IMPROVED LOCALITY FOR IRREGULAR SAMPLING ALGORITHMS

Hans G. Feichtinger and Tobias Werther [*]

NuHAG, Dept. of Mathematics
University of Vienna, Strudlhofg. 4
A-1090 Wien, Austria
fei@tyche.mat.univie.ac.at
werther@tyche.mat.univie.ac.at

ABSTRACT

This paper presents a method for the reconstruction of bandlimited functions from non-uniform samples by means of synthesis functions. We state explicit worst-case estimates which depend on the bandwidth and the sampling density. The proposed reconstruction method uses oversampling to deduce well-localized synthesis functions which improve the locality properties of the reconstruction. As a consequence, the resulting algorithm is suitable for local recovery of bandlimited signals from sufficiently dense local sampling information. The theoretical results are confirmed by numerical examples based on a finite-dimensional model described in the second part of this note.

1. INTRODUCTION

In the irregular sampling problem we ask under which conditions and how a bandlimited function f can be recovered from its values at irregularly distributed sampling points $\{t_n\}_{n\in\mathbb{Z}}$. In the case of equally spaced samples, the classical sampling theorem by Shannon, Whittaker and Kotel'nikov provides an explicit reconstruction. Given a bandlimited function $f \in L^2(\mathbb{R})$ with spectrum in $[-\omega_0, \omega_0]$, f can be represented as a cardinal series

$$f(t) = \sum_{n\in\mathbb{Z}} f(\frac{n}{2\omega_0}) sinc_{\omega_0}(t - \frac{n}{2\omega_0}),$$

where $sinc_{\omega_0}(t) = (\pi t)^{-1} sin(2\pi\omega_0 t)$ for $t \neq 0$ and $sinc_{\omega_0}(0) = 1$, cf. [3], [10]. The series converges uniformly and in the L^2-sense. In the case of oversampling, i.e., if sampling values $\{f(\nu n)\}_{n\in\mathbb{Z}}$ for some $\nu < 1/2\omega_0$ are known, the SINC-function can be replaced by sampling functions with better decay, cf. [2]. Such well-localized sampling functions improve the local behavior of the reconstruction in the sense that the value of a bandlimited function is essentially determined by the adjacent sampling values, and more distant sampling values have no influence.

The irregular sampling problem is more difficult. It has given rise to some deep theoretical but not constructive results, cf. [1]. On the other hand, several reconstruction algorithm have been developed [6], [8], [9]. Unfortunately, all of them lack locality since even for the case of regular sampling the poorly decaying SINC-function arises as building block.
In the late 80's and early 90's Feichtinger and Gröchenig developed a new approach using approximation operators in combination with convolution operators. One of the main results of their methods is that every bandlimited function f has a stable series expansion of the form

$$f = \sum_{n\in\mathbb{Z}} f(t_n)e_n,$$

where $\{t_n\}_{n\in\mathbb{Z}}$ is a sufficiently dense sampling set and the e_n are the corresponding *sampling atoms* of rapid decay [4],[5]. The decay is measured in form of weighted L^1-norms. The results hold true for a large class of bandlimited functions and are essentially based on norm-estimates of suitable operators. Because of the generality of the approach these estimates are not explicit but given as constants depending on the sampling rate and the bandwidth. It is the aim of this work to deduce explicit L^2-estimates and to establish an implementable reconstruction algorithm. The use of well-localized sampling atoms improves the locality properties of the proposed reconstruction method and provides well-convergent local solutions of the irregular sampling problems. For more details we refer to [11].

A different approach to localization in the irregular sampling problem is given in the recent paper [7].

---

[*] Partial support by FWF-project S-7001-MAT of the Austrian Science Foundation is gratefully acknowledged.

2. AN OPERATOR APPROACH

Most of the reconstruction methods make use of the orthogonal projection of approximations onto the starting space of bandlimited functions. High frequencies are discarded by applying the ideal low-pass filter, i.e., by using the convolution with the sinc-function. Under certain conditions on the sampling set we obtain approximation operators that are close to the identity operator on the Paley-Wiener space. Then, the inversion in sense of the Neumann series leads to a reconstruction method whose efficiency depends mainly on the sampling approximation. However, such reconstructions possess a bad localization due to the use of the sinc-function. Replacing the sinc-function by a rapidly decreasing reproducing function h which satisfies $f * h = f$ for all f of bandwidth $2\omega_0$, requires a different approach.

As usually, we present the approach for the continuous case and with respect to numerical experiments.

We use the following notation. Let $B^2_{\omega_0}(\mathbb{R})$ be the Paley-Wiener space of real-valued L^2-functions with $spec(f) \subseteq [-\omega_0, \omega_0]$. We denote by C_h the convolution operator of $h \in L^1(\mathbb{R})$ defined on L^2. The concept of the presented method is based on the definition of an approximation operator of C_h. Factorizing the convolution yields a continuous left inverse of this approximation operator restricted to $B^2_{\omega_0}(\mathbb{R})$ which is used to define the concentrated sampling atoms.

Given $g \in L^1_\alpha = \{f : f(1+|t|)^\alpha \in L^1(\mathbb{R})\}$ bandlimited we choose a bandlimited function $h \in L^1_\alpha$ with $\Omega_h = spec(h) \supseteq spec(g)$ and $\hat{h}(\omega) = 1$ on $spec(g)$. This entails the relation $C_g C_h = C_g$. Let A be a linear bounded operator such that $\|C_h - A\| < 1$ on $B^2_{\Omega_h}(\mathbb{R})$. Then $D = \sum_{k=0}^\infty (C_h - A)^k$ is well defined on $B^2_{\Omega_h}(\mathbb{R})$.

Lemma 1 [5] *Under the assumptions stated above, C_g factorizes as $C_g = DAC_g$ on all of L^2.*

Even though the proof of Lemma 1 is rather simple, it contains the basic idea of the presented algorithm.

3. METHOD OF ADAPTIVE WEIGHTS

We define a suitable approximation operator in the form of the sum of weighted translates of a reproducing function. For $\omega_1 > \omega_0 > 0$, we call $h_{\omega_1} \in L^1_\alpha$ a ω_0-reproducing function if $|\hat{h}_{\omega_1}(\omega)| \leq 1$, $\hat{h}_{\omega_1} = 1$ on $[-\omega_0, \omega_0]$, and $h_{\omega_1} \in B^2_{\omega_1}$.

Let $\{t_n\}_{n\in\mathbb{Z}}$ be a strictly increasing sampling set of density $\nu = \sup_n (t_{n+1} - t_n)$. Setting $w_n = (t_{n+1} - t_{n-1})/2$ for all $n \in \mathbb{Z}$, we define the adaptive weights operator A_w for all $f \in L^2$ by

$$A_w f(t) = \gamma \sum_{n\in\mathbb{Z}} f(t_n) w_n h_{\omega_1}(t - t_n),$$

where $\gamma = (1 + 4\nu^2\omega_1^2)^{-1}$. A_w is a bounded linear operator on L^2. Note that A_w depends on the sampling set and on the choice of the ω_0-reproducing function.

Lemma 2 *Assume that $2\nu\omega_1 < 1$. Then, the following holds true for all $f \in B^2_{\omega_1}(\mathbb{R})$.*

$$\|(C_{h_{\omega_1}} - A_w)f\|_2 \leq \frac{4\nu\omega_1}{1 + 4\nu^2\omega_1^2}\|f\|_2 < \|f\|_2$$

At this point it can be easily seen that for $2\nu\omega_1 < 1$ $D = \sum_{k=0}^\infty (C_{h_{\omega_1}} - A_w)^k$ is a continuous linear operator. Combining the norm estimate of Lemma 2 with the factorization of convolution we arrive at the main theorem.

Theorem 3 *Let ω_1, ω_0 be constants with $\omega_1 > \omega_0 > 0$. Choose $\nu > 0$ such that $2\nu\omega_1 < 1$. Then, any $f \in B^2_{\omega_0}(\mathbb{R})$ can be completely reconstructed from its sampled values on any ν-dense sequence $\{t_n\}_{n\in\mathbb{Z}}$ and there are functions $e_n \in L^2(\mathbb{R})$ with $supp(\hat{e}_n) \subseteq [-\omega_1, \omega_1]$, such that*

$$f = \sum_{n\in\mathbb{Z}} f(t_n) e_n \,, \tag{1}$$

where the sum converges uniformly and in L^2.

In the proof of Theorem 3 we obtain the sampling atoms e_n by choosing a ω_0-reproducing function h_{ω_1} and by setting $e_n(t) = D(h_{\omega_1}(t - t_n))$, for all $n \in \mathbb{Z}$. Following the definition of the operator D, we present the A_w-algorithm for iterative approximation of the sampling atoms.

Theorem 4 (The A_w-Algorithm)
Choose a ω_0-reproducing function h_{ω_1} and let $\{t_n\}_{n\in\mathbb{Z}}$ be a ν-dense sampling sequence . If $\nu < (2\omega_1)^{-1}$, the sampling atoms e_n, $n \in \mathbb{Z}$ can be computed by the following algorithm.

$$e_n^{(0)}(t) = w_n h_{\omega_1}(t - t_n)$$
$$e_n^{(m)} = e_n^{(0)} + (C_{h_{\omega_1}} - A_w) e_n^{(m-1)}$$

with $e_n = \lim_{m\to\infty} e_n^{(m)}$ uniformly and in L^2. Moreover,

$$\|e_n - e_n^{(m)}\|_2 \leq \left(\frac{4\nu\omega_1}{1 + 4\nu^2\omega_1^2}\right)^{m+1} \|e_n\|_2 \,.$$

In contrast to the general approach of [4] and [5], we have explicit norm estimates but no statement confirming the rapid decay of the sampling atoms. However, the use of a rapid decreasing reproducing function and the definition of both A_w and D encourage the assumption that the e_n are well-localized as we confirm by the numerical examples given below.

4. NUMERICAL RESULTS

For practical purposes we have to establish a finite-dimensional model for the problem to be accessible for numerical solutions.

We identify the index set $\{0, 1, \ldots, N-1\}$ with the cyclic group $\mathbb{Z}_N$, and all signals defined on $\mathbb{Z}_N$ are understood as periodic sequences with period N. The l^2-norm $\|f\|_2 = (\sum_{n=0}^{N-1} |f(n)|^2)^{1/2}$ represents the energy of the signal f. On $\mathbb{Z}_N$ the discrete Fourier transform is given by $\hat{f}(k) = (N)^{-1/2} \sum_{n=0}^{N-1} f(n) e^{-2\pi ikn/N}$. We define the space of discrete bandlimited functions of bandwidth $M_0 < N/2$ by

$$B_{M_0} = \{f(n) \in l^2(\mathbb{Z}_N) : \hat{f}(k) = 0 \text{ for } |k| > M_0\}.$$

The sampling set $\{n_i\}_{i=1}^r$ is an arbitrary sequence of integers satisfying $0 \le n_1 < n_2 < \cdots < n_r \le N-1$. The sampling density d is given by the maximal gap length $d = \max_{i=1,\ldots,r}(n_{i+1} - n_i)$, where $n_{r+1} = n_1 + N$ by periodicity.

Definition 5 *Given $0 < M_0 < M_1 < N/2$, the function $h_{M_1} \in l^2(\mathbb{Z})$ is called a discrete M_0-reproducing function if h_{M_1} is in B_{M_1} and has the following properties.*

(a) $0 \le |\hat{h}_{M_1}| \le 1$ *and* $\hat{h}_{M_1} \equiv 1$ *on* $[-M_0, M_0]$,

(b) h *is ϵ-concentrated on an interval I, i.e.,* $\|h - h\chi_I\|_2 \le \epsilon$ *for a small threshold* $\epsilon > 0$.

Given a sampling set $\{n_i\}_{i=1}^r$ we denote the midpoints of the sampling points by $l_i = [(n_i + n_{i+1})/2]$ for $i = 1, \ldots, r$, where $[q]$ is the largest integer $\le q \in \mathbb{R}$. Choosing a M_0-reproducing function h_{M_1} and setting $\Delta_i = l_{i+1} - l_i$, we define the A_w-operator for $f \in l^2(\mathbb{Z}_N)$ by

$$A_w f(n) = \frac{1}{1+\gamma^2} \sum_{i=1}^{r} f(n_i) \Delta_i h_{M_1}(n - n_i),$$

where $\gamma = (\sin \frac{\pi M_1}{N}) / (\sin \frac{\pi}{4[d/2]+2})$.
For sufficiently dense sampling sets the A_w-operator is close to the convolution operator $C_{h_{M_1}}$ on B_{M_1}, where

$$C_{h_{M_1}} f(k) = \sum_{n=0}^{N-1} f(n) h_{M_1}(k - n), \ k = 1, \ldots, N-1.$$

In the same way as in the previous section we can now proof a similar result.

Theorem 6 *Let $0 \le n_1 < n_2 < \cdots < n_r \le N-1$ be a sampling sequence of density d and M_1, M_0 be integers with $0 < M_0 < M_1 < N/2$. If $2M_1(2[d/2]+1) < N$, then there exist discrete sampling atoms $e_i \in B_{M_1}$ such that every $f \in B_{M_0}$ has a series representation of the form*

$$f(n) = \sum_{i=0}^{r} f(n_i) e_i(n), \qquad n = 0, 1 \ldots, N-1.$$

The sampling atoms can be approximated iteratively by the A_w-algorithm.

$$e_i^{(0)}(n) = \Delta_i h_{M_1}(n - n_i), \ n = 0, \ldots, N-1$$

$$e_i^{(m)} = e_i^{(0)} + (C_{h_{M_1}} - A_w) e_i^{(m-1)}$$

The algorithm converges to e_i geometrically with the rate $\lambda = 2\gamma(1+\gamma^2)^{-1}$.

The A_w-algorithm can be implemented in the same form as it appears in Theorem 6. It shows good performance also for sampling sets that do not satisfy the worst-case condition $2M_1(2[d/2]+1) < N$.

In the sequel we use the A_w-algorithm to compute the sampling atoms associated with a local set of sampling points in order to demonstrate local reconstruction of bandlimited signals from local samples by means of well-localized sampling atoms.

The starting point is a normalized discrete function of length $N = 512$ and bandwidth $M_0 = 10$ sampled at a random sequence of points between 150 and 350 with maximal gapsize $d = 10$.

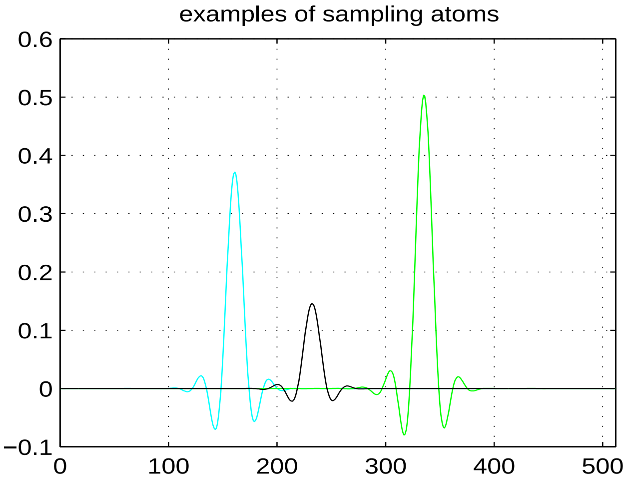

Figure 1: Examples of sampling atoms associated with the 2nd, the 15th, and the 29th sampling point.

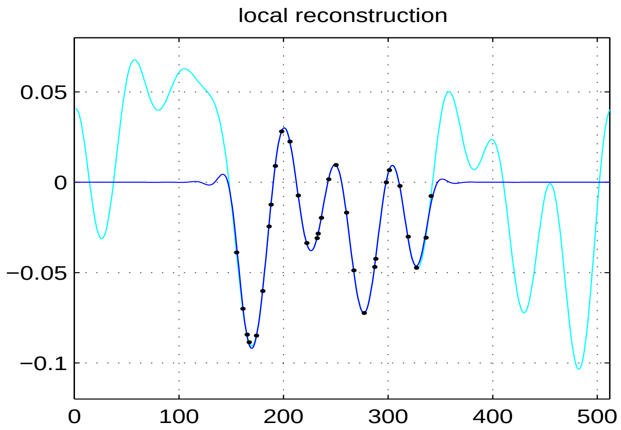

Figure 2: Example of local reconstruction using well-localized sampling atoms.

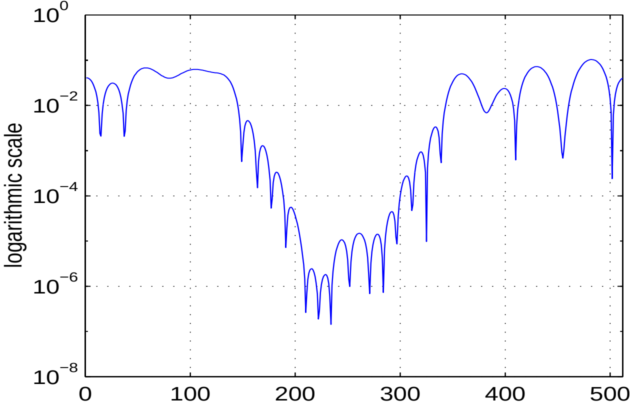

Figure 3: Error of reconstruction from local sampling information using well-localized sampling atoms.

Figure 3 shows the behavior of the local approximation on the selected interval. The best approximation is obtained in the middle of the interval. The local error can be improved by filling up the local sampling information in order to decrease the maximal gapsize as well as by adding random sampling points outside the interval without changing the sampling density.

5. SUMMARY

In Theorem 3 we show that the explicit worst-case estimate for the reconstruction of bandlimited L^2-functions from non-uniform samples by means of synthesis functions depends on the sampling density and the bandwidth of the auxiliary reproducing function. The results coincide qualitatively with the regular case.

For practical purposes it will be interesting to extend the presented results to higher dimensions.

REFERENCES

[1] J. J. Benedetto, "Irregular sampling and frames", in Wavelets: A Tutorial in Theory and Application, C. K. Chui, ed, Academic Press, 1992, pp. 445-507.

[2] L. Butzer, W. Engels, S. Ries, and R. L. Stens, "The Shannon sampling series and the reconstrution of signals in terms of linear, quadratic and cubic splines", SIAM J. Appl. Math., 46 (1986), pp. 299-323.

[3] L. Butzer, W. Splettstösser and R. L. Stens, "Sampling theorem and linear prediction in signal analysis", Jber.d.Dt.Math.-Verein, 90 (1988), pp. 1-70.

[4] H. G. Feichtinger and K. Gröchenig, "Iterative reconstruction of multivariate band-limited functions from irregular sampling values", SIAM J. Math. Anal., 23(1992), pp. 244-261.

[5] H. G. Feichtinger and K. Gröchenig, "Irregular sampling theorems and series expansion of band-limited functions", J. Math. Anal. Appl., 167 (1992), pp. 530-556.

[6] H. G. Feichtinger, K. Gröchenig, and T. Strohmer, "Efficient numerical methods in non-uniform sampling theory", Numer. Math., 69(1995), pp. 423-440.

[7] K. M. Flornes, Y. Lyubarskii, and K. Seip, "A direct interpolation method for irregular sampling", Appl. and Comp. Harm. Anal., 7(1999), pp. 305-314.

[8] K. Gröchenig, "Reconstruction algorithms in irregular sampling", Numer. Math., 59(1992), pp. 181-194.

[9] F. Marvasti and M. Analoui, "Recovery of signals from non-uniform samples using iterative methods", Proc. Int. Symp. Circuits Syst., Portland, OR, 1989.

[10] A. Papoulis, "Signal analysis", McGraw-Hill, NY, 1977.

[11] T. Werther, "Reconstruction from irregular samples with improved locality", Master's thesis, University of Vienna, Dec. 1999.