of the left side b:
 for j=1:J
   mi=ceil(xp(j))-S;
   ma=floor(xp(j))+S;
   for l = mi : ma
     b(l-mi+1) = b(l-mi+1) + xs(j)*gen(xp(j)-l);
   end
 end

% calculation of the matrix T:
 T=zeros(xp_max-xp_min+1+2*S,xp_max-xp_min+1+2*S);
 for j=1:J
   mi=ceil(xp(j))-S;
   ma=floor(xp(j))+S;
   for k=mi:ma
     for l=mi:ma
       T(k-mi+1,l-mi+1) = T(k-mi+1,l-mi+1) + gen(xp(j)-l)*gen(xp(j)-l);
     end
   end
 end

% calculation of the coefficients
  c_part = chol(T,b);       % solving the system T*c_part=b with a
                            %  banded Cholesky algorithm
  c(xp_min-S:xp_max+S) = c(xp_min-S:xp_max+S) + c_part;
  n(xp_min-S:xp_max+S) = n(xp_min-S:xp_max+S) + ones(xp_max-xp_min+1+2*S);
           %n ... normalization of coefficients because of overlapping
end

c = c ./ n;     %normalization of coefficients because of overlapping
% calculation of the reconstruction
for i = 1 : length(x_rec)
 for j = floor(x_rec(i)-S) : ceil(x_rec(i)+S)   % |x_rec-j| <= S
  f_rec(i) = f_rec(i) + gen(x_rec(i)-j) * c(j);
 end
end
```

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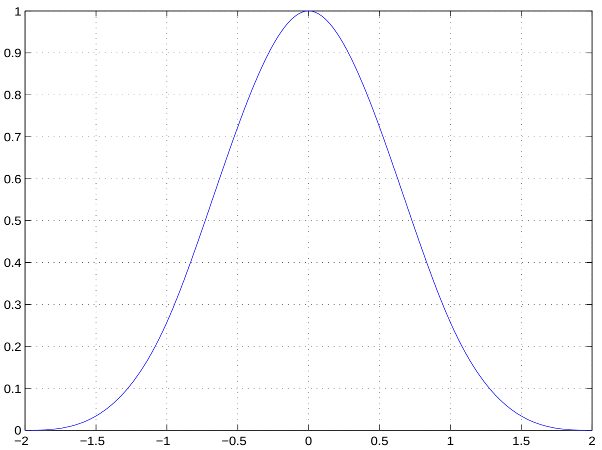

FIG. A.1. *Generator φ: a B-spline of order 3 with supp$(\varphi) \subseteq [-2, 2]$*

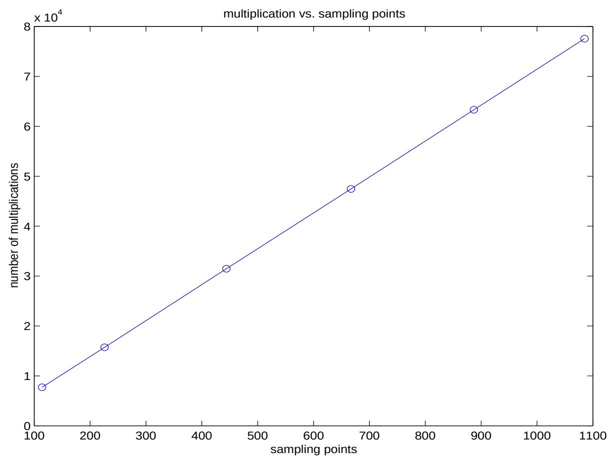

FIG. A.2. *Number of multiplications for reconstruction problems of different size. The operation count is* linear *in the number of samples.*

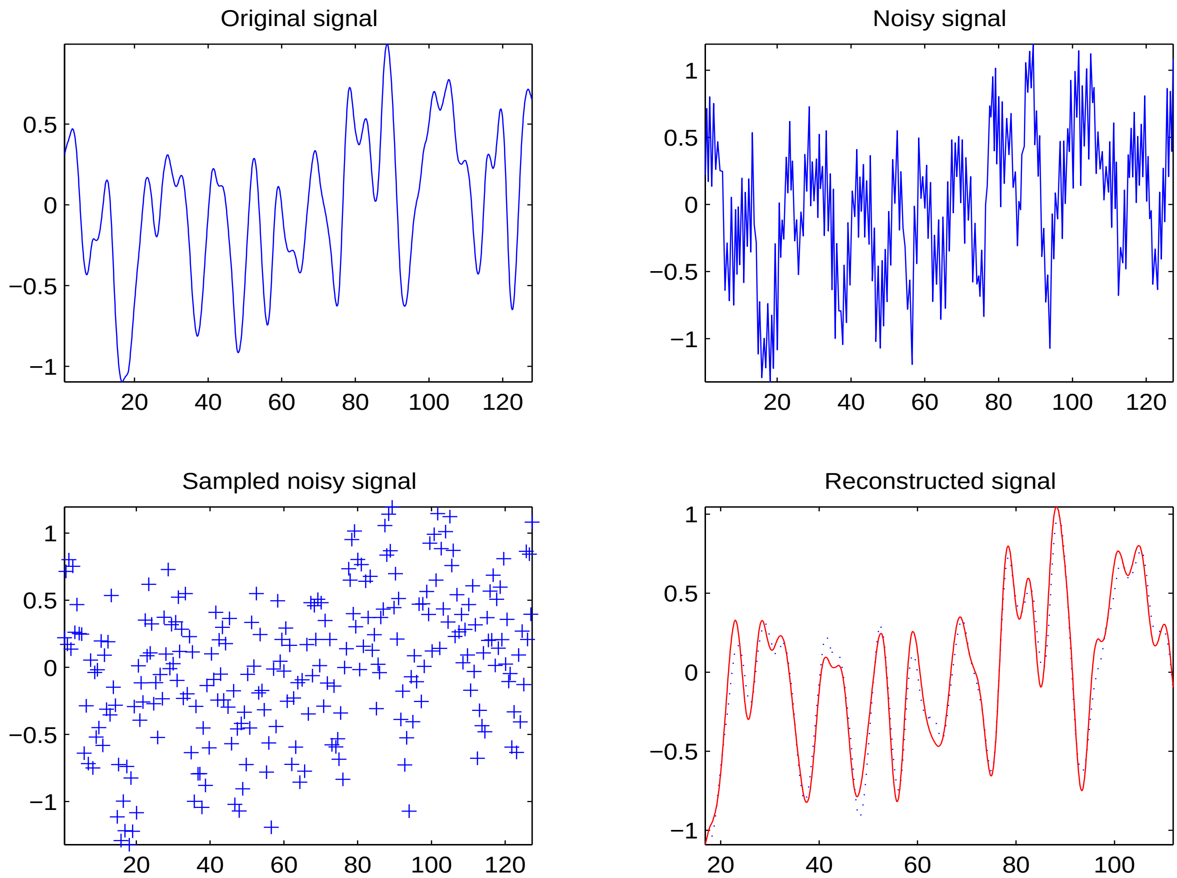

FIG. A.3. *Reconstruction with noisy nonuniform samples: The top right plot shows the signal with additive* $err_{samp} = 63.8\%$ *noise. Bottom left shows the noisy signal sampled on a nonuniform grid with maximal gap* ≈ 0.67. *Bottom right shows the reconstructed function (continuous line) and original function (dotted line).*

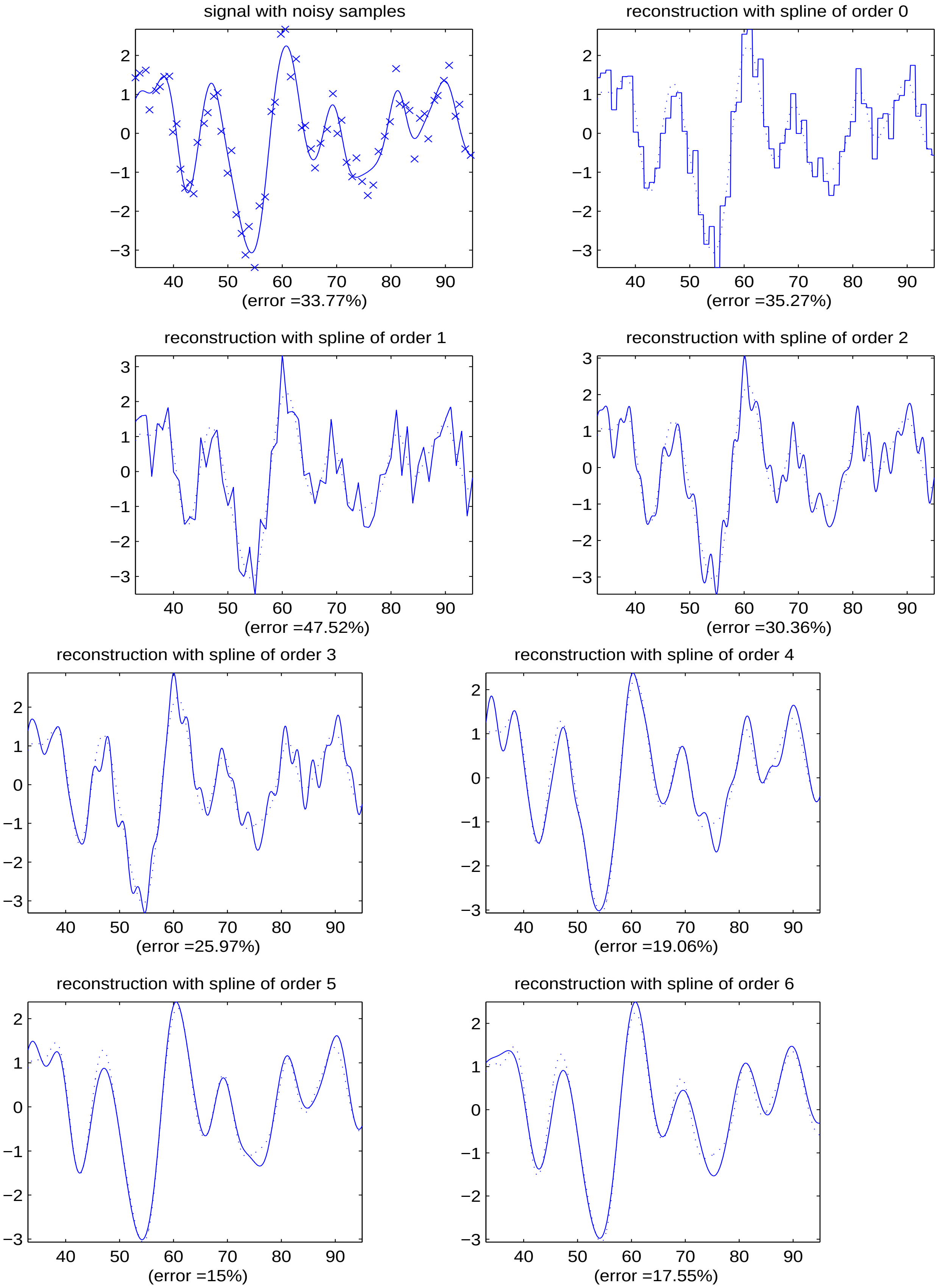

FIG. A.4. *Reconstruction of a signal from noisy samples in shift-invariant spaces with B-splines of different orders as generator. Top left: Original Signal (dotted line) and the noisy samples are marked* ($\times$)*. Other plots: Original Signal ... dotted line - Reconstruction ... solid line*

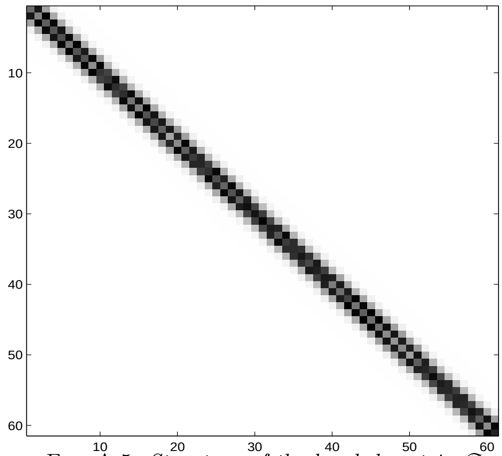

FIG. A.5. *Structure of the banded matrix* $\mathscr{T}$

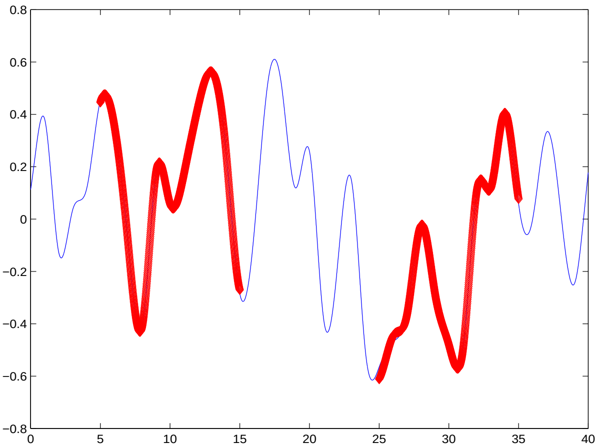

FIG. A.6. *Reconstruction of the function on disjoint intervals (without noise)*